The Brauer group of Kummer surfaces and torsion of elliptic curves

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Von einem *Ellipsoid* habe ich eine Vorstellung, vor einer *Kummerschen Fläche* nicht; aber durch passende Zeichnungen, durch Modelle oder durch theoretisch geleitete Bewegungen der Phantasie kann ich auch von ihr eine Vorstellung gewinnen.

Edmund Husserl, "Logische Untersuchungen" V, §44

Introduction

In this paper we are interested in computing the Brauer group of K3 surfaces. To an element of the Brauer–Grothendieck group $\mathrm{Br}(X)$ of a smooth projective variety X over a number field k class field theory associates the corresponding Brauer–Manin obstruction, which is a closed condition satisfied by k-points inside the topological space of adelic points of X, see [20], Chapter 5.2. If such a condition is non-trivial, X is a counterexample to weak approximation, and if no adelic point satisfies this condition, X is a counterexample to the Hasse principle. The computation of $\mathrm{Br}(X)$ is thus a first step in the computation of the Brauer–Manin obstruction on X.

Let k be an arbitrary field with a separable closure $\bar{k}$, $\Gamma = \mathrm{Gal}(\bar{k}/k)$. Recall that for a variety X over k the subgroup $\mathrm{Br}_0(X) \subset \mathrm{Br}(X)$ denotes the image of $\mathrm{Br}(k)$ in $\mathrm{Br}(X)$, and $\mathrm{Br}_1(X) \subset \mathrm{Br}(X)$ denotes the kernel of the natural map $\mathrm{Br}(X) \to \mathrm{Br}(\overline{X})$, where $\overline{X} = X \times_k \bar{k}$. In [22] we showed that if X is a K3 surface over a field k finitely generated over $\mathbb{Q}$, then $\mathrm{Br}(X)/\mathrm{Br}_0(X)$ is finite. No general approach to the computation of $\mathrm{Br}(X)/\mathrm{Br}_0(X)$ seems to be known; in fact until recently there was not a single K3 surface over a number field for which $\mathrm{Br}(X)/\mathrm{Br}_0(X)$ was known. One of the aims of this paper is to give examples of K3 surfaces X over $\mathbb{Q}$ such that $\mathrm{Br}(X) = \mathrm{Br}(\mathbb{Q})$.

We study a particular kind of K3 surfaces, namely Kummer surfaces $X = \mathrm{Kum}(A)$ constructed from abelian surfaces A. Let $\mathrm{Br}(X)_n$ denote the n-torsion subgroup of $\mathrm{Br}(X)$. Section 1 is devoted to the geometry of Kummer surfaces. We show that there is a natural isomorphism of Γ-modules $\mathrm{Br}(\overline{X}) \xrightarrow{\sim} \mathrm{Br}(\overline{A})$ (Proposition 1.3). When A is a product of two elliptic curves, the algebraic Brauer group $\mathrm{Br}_1(X)$ often coincides with $\mathrm{Br}(k)$, see Proposition 1.4.

Section 2 starts with a general remark on the étale cohomology of abelian varieties that may be of independent interest (Proposition 2.2). It implies that if n is an odd integer, then for any abelian variety A the group $\mathrm{Br}(A)_n/\mathrm{Br}_1(A)_n$ is canonically isomorphic to the quotient of $\mathrm{H}^2_{\text{ét}}(\bar{A}, \mu_n)^\Gamma$ by $\left(\mathrm{NS}(\bar{A})/n\right)^\Gamma$, where $\mathrm{NS}(\bar{A})$ is the Néron–Severi group (Corollary 2.3). For any $n \geqq 1$ we prove that $\mathrm{Br}(X)_n/\mathrm{Br}_1(X)_n$ is a subgroup of $\mathrm{Br}(A)_n/\mathrm{Br}_1(A)_n$, and this inclusion is an equality for odd n, see Theorem 2.4. We deduce that the subgroups of elements of odd order of the transcendental Brauer groups $\mathrm{Br}(X)/\mathrm{Br}_1(X)$ and $\mathrm{Br}(A)/\mathrm{Br}_1(A)$ are naturally isomorphic.

More precise results are obtained in Section 3 in the case when $A = E \times E'$ is a product of two elliptic curves. In this case for any $n \geqq 1$ we have

$$\mathrm{Br}(A)_n/\mathrm{Br}_1(A)_n = \mathrm{Hom}_\Gamma(E_n, E'_n)/\left(\mathrm{Hom}(\bar{E}, \bar{E'})/n\right)^\Gamma$$

(Proposition 3.3). This gives a convenient formula for $\mathrm{Br}(X)_n/\mathrm{Br}_1(X)_n$ when n is odd. See Proposition 3.7 for the case $n = 2$.

In Section 4 we find many pairs of elliptic curves E, E' over $\mathbb{Q}$ such that for $A = E \times E'$ the group $\mathrm{Br}(A)/\mathrm{Br}_1(A)$ is zero or a finite abelian 2-group. For example, if E is an elliptic curve over $\mathbb{Q}$ such that for all primes ℓ the representation $\Gamma \to \mathrm{Aut}(E_\ell) \simeq \mathrm{GL}(2, \mathbb{F}_\ell)$ is surjective, then for $A = E \times E$ we have $\mathrm{Br}(A) = \mathrm{Br}_1(A)$, whereas $\mathrm{Br}(\bar{A})^\Gamma \simeq \mathbb{Z}/2^m$ for some $m \geqq 1$ (Proposition 4.3). This shows, in particular, that the Hochschild–Serre spectral sequence $\mathrm{H}^p\left(k, \mathrm{H}^q_{\text{ét}}(\bar{A}, \mathbf{G}_m)\right) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(A, \mathbf{G}_m)$ does not degenerate. For this A the corresponding Kummer surface $X = \mathrm{Kum}(A)$ has trivial Brauer group $\mathrm{Br}(X) = \mathrm{Br}(\mathbb{Q})$ (whereas $\mathrm{Br}(\bar{X})^\Gamma \simeq \mathbb{Z}/2^m$ for some $m \geqq 1$). Note that by a theorem of W. Duke [2] most elliptic curves over $\mathbb{Q}$ have this property, see the remark after Proposition 4.3.

In Section 5 we discuss the resulting infinitely many Kummer surfaces X over $\mathbb{Q}$ such that $\mathrm{Br}(X) = \mathrm{Br}(\mathbb{Q})$, see (25)–(29) and Examples 3 and 4. In fact most Kummer surfaces $\mathrm{Kum}(E \times E')$ over $\mathbb{Q}$ have trivial Brauer group, see Example A2 in Section 4. We also exhibit Kummer surfaces X with an element of prime order $\ell \leqq 13$ in $\mathrm{Br}(X)$ which is not in $\mathrm{Br}_1(X)$. Finally, we discuss the Brauer group of Kummer surfaces that do not necessarily have rational points.

In the follow up paper [6] with Evis Ieronymou we give an upper bound on the size of $\mathrm{Br}(X)/\mathrm{Br}_0(X)$, where X is a smooth diagonal quartic surface in $\mathbb{P}^3_\mathbb{Q}$, and give examples when $\mathrm{Br}(X) = \mathrm{Br}(\mathbb{Q})$. The importance of K3 surfaces over $\mathbb{Q}$ such that $\mathrm{Br}(X) = \mathrm{Br}(\mathbb{Q})$ is that there is no Brauer–Manin obstruction, and so the Hasse principle and weak approximation for $\mathbb{Q}$-points can be tested by numerical experiments. It would be even more interesting to get theoretical evidence for or against the Hasse principle and weak approximation on such surfaces.

1. Picard and Brauer groups of Kummer surfaces over an algebraically closed field

Let k be a field of characteristic zero with an algebraic closure $\bar{k}$, and the absolute Galois group $\Gamma = \mathrm{Gal}(\bar{k}/k)$. Let X be a smooth and geometrically integral variety over k, and let $\bar{X} = X \times_k \bar{k}$. Let $\mathrm{Br}(X) = \mathrm{H}^2_{\text{ét}}(X, \mathbf{G}_m)$ be the Brauer group of X, and let

$\mathrm{Br}(\overline{X}) = \mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbf{G}_m)$ be the Brauer group of $\overline{X}$. For any prime number ℓ the Kummer sequence

$$1 \to \mu_{\ell^n} \to \mathbf{G}_m \to \mathbf{G}_m \to 1$$

gives rise to the exact sequence of abelian groups

$$0 \to \mathrm{Pic}(X) \otimes \mathbb{Z}/\ell^n \to \mathrm{H}^2_{\text{ét}}(X, \mu_{\ell^n}) \to \mathrm{Br}(X)_{\ell^n} \to 0,$$

and an exact sequence of Γ-modules

$$(1) \qquad 0 \to \mathrm{Pic}(\overline{X}) \otimes \mathbb{Z}/\ell^n \to \mathrm{H}^2_{\text{ét}}(\overline{X}, \mu_{\ell^n}) \to \mathrm{Br}(\overline{X})_{\ell^n} \to 0.$$

If X is projective, then $\mathrm{Pic}(\overline{X}) \otimes \mathbb{Z}/\ell^n = \mathrm{NS}(\overline{X}) \otimes \mathbb{Z}/\ell^n$, where $\mathrm{NS}(\overline{X})$ is the Néron–Severi group of $\overline{X}$. So in this case we have an exact sequence of Γ-modules

$$(2) \qquad 0 \to \mathrm{NS}(\overline{X}) \otimes \mathbb{Z}/\ell^n \to \mathrm{H}^2_{\text{ét}}(\overline{X}, \mu_{\ell^n}) \to \mathrm{Br}(\overline{X})_{\ell^n} \to 0.$$

Passing to the projective limit in (2) we obtain an embedding of Γ-modules

$$\mathrm{NS}(\overline{X}) \otimes \mathbb{Z}_\ell \hookrightarrow \mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(1)).$$

The Néron–Severi group of an abelian variety or a K3 surface is torsion free, so in these cases $\mathrm{NS}(\overline{X})$ is a submodule of $\mathrm{H}^2_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(1))$.

Remark. Let ρ be the rank of $\mathrm{NS}(\overline{X})$, and let b_2 be the second Betti number of $\overline{X}$. It is known ([3], Corollary 3.4, p. 82; [11], Chapter 5, Remark 3.29, pp. 216–217) that the ℓ-primary component $\mathrm{Br}(\overline{X})(\ell) \subset \mathrm{Br}(\overline{X})$ is an extension of $\mathrm{H}^3_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(1))_{\text{tors}}$ by $(\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{b_2 - \rho}$. By Poincaré duality, if X is a surface such that $\mathrm{NS}(\overline{X})$ has no ℓ-torsion, then $\mathrm{Br}(\overline{X})(\ell) \simeq (\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{b_2 - \rho}$. It follows that if X is an abelian variety or a K3 surface we have $\mathrm{Br}(\overline{X}) \simeq (\mathbb{Q}/\mathbb{Z})^{b_2 - \rho}$.

We write $k[X]$ for the k-algebra of regular functions on X, and $k[X]^*$ for the group of invertible regular functions. We state the following well known fact for future reference.

Lemma 1.1. *Let X be a smooth and geometrically integral variety over k, and let $U \subset X$ be an open subset whose complement in X has codimension at least 2. Then the natural restriction maps*

$$k[X] \to k[U], \quad \mathrm{Pic}(X) \to \mathrm{Pic}(U), \quad \mathrm{Br}(X) \to \mathrm{Br}(U)$$

are isomorphisms.

Proof. The first two statements are clear, and the last one follows from Grothendieck's purity theorem, see [3], Corollary 6.2, p. 136. $\square$

For an abelian variety A we denote by A_n the kernel of the multiplication by n map $[n] : A \to A$. Let ι be the antipodal involution on A, $\iota(x) = -x$. The set of fixed points of ι is A_2.

Assume now that A is an abelian surface. Let $A_0 = A \backslash A_2$ be the complement to A_2, and let $X_0 = A_0/\iota$. The surface X_0 is smooth and the morphism $A_0 \to X_0$ is a torsor under $\mathbb{Z}/2$. Let X be the surface obtained by blowing-up the singular points of A/ι. Then X can be viewed as a smooth compactification of X_0; the complement to X_0 in X is a closed sub-variety of dimension 1 which splits over $\bar{k}$ into a disjoint union of 16 smooth rational curves with self-intersection -2. We shall call X the Kummer surface attached to A, and write $X = \mathrm{Kum}(A)$.

Let A' be the surface obtained by blowing-up the subscheme A_2 in A, and let $\sigma : A' \to A$ be the resulting birational morphism. Let $\pi : A' \to X$ be the natural finite mor-phism of degree 2 ramified at $X \backslash X_0$ (cf. [14]). The set $A_2(\bar{k})$ is the disjoint union of Γ-orbits $\Lambda_1, \dots, \Lambda_r$. One may view each Λ_i as a closed point of A with residue field K_i. Then $M_i = \sigma^{-1}(\Lambda_i)$ in A' is the projective line $\mathbb{P}^1_{K_i}$ (cf. [9], Chapter III, Theorem 2.4 and Remark 2.5). It follows that $L_i = \pi(M_i)$ is also isomorphic to $\mathbb{P}^1_{K_i}$.

Since $\sigma : A' \to A$ is a monoidal transformation with smooth centre, the induced maps $\sigma^* : \mathrm{Br}(A) \to \mathrm{Br}(A')$ and $\sigma^* : \mathrm{Br}(\bar{A}) \to \mathrm{Br}(\bar{A}')$ are isomorphisms by [3], Corollary 7.2, p. 138. Let $Y \subset A'$ be an open subset containing A_0. The composition of injective maps

$$\mathrm{Br}(A) \xrightarrow{\sim} \mathrm{Br}(A') \to \mathrm{Br}(Y) \to \mathrm{Br}(A_0)$$

is an isomorphism by Lemma 1.1, and the same is true after the base change from k to $\bar{k}$. It follows that the following restriction maps are isomorphisms:

$$(3) \qquad \mathrm{Br}(A') \xrightarrow{\sim} \mathrm{Br}(Y), \quad \mathrm{Br}(\bar{A}') \xrightarrow{\sim} \mathrm{Br}(\bar{Y}).$$

This easily implies that the natural homomorphisms $\mathrm{Br}_1(A) \to \mathrm{Br}_1(A') \to \mathrm{Br}_1(Y)$ are iso-morphisms. We also obtain isomorphisms

$$(4) \qquad \mathrm{Br}(A)_n/\mathrm{Br}_1(A)_n \xrightarrow{\sim} \mathrm{Br}(A')_n/\mathrm{Br}_1(A')_n \xrightarrow{\sim} \mathrm{Br}(Y)_n/\mathrm{Br}_1(Y)_n.$$

Throughout the paper, we will freely use these isomorphisms, identifying the corresponding groups.

Proposition 1.2. *Let $X_1 \subset X$ be the complement to the union of some of the irreduc-ible components of $X \backslash X_0$ (that is, some of the lines L_i). Then there is an exact sequence*

$$(5) \qquad 0 \to \mathrm{Br}(X) \to \mathrm{Br}(X_1) \to \bigoplus_i K_i^*/K_i^{*2},$$

where the sum is over i such that $L_i \subset X \backslash X_1$. In particular, the restriction map $\mathrm{Br}(X) \to \mathrm{Br}(X_1)$ induces an isomorphism of the subgroups of elements of odd order. The restriction map $\mathrm{Br}(\bar{X}) \xrightarrow{\sim} \mathrm{Br}(\bar{X}_1)$ is an isomorphism of Γ-modules.

Proof. Let $Y = \pi^{-1}(X_1)$. From Grothendieck's exact sequence ([3], Corollary 6.2, p. 137) we obtain the following commutative diagram with exact rows:

$$(6)$$

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Br}(X) & \longrightarrow & \mathrm{Br}(X_1) & \longrightarrow & \bigoplus_i \mathrm{H}^1(L_i, \mathbb{Q}/\mathbb{Z}) \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Br}(A') & \xrightarrow{\sim} & \mathrm{Br}(Y) & \longrightarrow & \bigoplus_i \mathrm{H}^1(M_i, \mathbb{Q}/\mathbb{Z})
\end{array}
$$

where both sums are over i such that $L_i \subset X \backslash X_1$. Recall that the restriction map $\mathrm{Br}(A') \to \mathrm{Br}(Y)$ is an isomorphism by (3), hence the right bottom arrow is zero. Let $\mathrm{res}_{L_i} : \mathrm{Br}(X_1) \to \mathrm{H}^1(L_i, \mathbb{Q}/\mathbb{Z})$ and $\mathrm{res}_{M_i} : \mathrm{Br}(Y) \to \mathrm{H}^1(M_i, \mathbb{Q}/\mathbb{Z})$ be the residue maps from (6). The double covering $\pi : A' \to X$ is ramified at L_i, thus $\mathrm{res}_{M_i}(\pi^*\alpha) = 2\,\mathrm{res}_{L_i}(\alpha)$. But this is zero, so that $\mathrm{res}_{L_i}(\alpha)$ belongs to the injective image of $\mathrm{H}^1_{\text{ét}}(\mathbb{P}^1_{K_i}, \mathbb{Z}/2)$ in $\mathrm{H}^1_{\text{ét}}(\mathbb{P}^1_{K_i}, \mathbb{Q}/\mathbb{Z})$. Since $\mathrm{H}^1_{\text{ét}}(\mathbb{P}^1_{\bar{k}}, \mathbb{Z}/2) = 0$ we deduce from the Hochschild–Serre spectral sequence

$$\mathrm{H}^p\big(K_i, \mathrm{H}^q_{\text{ét}}(\mathbb{P}^1_{\bar{k}}, \mathbb{Z}/2)\big) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(\mathbb{P}^1_{K_i}, \mathbb{Z}/2)$$

that $\mathrm{H}^1(\mathbb{P}^1_{K_i}, \mathbb{Z}/2) = K_i^*/K_i^{*2}$. This establishes the exact sequence (5).

The same theorem of Grothendieck [3], Corollary 6.2, gives an exact sequence

$$0 \to \mathrm{Br}(\overline{X}) \to \mathrm{Br}(\overline{X}_1) \to \bigoplus_i \mathrm{H}^1(L_i \times_k \bar{k}, \mathbb{Q}/\mathbb{Z}) = 0.$$

Since $L_i \times_k \bar{k}$ is a disjoint union of finitely many copies of $\mathbb{P}^1_{\bar{k}}$, and $\mathrm{H}^1_{\text{ét}}(\mathbb{P}^1_{\bar{k}}, \mathbb{Q}/\mathbb{Z}) = 0$, this implies the last statement of the proposition. $\square$

Proposition 1.3. *The natural map* $\pi^* : \mathrm{Br}(\overline{X}_0) \to \mathrm{Br}(\overline{A}_0)$ *is an isomorphism, so that the composed map* $(\sigma^*)^{-1}\pi^* : \mathrm{Br}(\overline{X}) \to \mathrm{Br}(\overline{A})$ *is an isomorphism of* Γ-modules.

Proof. Since $\pi : A_0 \to X_0$ is a torsor under $\mathbb{Z}/2$ we have the Hochschild–Serre spectral sequence [11], Theorem III.2.20,

$$\mathrm{H}^p\big(\mathbb{Z}/2, \mathrm{H}^q_{\text{ét}}(\overline{A}_0, \mathbf{G}_m)\big) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(\overline{X}_0, \mathbf{G}_m).$$

Let us compute a first few terms of this sequence. By Lemma 1.1 we have

$$\bar{k}[A_0]^* = \bar{k}^*, \quad \mathrm{Pic}(\overline{A}_0) = \mathrm{Pic}(\overline{A}), \quad \mathrm{Br}(\overline{A}_0) = \mathrm{Br}(\overline{A}).$$

Since k has characteristic 0, and $\mathbb{Z}/2$ acts trivially on $\bar{k}^*$, the Tate cohomology group $\hat{\mathrm{H}}^0(\mathbb{Z}/2, \bar{k}^*)$ is trivial. We have $\mathrm{H}^1(\mathbb{Z}/2, \bar{k}^*) = \mathbb{Z}/2$. By the periodicity of group cohomology of cyclic groups we obtain $\mathrm{H}^2(\mathbb{Z}/2, \bar{k}^*) = 0$.

We have an exact sequence of Γ-modules

$$0 \to A^t(\bar{k}) \to \mathrm{Pic}(\overline{A}) \to \mathrm{NS}(\overline{A}) \to 0,$$

where A^t is the dual abelian surface. The torsion-free abelian group $\mathrm{NS}(\overline{A})$ embeds into $\mathrm{H}^2_{\text{ét}}(\overline{A}, \mathbb{Z}_\ell(1))$, and since the antipodal involution ι acts trivially on $\mathrm{H}^2_{\text{ét}}(\overline{A}, \mathbb{Z}_\ell(1))$, it acts trivially on $\mathrm{NS}(\overline{A})$, too. Thus $\mathrm{H}^1(\mathbb{Z}/2, \mathrm{NS}(\overline{A})) = 0$, so that $\mathrm{H}^1(\mathbb{Z}/2, \mathrm{Pic}(\overline{A}))$ is the image of $\mathrm{H}^1(\mathbb{Z}/2, A^t(\bar{k}))$. Since ι acts on A^t as multiplication by -1, we have

$$\mathrm{H}^1\big(\mathbb{Z}/2, A^t(\bar{k})\big) = A^t(\bar{k})/(1-\iota)A^t(\bar{k}) = 0.$$

Putting all this into the spectral sequence and using Proposition 1.2 with $X_1 = X_0$ we obtain an embedding $\mathrm{Br}(\overline{X}) \hookrightarrow \mathrm{Br}(\overline{A})$. In order to prove that this is an isomorphism, it suffices

to check that the corresponding embeddings $\mathrm{Br}(\overline{X})_{\ell^n} \hookrightarrow \mathrm{Br}(\overline{A})_{\ell^n}$ are isomorphisms for all primes ℓ and all positive integers n. It is well known that $b_2(\overline{X}) = 22$, $b_2(\overline{A}) = 6$ and $\rho(\overline{X}) = \rho(\overline{A}) + 16$, see, e.g., [15] or [14]. From this and the remark before Lemma 1.1 it follows that the orders of $\mathrm{Br}(\overline{X})_{\ell^n}$ and $\mathrm{Br}(\overline{A})_{\ell^n}$ are the same. This finishes the proof. $\qquad\square$

Remark 1. The same spectral sequence gives an exact sequence of Γ-modules

$$(7) \qquad 0 \to \mathbb{Z}/2 \to \mathrm{Pic}(\overline{X}_0) \to \mathrm{Pic}(\overline{A})^\iota \to 0,$$

where $\mathrm{Pic}(\overline{A})^\iota$ is the ι-invariant subgroup of $\mathrm{Pic}(\overline{A})$. From (7) we deduce the exact sequence

$$(8) \qquad 0 \to \mathbb{Z}/2 \to \mathrm{Pic}(\overline{X}_0)_{\mathrm{tors}} \to A_2^\iota \to 0.$$

Let $\mathbb{Z}^{16} \subset \mathrm{Pic}(\overline{X}) = \mathrm{NS}(\overline{X})$ be the lattice generated by the classes of the 16 lines. Its saturation Π in $\mathrm{NS}(\overline{X})$ is called the Kummer lattice. In other words, Π is the subgroup of $\mathrm{NS}(\overline{X})$ consisting of linear combinations of the classes of the 16 lines with coefficients in $\mathbb{Q}$. Since $\mathrm{NS}(\overline{X})/\mathbb{Z}^{16} = \mathrm{Pic}(\overline{X}_0)$, we have $\Pi/\mathbb{Z}^{16} = \mathrm{Pic}(\overline{X}_0)_{\mathrm{tors}}$. It follows from (8) that $[\Pi : \mathbb{Z}^{16}] = 2^5$. Since the 16 lines are disjoint, and each of them has self-intersection -2, the discriminant of $\mathbb{Z}^{16}$ is 2^{16}. Thus the discriminant of Π is 2^6, as was first observed in [15], Lemma 4 on p. 555, see also [14].

Remark 2. Considering (7) modulo torsion and taking into account that $\mathrm{NS}(\overline{A})^\iota = \mathrm{NS}(\overline{A})$ and $\mathrm{H}^1\big(\mathbb{Z}/2, A^\iota(\overline{k})\big) = 0$, we obtain an isomorphism

$$\mathrm{NS}(\overline{X})/\Pi = \mathrm{Pic}(\overline{X}_0)/\mathrm{Pic}(\overline{X}_0)_{\mathrm{tors}} \xrightarrow{\sim} \mathrm{NS}(\overline{A}).$$

In other words, we have an exact sequence of Γ-modules

$$(9) \qquad 0 \longrightarrow \Pi \longrightarrow \mathrm{NS}(\overline{X}) \xrightarrow{\sigma_* \pi^*} \mathrm{NS}(\overline{A}) \longrightarrow 0.$$

Remark 3. Recall that A_2 acts on X and X_0 compatibly with its action on A by translations, moreover, the morphisms π and σ are A_2-equivariant. Thus the isomorphism $(\sigma^*)^{-1}\pi^* : \mathrm{Br}(\overline{X}) \xrightarrow{\sim} \mathrm{Br}(\overline{A})$ is also A_2-equivariant. Since translations of an abelian variety act trivially on its cohomology, the exact sequence (2) shows that the induced action of A_2 on $\mathrm{Br}(\overline{A})$ is trivial. We conclude that the induced action of A_2 on $\mathrm{Br}(\overline{X})$ is also trivial.

Let us now assume that X is the Kummer surface constructed from the abelian surface $A = E \times E'$, where E and E' are elliptic curves. For a divisor D we write $[D]$ for the class of D in the Picard group.

Let $C \subset \overline{A}$ be a curve, and let $p : C \to E$, $p' : C \to E'$ be the natural projections. Then $p'_* p^* : \mathrm{Pic}^0(\overline{E}) \to \mathrm{Pic}^0(\overline{E'})$ defines a homomorphism $\overline{E} \to \overline{E'}$. This gives a well known isomorphism of Galois modules

$$(10) \qquad \mathrm{NS}(\overline{A}) = \mathbb{Z}[e] \oplus \mathbb{Z}[e'] \oplus \mathrm{Hom}(\overline{E}, \overline{E'}),$$

where $e = E \times \{0\}$ and $e' = \{0\} \times E'$, and the Γ-module $\mathrm{Hom}(\overline{E}, \overline{E'})$ is realised inside $\mathrm{NS}(\overline{A})$ as the orthogonal complement to $\mathbb{Z}[e] \oplus \mathbb{Z}[e']$ with respect to the intersection pairing.

For a curve $C \subset A$ we denote by $\sigma^{-1}C \subset A'$ its strict transform in A' (i.e. the Zariski closure of $C \cap A_0$ in A'). In particular, if C does not contain a point of order 2 in A, then $\sigma^{-1}C$ does not meet the corresponding line in A', and hence $\pi(\sigma^{-1}C)$ does not meet the corresponding line in X.

We write the $\bar{k}$-points in E_2 as $\{o, 1, 2, 3\}$ with the convention that o is the origin of the group law, and similarly for E_2'. The divisors $\{i\} \times E'$, $E \times \{j\}$, where $i \in E_2$, $j \in E_2'$, are ι-invariant, thus there are rational curves s_j and l_i in $\overline{X}$ such that π induces double coverings

$$\sigma^{-1}(E \times \{j\}) \to s_j, \quad \sigma^{-1}(\{i\} \times E') \to l_i.$$

Let l_{ij} be the line in $\overline{X}$ corresponding to the 2-division point $(i, j) \in A_2$. Note that $\sigma_* \pi^*$ sends $[s_j]$, $[l_i]$, $[l_{ij}]$ to $[e]$, $[e']$, 0, respectively. Finally, let

$$f_1 = 2l_o + l_{oo} + l_{o1} + l_{o2} + l_{o3}, \quad f_2 = 2s_o + l_{oo} + l_{1o} + l_{2o} + l_{3o}.$$

Consider the following 9-element Galois-invariant subsets of $\mathrm{NS}(\overline{X})$:

$$\Lambda = \{[l_{ij}]\}, \quad \Sigma = \{[f_1], [f_2], [l_o], [l_i], [s_j]\},$$

where i and j take all values in $\{1, 2, 3\}$. Let N_Λ (resp. N_Σ) be the subgroup of $\mathrm{NS}(\overline{X})$ generated by Λ (resp. by Σ).

Proposition 1.4. *Let $A = E \times E'$, where E and E' are elliptic curves, and let $X = \mathrm{Kum}(A)$.*

(i) *The set Λ (resp. Σ) is a Γ-invariant basis of N_Λ (resp. of N_Σ). There is an exact sequence of Γ-modules*

(11) $$0 \to N_\Lambda \oplus N_\Sigma \to \mathrm{NS}(\overline{X}) \to \mathrm{Hom}(\overline{E}, \overline{E'}) \to 0.$$

(ii) *We have $\mathrm{Br}_1(X) = \mathrm{Br}(k)$ in each of the following cases*:

E and E' are not isogenous over $\bar{k}$.

$E = E'$ is an elliptic curve without complex multiplication over $\bar{k}$.

$E = E'$ is an elliptic curve which, over $\bar{k}$, has complex multiplication by an order $\mathcal{O}$ of an imaginary quadratic field K, that is, $\mathrm{End}(\overline{E}) = \mathcal{O}$, and, moreover, $\mathrm{H}^1(k, \mathcal{O}) = 0$ (for example, $K \subset k$).

Proof. (i) Relations (3.8) on p. 3217 of [5] easily imply that all 16 classes $[l_{ij}]$ are in $N_\Lambda + N_\Sigma$. Then relation (3.9) of *loc. cit.* shows that $[s_o]$ also belongs to $N_\Lambda + N_\Sigma$. Recall that $[\Pi : \mathbb{Z}^{16}] = 2^5$, see Remark 1 above. It is easy to deduce that the Kummer lattice Π is generated by the 16 classes $[l_{ij}]$, together with the differences $[s_i] - [s_j]$ and $[l_i] - [l_j]$ for all possible pairs (i, j). Thus $\Pi \subset N_\Lambda + N_\Sigma$.

A straightforward computation of the intersection pairing shows that the lattice generated by $[f_1]$, $[f_2]$ and the classes of 16 lines is freely generated by these elements, and so is of rank 18. This lattice is contained in $N_\Lambda + N_\Sigma$, hence $N_\Lambda + N_\Sigma$ is also of rank 18 and is freely generated by $\Lambda \cup \Sigma$, so that $N_\Lambda + N_\Sigma = N_\Lambda \oplus N_\Sigma$. We have $\sigma_* \pi^*([s_j]) = [e]$ and $\sigma_* \pi^*([l_i]) = [e']$ for any i and j. It follows from the exact sequence (9) that $(N_\Lambda \oplus N_\Sigma)/\Pi$ is a Γ-submodule of $\mathrm{NS}(\bar{A})$ generated by $[e]$ and $[e']$. We now obtain (11) from (10).

(ii) If $\bar{E}$ and $\overline{E'}$ are not isogenous, then $\mathrm{Hom}(\bar{E}, \overline{E'}) = 0$. If $E = E'$ is an elliptic curve without complex multiplication, then $\mathrm{Hom}(\bar{E}, \overline{E'}) = \mathbb{Z}$ is a trivial Γ-module. In the last case $\mathrm{Hom}(\bar{E}, \overline{E'}) = 0$, so in all cases we have $\mathrm{H}^1\big(k, \mathrm{Hom}(\bar{E}, \overline{E'})\big) = 0$. By Shapiro's lemma $\mathrm{H}^1(k, N_\Lambda) = \mathrm{H}^1(k, N_\Sigma) = 0$, thus $\mathrm{H}^1\big(k, \mathrm{NS}(\overline{X})\big) = 0$ follows from the long exact sequence of Galois cohomology attached to (11).

The surface X has k-points, for example, on $l_{oo} \simeq \mathbb{P}^1_k$. This implies that the natural map $\mathrm{Br}(k) \to \mathrm{Br}(X)$ has a retraction, and hence is injective. The same holds for the natural map $\mathrm{H}^3_{\text{ét}}(k, \mathbf{G}_m) \to \mathrm{H}^3_{\text{ét}}(X, \mathbf{G}_m)$. Now from the Hochschild–Serre spectral sequence $\mathrm{H}^p\big(k, \mathrm{H}^q_{\text{ét}}(\overline{X}, \mathbf{G}_m)\big) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(X, \mathbf{G}_m)$ we obtain a split exact sequence

$$0 \to \mathrm{Br}(k) \to \mathrm{Br}_1(X) \to \mathrm{H}^1\big(k, \mathrm{Pic}(\overline{X})\big) \to 0.$$

Since $\mathrm{Pic}(\overline{X}) = \mathrm{NS}(\overline{X})$, this finishes the proof. $\square$

2. On étale cohomology of abelian varieties and Kummer surfaces

We refer to [20], Chapter 2, for a general introduction to torsors.

Let A be an abelian variety over k, and let $n \geq 1$. Let $\mathscr{T}$ be the A-torsor with structure group A_n defined by the multiplication by n map $[n] : A \to A$. Let $[\mathscr{T}]$ be the class of $\mathscr{T}$ in $\mathrm{H}^1_{\text{ét}}(A, A_n)$, and let $[\overline{\mathscr{T}}]$ be the image of $[\mathscr{T}]$ under the natural map $\mathrm{H}^1_{\text{ét}}(A, A_n) \to \mathrm{H}^1_{\text{ét}}(\bar{A}, A_n)^\Gamma$. The cup-product defines a Galois-equivariant bilinear pairing

$$\mathrm{H}^1_{\text{ét}}(\bar{A}, A_n) \times \mathrm{Hom}(A_n, \mathbb{Z}/n) \to \mathrm{H}^1_{\text{ét}}(\bar{A}, \mathbb{Z}/n).$$

Pairing with $[\overline{\mathscr{T}}]$ gives a homomorphism of Γ-modules

$$\tau_A : \mathrm{Hom}(A_n, \mathbb{Z}/n) \to \mathrm{H}^1_{\text{ét}}(\bar{A}, \mathbb{Z}/n).$$

The following lemma is certainly well known, and is proved here for the convenience of the reader.

Lemma 2.1. τ_A *is an isomorphism of Γ-modules.*

Proof.   The two groups have the same number of elements, hence it is enough to prove the injectivity. A non-zero homomorphism $\alpha : A_n \to \mathbb{Z}/n$ can be written as the composition of a surjection $\beta : A_n \to \mathbb{Z}/m$ where $m \mid n$, $m \neq 1$, followed by the injection $\mathbb{Z}/m \hookrightarrow \mathbb{Z}/n$. The induced map $\mathrm{H}^1_{\text{ét}}(\bar{A}, \mathbb{Z}/m) \to \mathrm{H}^1_{\text{ét}}(\bar{A}, \mathbb{Z}/n)$ is injective, hence if

$[\overline{\mathcal{T}}] \cup \alpha = [\alpha_* \overline{\mathcal{T}}] = 0$, then the $\bar{A}$-torsor $\beta_* \overline{\mathcal{T}}$ under $\mathbb{Z}/m$ is trivial. A trivial $\bar{A}$-torsor under a finite group is not connected, whereas the push-forward $\beta_* \overline{\mathcal{T}}$ is canonically isomorphic to the quotient $\bar{A}/\mathrm{Ker}(\beta)$, which is connected. This contradiction shows that sending α to $[\alpha_* \overline{\mathcal{T}}]$ defines an injective homomorphism of abelian groups $\mathrm{Hom}(A_n, \mathbb{Z}/n) \xrightarrow{\sim} \mathrm{H}^1_{\text{ét}}(\bar{A}, \mathbb{Z}/n)$. The lemma is proved. $\square$

Proposition 2.2. *Let A be an abelian variety over k, and let m, $n \geq 1$ and $q \geq 0$ be integers such that $(n, q!) = 1$. Then the natural group homomorphism*

$$\mathrm{H}^q_{\text{ét}}(A, \mu_n^{\otimes m}) \to \mathrm{H}^q_{\text{ét}}(\bar{A}, \mu_n^{\otimes m})^{\Gamma}$$

has a section, and hence is surjective.

Proof. We break the proof into three steps.

Step 1. Let M be a free $\mathbb{Z}/n$-module of rank d with a basis $\{e_i\}_{i=1}^d$, and let M^* be the dual $\mathbb{Z}/n$-module with the dual basis $\{f_i\}_{i=1}^d$. For each $q \geq 1$ we have the identity map

$$\mathrm{Id}_{\bigwedge^q M} \in \mathrm{End}_{\mathbb{Z}/n}(\textstyle\bigwedge^q M) = \bigwedge^q M \otimes_{\mathbb{Z}/n} \bigwedge^q M^*,$$

$$\mathrm{Id}_{\bigwedge^q M} = \sum (e_{i_1} \wedge \cdots \wedge e_{i_q}) \otimes (f_{i_1} \wedge \cdots \wedge f_{i_q}),$$

where $i_1 < \cdots < i_q$. The multiplication law $(a \otimes b) \cdot (a' \otimes b') = (a \wedge a') \otimes (b \wedge b')$ turns the ring

$$\bigoplus_{q \geq 0} \bigwedge^q M \otimes_{\mathbb{Z}/n} \bigwedge^q M^*$$

into a commutative $\mathbb{Z}/n$-algebra. A straightforward calculation shows that

$$(12) \qquad\qquad\qquad (\mathrm{Id}_M)^q = q! \, \mathrm{Id}_{\bigwedge^q M}.$$

Step 2. Recall that the cup-product defines a canonical isomorphism

$$\bigwedge^q \mathrm{H}^1_{\text{ét}}(\bar{A}, \mathbb{Z}/n) \xrightarrow{\sim} \mathrm{H}^q_{\text{ét}}(\bar{A}, \mathbb{Z}/n).$$

We have a natural homomorphism of $\mathbb{Z}/n$-modules

$$\mathrm{H}^q_{\text{ét}}(\bar{A}, \mathbb{Z}/n) \otimes_{\mathbb{Z}/n} \bigwedge^q (A_n) \to \mathrm{H}^q_{\text{ét}}\big(\bar{A}, \textstyle\bigwedge^q (A_n)\big),$$

and since the abelian group $\bigwedge^q (A_n)$ is a product of copies of $\mathbb{Z}/n$, this is clearly an isomorphism.

Write $M = \mathrm{H}^1_{\text{ét}}(\bar{A}, \mathbb{Z}/n)$ and use Lemma 2.1 to identify A_n with M^*. We obtain an isomorphism of $\mathbb{Z}/n$-modules

$$(13) \qquad \mathrm{H}^q_{\text{ét}}\big(\bar{A}, \textstyle\bigwedge^q (A_n)\big) = \mathrm{H}^q_{\text{ét}}(\bar{A}, \mathbb{Z}/n) \otimes_{\mathbb{Z}/n} \bigwedge^q (A_n) = \bigwedge^q M \otimes_{\mathbb{Z}/n} \bigwedge^q M^*.$$

The cup-product in étale cohomology gives rise to the map

$$H^1_{\text{ét}}(\bar{A}, A_n)^{\otimes q} \to H^q_{\text{ét}}(\bar{A}, A_n^{\otimes q}) \to H^q_{\text{ét}}\big(\bar{A}, \textstyle\bigwedge^q(A_n)\big),$$

and we denote by $\bigwedge^q[\overline{\mathscr{T}}] \in H^q_{\text{ét}}\big(\bar{A}, \bigwedge^q(A_n)\big)^\Gamma$ the image of the product of q copies of $[\overline{\mathscr{T}}]$. Now (12) says that the isomorphism (13) identifies $\bigwedge^q[\overline{\mathscr{T}}]$ with $q!\,\mathrm{Id}_{\bigwedge^q M}$.

We have an obvious commutative diagram of Γ-equivariant pairings, where the vertical arrows are isomorphisms:

$$
\begin{array}{ccccc}
\textstyle\bigwedge^q M \otimes_{\mathbb{Z}/n} \bigwedge^q M^* & \times\ \mathrm{Hom}(\bigwedge^q M^*, \mathbb{Z}/n) & \longrightarrow & \bigwedge^q(M) \\
\downarrow & \| & & \downarrow \\
H^q_{\text{ét}}(\bar{A}, \mathbb{Z}/n) \otimes_{\mathbb{Z}/n} \bigwedge^q M^* & \times\ \mathrm{Hom}(\bigwedge^q M^*, \mathbb{Z}/n) & \longrightarrow & H^q_{\text{ét}}(\bar{A}, \mathbb{Z}/n) \\
\downarrow & \downarrow & & \| \\
H^q_{\text{ét}}\big(\bar{A}, \bigwedge^q(A_n)\big) & \times\ \mathrm{Hom}\big(\bigwedge^q(A_n), \mathbb{Z}/n\big) & \longrightarrow & H^q_{\text{ét}}(\bar{A}, \mathbb{Z}/n).
\end{array}
$$

The pairing with the Γ-invariant element $\bigwedge^q[\overline{\mathscr{T}}]$ gives a homomorphism of Γ-modules

$$\mathrm{Hom}\big(\textstyle\bigwedge^q(A_n), \mathbb{Z}/n\big) \xrightarrow{\sim} H^q_{\text{ét}}(\bar{A}, \mathbb{Z}/n),$$

which is $q!$ times the canonical isomorphism $\mathrm{Hom}(\bigwedge^q M^*, \mathbb{Z}/n) \xrightarrow{\sim} \bigwedge^q M$. By assumption $q!$ is invertible in $\mathbb{Z}/n$, so this is an isomorphism of Γ-modules. Tensoring with the Γ-module $\mu_n^{\otimes m}$ we obtain an isomorphism of Γ-modules

$$\mathrm{Hom}\big(\textstyle\bigwedge^q(A_n), \mu_n^{\otimes m}\big) \xrightarrow{\sim} H^q_{\text{ét}}(\bar{A}, \mu_n^{\otimes m}).$$

Therefore, pairing with $\bigwedge^q[\overline{\mathscr{T}}]$ gives rise to an isomorphism of abelian groups

$$(14) \qquad \mathrm{Hom}_\Gamma\big(\textstyle\bigwedge^q(A_n), \mu_n^{\otimes m}\big) \xrightarrow{\sim} H^q_{\text{ét}}(\bar{A}, \mu_n^{\otimes m})^\Gamma.$$

Step 3. The cup-product in étale cohomology gives rise to the map

$$H^1_{\text{ét}}(A, A_n)^{\otimes q} \to H^q_{\text{ét}}(A, A_n^{\otimes q}) \to H^q_{\text{ét}}\big(A, \textstyle\bigwedge^q(A_n)\big),$$

and we denote by $\bigwedge^q[\mathscr{T}] \in H^q_{\text{ét}}\big(A, \bigwedge^q(A_n)\big)$ the image of the product of q copies of $[\mathscr{T}]$. There is a natural pairing of abelian groups

$$H^q_{\text{ét}}\big(A, \textstyle\bigwedge^q(A_n)\big) \times \mathrm{Hom}_\Gamma\big(\textstyle\bigwedge^q(A_n), \mu_n^{\otimes m}\big) \to H^q_{\text{ét}}(A, \mu_n^{\otimes m}).$$

Pairing with $\bigwedge^q[\mathscr{T}]$ induces a map $\mathrm{Hom}_\Gamma\big(\bigwedge^q(A_n), \mu_n^{\otimes m}\big) \to H^q_{\text{ét}}(A, \mu_n^{\otimes m})$ such that the composition

$$\mathrm{Hom}_\Gamma\big(\textstyle\bigwedge^q(A_n), \mu_n^{\otimes m}\big) \to H^q_{\text{ét}}(A, \mu_n^{\otimes m}) \to H^q_{\text{ét}}(\bar{A}, \mu_n^{\otimes m})^\Gamma$$

is the isomorphism (14). This proves the proposition. $\square$

In some cases the condition $(n, q!) = 1$ can be dropped, see Corollary 3.2 below.

Corollary 2.3. *Let n be an odd integer. Then the images of the groups $\mathrm{Br}(A)_n$ and $\mathrm{H}^2_{\text{ét}}(\bar{A}, \mu_n)^{\Gamma}$ in $\mathrm{Br}(\bar{A})^{\Gamma}_n$ coincide, so that we have an isomorphism*

$$\mathrm{Br}(A)_n/\mathrm{Br}_1(A)_n \simeq \mathrm{H}^2_{\text{ét}}(\bar{A}, \mu_n)^{\Gamma}/(\mathrm{NS}(\bar{A})/n)^{\Gamma}.$$

Proof. The Kummer sequences for A and $\bar{A}$ give rise to the following obvious commutative diagram with exact rows, cf. (2):

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \left(\mathrm{NS}(\bar{A})/n\right)^{\Gamma} & \longrightarrow & \mathrm{H}^2_{\text{ét}}(\bar{A}, \mu_n)^{\Gamma} & \longrightarrow & \mathrm{Br}(\bar{A})^{\Gamma}_n \\
& & & & \Big\Vert\Big\uparrow & & \Big\uparrow \\
& & & & \mathrm{H}^2_{\text{ét}}(A, \mu_n) & \longrightarrow & \mathrm{Br}(A)_n & \longrightarrow & 0.
\end{array}
$$

The downward arrow is the section of Proposition 2.2. Both statements follow from this diagram. $\square$

Theorem 2.4. *Let A be an abelian surface, and let $X = \mathrm{Kum}(A)$. Then π^* defines an embedding*

$$\mathrm{Br}(X)_n/\mathrm{Br}_1(X)_n \hookrightarrow \mathrm{Br}(A)_n/\mathrm{Br}_1(A)_n,$$

which is an isomorphism if n is odd. The subgroups of elements of odd order of the transcendental Brauer groups $\mathrm{Br}(X)/\mathrm{Br}_1(X)$ and $\mathrm{Br}(A)/\mathrm{Br}_1(A)$ are isomorphic.

Proof. By Proposition 1.3 we have the commutative diagram

$$
(15) \qquad
\begin{array}{ccc}
\mathrm{Br}(X)_n & \longrightarrow & \mathrm{Br}(A)_n \\
\downarrow & & \downarrow \\
\mathrm{Br}(\bar{X})_n & \xrightarrow{\ \sim\ } & \mathrm{Br}(\bar{A})_n
\end{array}
$$

which implies the desired embedding. Now assume that n is odd. We can write

$$\mathrm{Br}(A)_n = \mathrm{Br}(A)^+_n \oplus \mathrm{Br}(A)^-_n,$$

where $\mathrm{Br}(A)^+_n$ (resp. $\mathrm{Br}(A)^-_n$) is the ι-invariant (resp. ι-antiinvariant) subgroup of $\mathrm{Br}(A)_n$. The involution ι acts trivially on $\mathrm{H}^2_{\text{ét}}(\bar{A}, \mu_{\ell^m})$ for any ℓ and m, hence by (2) it also acts trivially on $\mathrm{Br}(\bar{A})$. It follows that for odd n the image of $\mathrm{Br}(A)^-_n$ in $\mathrm{Br}(\bar{A})$ is zero. This gives an isomorphism

$$\mathrm{Br}(A)_n/\mathrm{Br}_1(A)_n = \mathrm{Br}(A)^+_n/\mathrm{Br}_1(A)^+_n.$$

[6], Theorem 1.4 states that if $Y \to X$ is a finite flat Galois covering of smooth geometrically irreducible varieties with Galois group G, and n is coprime to $|G|$, then the natural map $\mathrm{Br}(X)_n \to \mathrm{Br}(Y)^G_n$ is an isomorphism. We apply this to the double covering

$\pi : A' \to X$. Taking into account the isomorphism $\mathrm{Br}(A) = \mathrm{Br}(A')$ we obtain the following commutative diagram

(16)
$$
\begin{array}{ccc}
\mathrm{Br}(X)_n & \xrightarrow{\;\sim\;} & \mathrm{Br}(A)_n^+ \\
\downarrow & & \downarrow \\
\mathrm{Br}(\overline{X})_n & \xrightarrow{\;\sim\;} & \mathrm{Br}(\overline{A})_n.
\end{array}
$$

Our first statement follows. The second statement follows from the first one once we note that an element of odd order in $\mathrm{Br}(X)/\mathrm{Br}_1(X)$ comes from $\mathrm{Br}(X)_n$ for some odd n. □

3. The case of product of two elliptic curves

We now assume that $A = E \times E'$ is the product of two elliptic curves. In this case we can prove the same statement as in Corollary 2.3 but without the assumption on n.

The Künneth formula (see [11], Corollary VI.8.13) gives a direct sum decomposition of Γ-modules

$$
\mathrm{H}^2_{\text{ét}}(\overline{A}, \mathbb{Z}/n) = \mathrm{H}^2_{\text{ét}}(\overline{E}, \mathbb{Z}/n) \oplus \mathrm{H}^2_{\text{ét}}(\overline{E'}, \mathbb{Z}/n) \oplus \mathrm{H}^2_{\text{ét}}(\overline{A}, \mathbb{Z}/n)_{\text{prim}},
$$

where

$$
\mathrm{H}^2_{\text{ét}}(\overline{A}, \mathbb{Z}/n)_{\text{prim}} = \mathrm{H}^1_{\text{ét}}(\overline{E}, \mathbb{Z}/n) \otimes \mathrm{H}^1_{\text{ét}}(\overline{E'}, \mathbb{Z}/n)
$$

is the *primitive* subgroup of $\mathrm{H}^2_{\text{ét}}(\overline{A}, \mathbb{Z}/n)$. On twisting with μ_n we obtain the decomposition of Γ-modules

$$
\mathrm{H}^2_{\text{ét}}(\overline{A}, \mu_n) = \mathbb{Z}/n \oplus \mathbb{Z}/n \oplus \mathrm{H}^2_{\text{ét}}(\overline{A}, \mu_n)_{\text{prim}},
$$

where

$$
\mathrm{H}^2_{\text{ét}}(\overline{A}, \mu_n)_{\text{prim}} = \mathrm{H}^1_{\text{ét}}(\overline{E}, \mathbb{Z}/n) \otimes \mathrm{H}^1_{\text{ét}}(\overline{E'}, \mu_n).
$$

The canonical isomorphism $\tau_E : \mathrm{Hom}(E_n, \mathbb{Z}/n) \xrightarrow{\sim} \mathrm{H}^1_{\text{ét}}(\overline{E}, \mathbb{Z}/n)$ from Lemma 2.1 gives an isomorphism of Γ-modules

$$
\mathrm{H}^2_{\text{ét}}(\overline{A}, \mu_n)_{\text{prim}} = \mathrm{Hom}(E_n \otimes E'_n, \mu_n).
$$

Using the Weil pairing we obtain a canonical isomorphism

$$
\mathrm{H}^1_{\text{ét}}(\overline{E'}, \mu_n) = \mathrm{Hom}(E'_n, \mu_n) = E'_n.
$$

Combining all this gives canonical isomorphisms of Γ-modules

(17) $\mathrm{H}^2_{\text{ét}}(\overline{A}, \mu_n) = \mathbb{Z}/n \oplus \mathbb{Z}/n \oplus \mathrm{H}^2_{\text{ét}}(\overline{A}, \mu_n)_{\text{prim}} = \mathbb{Z}/n \oplus \mathbb{Z}/n \oplus \mathrm{Hom}(E_n, E'_n).$

Let $p : A \to E$ and $p' : A \to E'$ be the natural projections. The multiplication by n map $E \to E$ defines an E-torsor $\mathscr{T}$ with structure group E_n. We define the E'-torsor $\mathscr{T}'$ similarly. The pullbacks $p^*\mathscr{T}$ and $p'^*\mathscr{T}$ are A-torsors with structure groups E_n and E'_n, respectively. Let $[\mathscr{T}] \boxtimes [\mathscr{T}']$ be the product of $p^*[\mathscr{T}]$ and $p'^*[\mathscr{T}']$ under the pairing

$$\mathrm{H}^1_{\text{ét}}(A, E_n) \times \mathrm{H}^1_{\text{ét}}(A, E'_n) \to \mathrm{H}^2_{\text{ét}}(A, E_n \otimes E'_n).$$

Consider the natural pairing

$$\mathrm{H}^2_{\text{ét}}(A, E_n \otimes E'_n) \times \mathrm{Hom}_\Gamma(E_n \otimes E'_n, \mu_n) \to \mathrm{H}^2_{\text{ét}}(A, \mu_n).$$

Let

$$\xi : \mathrm{Hom}_\Gamma(E_n \otimes E'_n, \mu_n) \to \mathrm{H}^2_{\text{ét}}(A, \mu_n)$$

be the map defined by pairing with $[\mathscr{T}] \boxtimes [\mathscr{T}']$.

The following map is defined by the base change from k to $\bar{k}$ followed by the Künneth projector to the primitive subgroup:

$$\eta : \mathrm{H}^2_{\text{ét}}(A, \mu_n) \to \mathrm{H}^2_{\text{ét}}(\bar{A}, \mu_n)^\Gamma \to \mathrm{H}^2_{\text{ét}}(\bar{A}, \mu_n)^\Gamma_{\text{prim}} = \mathrm{Hom}_\Gamma(E_n \otimes E'_n, \mu_n).$$

Lemma 3.1. *We have $\eta \circ \xi = \mathrm{Id}$. In particular, η has a section, and hence is surjective.*

Proof. We must check that the composed map

$$(18) \qquad \mathrm{Hom}(E_n \otimes E'_n, \mathbb{Z}/n) \to \mathrm{H}^2_{\text{ét}}(\bar{A}, \mathbb{Z}/n) \to \mathrm{H}^1_{\text{ét}}(\bar{E}, \mathbb{Z}/n) \otimes \mathrm{H}^1_{\text{ét}}(\overline{E'}, \mathbb{Z}/n)$$

defined by pairing with the image of $[\mathscr{T}] \boxtimes [\mathscr{T}']$ in $\mathrm{H}^2_{\text{ét}}(\bar{A}, E_n \otimes E'_n)$ followed by the Künneth projector to the primitive subgroup, is the isomorphism

$$\tau_E \otimes \tau_{E'} : \mathrm{Hom}(E_n \otimes E'_n, \mathbb{Z}/n) \xrightarrow{\sim} \mathrm{H}^1_{\text{ét}}(\bar{E}, \mathbb{Z}/n) \otimes \mathrm{H}^1_{\text{ét}}(\overline{E'}, \mathbb{Z}/n)$$

(cf. Lemma 2.1). Note that the first arrow in (18) is $\tau_E \otimes \tau_{E'}$ followed by the composed map

$$(19) \qquad \mathrm{H}^1_{\text{ét}}(\bar{E}, \mathbb{Z}/n) \otimes \mathrm{H}^1_{\text{ét}}(\overline{E'}, \mathbb{Z}/n) \to \mathrm{H}^1_{\text{ét}}(\bar{A}, \mathbb{Z}/n) \otimes \mathrm{H}^1_{\text{ét}}(\bar{A}, \mathbb{Z}/n) \to \mathrm{H}^2_{\text{ét}}(\bar{A}, \mathbb{Z}/n),$$

where the first arrow is $p^* \otimes p'^*$, and the second one is the cup-product. By [11], Corollary VI.8.13, the composition of (19) with the Künneth projector is the identity, hence the composed map in (18) is $\tau_E \otimes \tau_{E'}$. $\quad\square$

Corollary 3.2. *For $A = E \times E'$ and any $n \geqq 1$ the natural map*

$$\mathrm{H}^2_{\text{ét}}(A, \mu_n) \to \mathrm{H}^2_{\text{ét}}(\bar{A}, \mu_n)^\Gamma$$

has a section, and hence is surjective.

Proof.   We have a canonical map $p^* : \mathrm{H}^2_{\text{ét}}(E, \mu_n) \to \mathrm{H}^2_{\text{ét}}(A, \mu_n)$. By Künneth decomposition and Lemma 3.1 it is enough to check that

$$\mathrm{H}^2_{\text{ét}}(E, \mu_n) \to \mathrm{H}^2_{\text{ét}}(\bar{E}, \mu_n)^\Gamma$$

has a section (and similarly for E'). The Kummer sequences for E and $\bar{E}$ give a commutative diagram

$$
\begin{array}{ccccc}
0 & \longrightarrow & \mathbb{Z}/n & \stackrel{\sim}{\longrightarrow} & \mathrm{H}^2_{\text{ét}}(\bar{E}, \mu_n) \\
& & \big\uparrow & & \big\uparrow \\
0 & \longrightarrow & \mathrm{Pic}(E)/n & \longrightarrow & \mathrm{H}^2_{\text{ét}}(E, \mu_n).
\end{array}
$$

The left vertical arrow is given by the degree map $\mathrm{Pic}(E) \to \mathbb{Z}$. It has a section that sends $1 \in \mathbb{Z}/n$ to the class of the neutral element of E in $\mathrm{Pic}(E)/n$. $\square$

Remark.   This shows that if an abelian variety A is a product of elliptic curves, then the condition on n in Proposition 2.2 is superfluous.

Recall that the natural map $\mathrm{Hom}(\bar{E}, \bar{E'})/n \to \mathrm{Hom}(E_n, E'_n)$ is injective [12], p. 124. Write $\mathrm{Hom}(E, E') = \mathrm{Hom}_\Gamma(\bar{E}, \bar{E'})$ for the group of homomorphisms $E \to E'$.

Proposition 3.3.   *For $A = E \times E'$ we have a canonical isomorphism of Γ-modules*

$$\mathrm{Br}(\bar{A})_n = \mathrm{Hom}(E_n, E'_n)/\big(\mathrm{Hom}(\bar{E}, \bar{E'})/n\big),$$

and a canonical isomorphism of abelian groups

$$\mathrm{Br}(A)_n/\mathrm{Br}_1(A)_n = \mathrm{Hom}_\Gamma(E_n, E'_n)/\big(\mathrm{Hom}(\bar{E}, \bar{E'})/n\big)^\Gamma.$$

Proof.   The Kummer sequences for A and $\bar{A}$ give rise to the commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{NS}(\bar{A})/n & \longrightarrow & \mathrm{H}^2_{\text{ét}}(\bar{A}, \mu_n) & \longrightarrow & \mathrm{Br}(\bar{A})_n & \longrightarrow & 0 \\
& & & & \big\uparrow & & \big\uparrow & & \\
& & & & \mathrm{H}^2_{\text{ét}}(A, \mu_n) & \longrightarrow & \mathrm{Br}(A)_n & \longrightarrow & 0.
\end{array}
$$

Using (10) and (17) we rewrite this diagram as follows:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}(\bar{E}, \bar{E'})/n & \longrightarrow & \mathrm{Hom}(E_n, E'_n) & \longrightarrow & \mathrm{Br}(\bar{A})_n & \longrightarrow & 0 \\
& & & & \big\uparrow & & \big\uparrow & & \\
& & & & \mathrm{H}^2_{\text{ét}}(A, \mu_n) & \longrightarrow & \mathrm{Br}(A)_n & \longrightarrow & 0.
\end{array}
$$

The upper row here is the first isomorphism of the proposition. From Lemma 3.1 we deduce the commutative diagram

$$0 \longrightarrow \left(\mathrm{Hom}(\bar{E}, \bar{E}')/n\right)^{\Gamma} \longrightarrow \mathrm{Hom}_{\Gamma}(E_n, E_n') \longrightarrow \mathrm{Br}(\bar{A})_n^{\Gamma}$$

$$\mathrm{H}^2_{\text{ét}}(A, \mu_n) \longrightarrow \mathrm{Br}(A)_n \longrightarrow 0$$

where the left upward arrow is η, and the downward arrow is ξ. The second isomorphism of the proposition is a consequence of the commutativity of this diagram. $\square$

Until the end of this section we assume that $n = 2$ and the points of order 2 of E and E' are defined over k, i.e. $E_2 \subset E(k)$ and $E_2' \subset E'(k)$. The above considerations can then be made more explicit. (This construction was previously used in [21], Appendix A.2, see also [5], Section 3.2.) In this case

$$\mathrm{Br}(\bar{A})_2 = \mathrm{Br}(\bar{A})_2^{\Gamma} = \mathrm{Br}(A)_2/\mathrm{Br}_1(A)_2.$$

Using the Weil pairing the map ξ gives rise to the map $E_2 \otimes E_2' \to \mathrm{Br}(A)_2$ whose image maps surjectively onto $\mathrm{Br}(\bar{A})_2$. The elements of $\mathrm{Br}(A)_2$ obtained in this way can be given by symbols as follows. The curves E and E' can be given by their respective equations

$$y^2 = x(x-a)(x-b), \quad v^2 = u(u-a')(u-b'),$$

where a and b are distinct non-zero elements of k, and similarly for a' and b'. The multiplication by 2 torsor $E \to E$ corresponds to the biquadratic extension of the function field $k(E)$ given by the square roots of $(x-a)(x-b)$ and $x(x-b)$, see, e.g., [8], Theorem 4.2, p. 85. We choose $e_1 = (0,0)$ and $e_2 = (a,0)$ as a basis of E_2, and $e_1' = (0,0)$ and $e_2' = (a',0)$ as a basis of E_2'; this gives rise to an obvious basis of $E_2 \otimes E_2'$. The four resulting Azumaya algebras on A are written as follows:

$$(20) \qquad \left((x-\mu)(x-b), (u-v)(u-b')\right), \quad \mu \in \{0, a\}, v \in \{0, a'\}.$$

We note that the specialisation of any of these algebras at the neutral element of A is $0 \in \mathrm{Br}(k)$. By the above, the classes of the algebras (20) in $\mathrm{Br}(\bar{A})$ generate $\mathrm{Br}(\bar{A})_2$.

The antipodal involution ι sends (x, y) to $(x, -y)$, and (u, v) to $(u, -v)$, hence the Kummer surface $X = \mathrm{Kum}(A)$ is given by the affine equation

$$(21) \qquad z^2 = x(x-a)(x-b)y(y-a')(y-b').$$

We denote by $A_{\mu, v}$ the class in $\mathrm{Br}(k(X))$ given by the corresponding symbol (20).

For $A = E \times E'$ it is convenient to replace $X_0 \subset X$ by a larger open subset. Let us denote by $E_2^{\#}$ the set of $\bar{k}$-points of E of *exact* order 2; in other words, E_2 is the disjoint union of $\{0\}$ and $E_2^{\#}$. Define $W \subset X$ as the complement to the 9 lines that correspond to the points of $E_2^{\#} \times E_2'^{\#}$. The line $l_{oo} = \pi(\sigma^{-1}(0))$, where $0 \in A(k)$ is the neutral element, is contained in W. Choose a k-point Q on l_{oo}, and denote by $\mathrm{Br}(W)^0$ the subgroup of $\mathrm{Br}(W)$ consisting of the elements that specialise to 0 at Q. Since $\mathrm{Br}(\mathbb{P}_k^1) = \mathrm{Br}(k)$, we see that

$\mathrm{Br}(W)^0$ is the kernel of the restriction map $\mathrm{Br}(W) \to \mathrm{Br}(l_{oo})$, hence $\mathrm{Br}(W)^0$ does not depend on the choice of Q.

Lemma 3.4. *We have $A_{\mu,v} \in \mathrm{Br}(W)_2^0$ for any $\mu \in \{0, a\}$ and $v \in \{0, a'\}$.*

Proof. We have $A_{\mu,v} \in \mathrm{Br}(W)$ by [21], Lemma A.2. Every $A_{\mu,v}$ lifts to an element of $\mathrm{Br}(A)$ with value 0 at the neutral element of A, hence $A_{\mu,v} \in \mathrm{Br}(W)^0$. Since $A_{\mu,v}$ is the class of a quaternion algebra, we have $A_{\mu,v} \in \mathrm{Br}(W)_2^0$. $\square$

The map $(\mu, v) \mapsto A_{\mu,v}$ defines a group homomorphism $\omega : E_2 \otimes E_2' \to \mathrm{Br}(W)_2^0$.

Proposition 3.5. *Assume one of the conditions of Proposition 1.4 (ii). Then we have*

(i) $\mathrm{Br}_1(W) = \mathrm{Br}(k)$;

(ii) $\mathrm{Im}(\omega) = \mathrm{Br}(W)_2^0$;

(iii) $\mathrm{Ker}(\omega) = \mathrm{Hom}(\bar{E}, \overline{E'})/2$.

Proof (cf. [21], Appendix A2). (i) We have $\bar{k}[W]^* = \bar{k}^*$, as it follows from $\bar{k}[A_0]^* = \bar{k}^*$. We also have $\mathrm{Br}_0(W) = \mathrm{Br}(k)$ since W has a k-point. Then the Hochschild–Serre spectral sequence $\mathrm{H}^p(k, \mathrm{H}_{\text{ét}}^q(\overline{W}, \mathbf{G}_m)) \Rightarrow \mathrm{H}_{\text{ét}}^{p+q}(W, \mathbf{G}_m)$ shows that it is enough to prove $\mathrm{H}^1(k, \mathrm{Pic}(\overline{W})) = 0$. In the notation of Proposition 1.4 we have

$$\mathrm{Pic}(\overline{W}) = \mathrm{Pic}(\overline{X})/N_\Lambda,$$

hence there is an exact sequence of Γ-modules analogous to (11):

$$0 \to N_\Sigma \to \mathrm{Pic}(\overline{W}) \to \mathrm{Hom}(\bar{E}, \overline{E'}) \to 0.$$

By Shapiro's lemma $\mathrm{H}^1(k, N_\Lambda) = 0$, thus, under the assumptions of Proposition 1.4 (ii), $\mathrm{H}^1(k, \mathrm{Pic}(\overline{W})) = 0$ follows from the long exact sequence of Galois cohomology.

(ii) Let $\mathfrak{A} \subset \mathrm{Br}(W)$ be the four-element set $\{A_{\mu,v}\}$, and let $\bar{\mathfrak{A}}$ be the image of $\mathfrak{A}$ in $\mathrm{Br}(\overline{W})$. By Proposition 1.2 we have $\mathrm{Br}(\overline{W}) = \mathrm{Br}(\overline{X})$, thus we can think of $\bar{\mathfrak{A}}$ as a subset of $\mathrm{Br}(\overline{X})$. The image of $\bar{\mathfrak{A}}$ under the isomorphism $(\sigma^*)^{-1}\pi^* : \mathrm{Br}(\overline{X}) \to \mathrm{Br}(\overline{A})$ from Proposition 1.3 generates $\mathrm{Br}(\overline{A})_2$, hence $\bar{\mathfrak{A}}$ generates $\mathrm{Br}(\overline{W})_2$. Therefore, any $\alpha \in \mathrm{Br}(W)_2^0$ can be written as

$$\alpha = \beta + \sum \delta_{\mu,v} A_{\mu,v},$$

where $\delta_{\mu,v} \in \{0, 1\}$, and $\beta \in \mathrm{Br}_1(W)$ has value zero at Q. It remains to apply (i).

(iii) By part (i) the natural map $\mathrm{Br}(W)^0 \to \mathrm{Br}(\overline{W})$ is injective, and we have just seen that the latter group is naturally isomorphic to $\mathrm{Br}(\overline{A})$. Now our statement follows from the first formula of Proposition 3.3. $\square$

We now calculate the residues of the $A_{\mu,v}$ at the 9 lines of $X \backslash W$ (cf. Proposition 1.2 and its proof).

Lemma 3.6.   *The residues of $A_{a,a'}$, $A_{a,0}$, $A_{0,a'}$, A_{00}, at the lines l_{00}, $l_{0,a'}$, $l_{a,0}$, $l_{a,a'}$, written in this order, are the classes in k^*/k^{*2} represented by the entries of the following matrix:*

$$(22) \qquad \begin{pmatrix} 1 & ab & a'b' & -aa' \\ ab & 1 & aa' & a'(a'-b') \\ a'b' & aa' & 1 & a(a-b) \\ -aa' & a'(a'-b') & a(a-b) & 1 \end{pmatrix}.$$

For any $\mu \in \{0, a\}$ and $v \in \{0, a'\}$ the product of residues of $A_{\mu v}$ at the three lines l_{ij}, $i \neq 0$, $j \neq 0$, with fixed first or second index, is $1 \in k^*/k^{*2}$.

Proof.   We write res_{ij} for the residue at l_{ij}. The local ring $O \subset k(X)$ of l_{ij} is a discrete valuation ring with valuation $\mathrm{val} : k(X)^* \to \mathbb{Z}$. For $f, g \in O \backslash \{0\}$ the residue of (f, g) at l_{ij} is computed by the following rule: if $\mathrm{val}(f) = \mathrm{val}(g) = 0$, then $\mathrm{res}_{ij}((f, g))$ is trivial, and if $\mathrm{val}(f) = 0$, $\mathrm{val}(g) = 1$, then $\mathrm{res}_{ij}((f, g))$ is the class in $k(l_{ij})^*/k(l_{ij})^{*2}$ of the reduction of f modulo the maximal ideal of O. In our case this class will automatically be in k^*/k^{*2}.

Let us calculate the residues of $A_{00} = (x(x-b), x(y-b'))$. Using the above rule we obtain

$$\mathrm{res}_{0,a'}(A_{00}) = a'(a'-b'), \quad \mathrm{res}_{a,0}(A_{00}) = a(a-b), \quad \mathrm{res}_{a,a'}(A_{00}) = 1.$$

Using equation (21) and the relation $(r, -r) = 0$ for any $r \in k(X)^*$ we can write

$$A_{00} = (x(x-b), -(x-a)(y-a')).$$

The residue of A_{00} at l_{00} is then the value of $-(x-a)(y-a')$ at $x = y = 0$, that is, $-aa'$. We thus checked the last row of (22). The residue of A_{00} at $l_{0,b'}$ is the class of $a(b'-a')$, which shows that the product of residues of A_{00} at l_{00}, $l_{0,a'}$ and $l_{0,b'}$ is 1. The calculations in all other cases are quite similar. $\square$

Question.   Is there a conceptual explanation of the symmetry of (22)?

Let r be the rank of $\mathrm{Hom}(\bar{E}, \bar{E'})$, and let d be the dimension of the kernel of the homomorphism

$$\mathrm{Hom}(E_2, E_2') = E_2 \otimes E_2' \simeq (\mathbb{Z}/2)^4 \to (k^*/k^{*2})^4$$

given by the matrix (22).

Proposition 3.7.   *Let $X = \mathrm{Kum}(E \times E')$, where E and E' are elliptic curves with rational 2-torsion points. Assume one of the conditions of Proposition 1.4 (ii). Then*

$$\dim_{\mathbb{F}_2} \mathrm{Br}(X)_2 / \mathrm{Br}(k)_2 = d - r.$$

In particular, if $E = E'$ and $d = 1$, then $\mathrm{Br}(X)_2 = \mathrm{Br}(k)_2$.

Proof. This follows from Proposition 3.5 (ii) and (iii), and Lemma 3.6. $\square$

Note that if $E = E'$ and $d = 1$, then $\mathrm{End}(\bar{E})$ has rank 1, so that E has no complex multiplication over $\bar{k}$.

4. Brauer groups of abelian surfaces

In the rest of this paper we discuss abelian surfaces of the following types:

(A) $A = E \times E'$, where the elliptic curves E and E' are not isogenous over $\bar{k}$.

(B) $A = E \times E$, where E has no complex multiplication over $\bar{k}$.

(C) $A = E \times E$, where E has complex multiplication over $\bar{k}$.

Case (A). In case (A), the Néron–Severi group $\mathrm{NS}(\bar{A})$ is freely generated by the classes $E \times \{0\}$ and $\{0\} \times E'$, hence $\mathrm{H}^2_{\text{ét}}(\bar{A}, \mu_n)$ is the direct sum of Γ-modules $\mathrm{NS}(\bar{A})/n \oplus \mathrm{Br}(\bar{A})_n$, and we have

$$(23) \qquad\qquad \mathrm{Br}(\bar{A})_n = \mathrm{Hom}(E_n, E'_n).$$

Proposition 4.1. *Let E be an elliptic curve such that the representation of Γ in E_ℓ is a surjection $\Gamma \to \mathrm{GL}(E_\ell)$ for every prime ℓ. Let E' be an elliptic curve with complex multiplication over $\bar{k}$, which has a k-point of order 6. Then for $A = E \times E'$ we have $\mathrm{Br}(\bar{A})^\Gamma = 0$.*

Proof. Since $\mathrm{Br}(\bar{A})$ is a torsion group it is enough to prove that for every prime ℓ we have $\mathrm{Br}(\bar{A})^\Gamma_\ell = \mathrm{Hom}_\Gamma(E_\ell, E'_\ell) = 0$.

By assumption E' has complex multiplication by some imaginary quadratic field K. Thus there exists an extension k'/k of degree at most 2 such that the image of $\mathrm{Gal}(\bar{k}/k')$ in $\mathrm{Aut}(E'_\ell)$ is abelian. Thus the image of Γ in $\mathrm{Aut}(E'_\ell)$ is a solvable group. We note that for $\ell \geq 5$ the group $\mathrm{GL}(2, \mathbb{F}_\ell)$ is not solvable. This implies that E has no complex multiplication over $\bar{k}$. It follows that E and E' are not isogenous over $\bar{k}$.

The Γ-module E_ℓ is simple, hence any non-zero homomorphism of Γ-modules $E_\ell \to E'_\ell$ must be an isomorphism. This gives a contradiction for $\ell \geq 5$. If $\ell = 2$ or $\ell = 3$, the curve E' has a k-point of order ℓ, so that E'_ℓ is not a simple Γ-module, which is again a contradiction. $\square$

Example (A1). Let $k = \mathbb{Q}$, let E be the curve $y^2 = x^3 + 6x - 2$ of conductor $2^6 3^3$, and let E' be the curve $y^2 = x^3 + 1$ with the point $(2, 3)$ of order 6. It follows from [18], 5.9.2, p. 318, that the conditions of Proposition 4.1 are satisfied.

Example (A2). Here is a somewhat different construction for case (A), again over $k = \mathbb{Q}$. Let us call a pair of elliptic curves (E, E') *non-exceptional* if for all primes ℓ the image of the Galois group $\Gamma = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ in $\mathrm{Aut}(E_\ell) \times \mathrm{Aut}(E'_\ell)$ is as large as it can possibly be, that is, it is the subgroup of $\mathrm{GL}(2, \mathbb{F}_\ell) \times \mathrm{GL}(2, \mathbb{F}_\ell)$ given by the condition

$\det(x) = \det(x')$. This implies $\mathrm{Hom}_\Gamma(E_\ell, E'_\ell) = 0$, so that $\mathrm{Br}(\bar{A})^\Gamma = 0$, where $A = E \times E'$. For example, let E be the curve $y^2 + y = x^3 - x$ of conductor 37, and let E' be the curve $y^2 + y = x^3 + x^2$ of conductor 43. The curve E has multiplicative reduction at 37, whereas E' has good reduction, therefore E and E' are not isogenous over $\bar{\mathbb{Q}}$. By [18], the remark on page 329, the pair (E, E') is non-exceptional. In fact, most pairs (E, E') are non-exceptional in a similar sense to the remark after Proposition 4.3 (Nathan Jones, see [7]).

We now explore some other constructions providing an infinite series of examples when $\mathrm{Br}(\bar{A})^\Gamma$ has no elements of odd order. Later we shall show that for such abelian surfaces A we often have $\mathrm{Br}(\mathrm{Kum}(A)) = \mathrm{Br}(\mathbb{Q})$, see Example 3 in Section 5.

Proposition 4.2. *Let E be an elliptic curve over $\mathbb{Q}$ such that $\mathrm{val}_5\big(j(E)\big) = -2^m$ and $\mathrm{val}_7\big(j(E)\big) = -2^n$, where m and n are non-negative integers. Let E' be an elliptic curve over $\mathbb{Q}$ with good reduction at 5 and 7, and with rational 2-torsion, i.e. $E'_2 \subset E'(\mathbb{Q})$. Then E and E' are not isogenous over $\bar{\mathbb{Q}}$, and $\mathrm{Hom}_\Gamma(E_\ell, E'_\ell) = 0$ for any prime $\ell \neq 2$. If $A = E \times E'$, then $\mathrm{Br}(\bar{A})^\Gamma$ is a finite abelian 2-group.*

Proof. Since $j(E)$ is not a 5-adic integer, E has potential multiplicative reduction at 5. But E' has good reduction at 5, so E and E' are not isogenous over $\bar{\mathbb{Q}}$.

Let $p = 5$. Our assumption implies that there exists a *Tate curve* $\tilde{E}$ over $\mathbb{Q}_p$ such that $E \times_\mathbb{Q} \mathbb{Q}_p$ is the twist of $\tilde{E}$ by a quadratic or trivial character

$$\chi : \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \to \{\pm 1\}.$$

Consider the case when $\ell \neq 5$. Let K be the extension of $\mathbb{Q}_p$ defined as follows: if χ is trivial or unramified, then $K = \mathbb{Q}_p$, and if χ is ramified, then $K \subset \bar{\mathbb{Q}}_p$ is the invariant subfield of $\mathrm{Ker}(\chi)$. Let $\mathfrak{p}$ be the maximal ideal of the ring of integers of K. We note that in both cases the residue field of K is $\mathbb{F}_p$.

Since $\tilde{E}$ is a Tate curve, the ℓ-torsion $\tilde{E}_\ell$ contains a Galois submodule isomorphic to μ_ℓ. Then the quotient $\tilde{E}_\ell/\mu_\ell$ is isomorphic to the trivial Galois module $\mathbb{Z}/\ell$. Hence there is a basis of E_ℓ such that the image of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/K)$ in $\mathrm{Aut}(E_\ell) \simeq \mathrm{GL}(2, \mathbb{F}_\ell)$ is contained in the subgroup of upper-triangular matrices. Let q_E be the multiplicative period of $\tilde{E}$. Since $\mathrm{val}_\mathfrak{p}(q_E) = -\mathrm{val}_\mathfrak{p}(j(E))$ is not divisible by the odd prime ℓ, the image of the inertia group $I(\mathfrak{p})$ in $\mathrm{Aut}(E_\ell)$ contains $\mathrm{Id} + N$ for some nilpotent $N \neq 0$, see [17], Chapter IV, Section 3.2, Lemma 1. Thus E_ℓ has exactly one non-zero $\mathrm{Gal}(\bar{\mathbb{Q}}_p/K)$-invariant subgroup $C \neq E_\ell$. As a $\mathrm{Gal}(\bar{\mathbb{Q}}_p/K)$-module, C is isomorphic to μ_ℓ if $K \neq \mathbb{Q}_p$, and to μ_ℓ twisted by the unramified character χ if $K = \mathbb{Q}_p$. The $\mathrm{Gal}(\bar{\mathbb{Q}}_p/K)$-module E_ℓ/C is isomorphic to $\mathbb{Z}/\ell$ if $K \neq \mathbb{Q}_p$, and to $\mathbb{Z}/\ell$ twisted by χ if $K = \mathbb{Q}_p$. In particular, the $\mathrm{Gal}(\bar{\mathbb{Q}}_p/K)$-modules C and E_ℓ/C are isomorphic if and only if K contains a primitive ℓ-th root of unity.

Suppose that there exists a non-zero homomorphism of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/K)$-modules $\phi : E_\ell \to E'_\ell$. Since E' has good reduction at p, the inertia $I(\mathfrak{p})$ acts trivially on E'_ℓ; in particular ϕ is not an isomorphism. Then $\mathrm{Ker}(\phi) = C$, and E'_ℓ contains a $\mathrm{Gal}(\bar{\mathbb{Q}}_p/K)$-submodule isomorphic to $\mathbb{Z}/\ell$ (when χ is ramified) or $\mathbb{Z}/\ell$ twisted by χ (when χ is unramified or trivial). In the first case let $E'' = E'$, and in the second case let E'' be the quadratic twist of E' by χ. Then $E''(K)$ contains a point of order ℓ, so that $E''(K)$ contains a finite

subgroup of order 4ℓ. Since E' has good reduction at $\mathfrak{p}$, the curve E'' also has good reduction at $\mathfrak{p}$. Then the group of $\mathbb{F}_5$-points on the reduction has at least 12 elements, which contradicts the Hasse bound, according to which an elliptic curve over $\mathbb{F}_p$ cannot have more than $p + 2\sqrt{p} + 1$ points, see [8].

It remains to consider the case $\ell = 5$. The above arguments work equally well with $p = 7$. We obtain a contradiction with the Hasse bound since no elliptic curve over $\mathbb{F}_7$ can contain as many as $4\ell = 20$ rational points.

The last statement of (i) follows from formula (23). □

Example (A3). The curve E in this proposition can be any curve with equation $y^2 = x(x - a)(x - b)$, where a and b are distinct non-zero integers such that exactly one of the numbers a, b, $a - b$ is divisible by 5, exactly one is divisible by 7, and none are divisible by 25 or 49. (For example, $a = 5 + 35m$, $b = 7 + 35n$, where $m \not\equiv 2 + 5k$ and $n \not\equiv 4 + 7k$.) The modular invariant of E is

$$j(E) = 2^8 \frac{(a^2 + b^2 - ab)^3}{a^2 b^2 (a - b)^2},$$

which is immediate from the standard formula

$$j(E) = 1728 \frac{4p^3}{4p^3 + 27q^2}$$

for E written in the Weierstrass form $y^2 = x^3 + px + q$, see [8], Chapter III, Section 2. It follows that $\mathrm{val}_5\big(j(E)\big) = \mathrm{val}_7\big(j(E)\big) = -2$. The curve E' can be any curve with equation $y^2 = x(x - a')(x - b')$, where a' and b' are distinct non-zero integers such that a', b', $a' - b'$ are coprime to 35, e.g., $a' = 35m' + 1$, $b' = 35n' + 2$ for any $m', n' \in \mathbb{Z}$.

Case (B). In case (B), the group $\mathrm{NS}(\bar{A})$ is freely generated by the classes of the curves $E \times \{0\}$, $\{0\} \times E$ and the diagonal. The image of the class of the diagonal under the map $\mathrm{NS}(\bar{A}) \to \mathrm{End}(\bar{E}) \to \mathrm{End}(E_n)$ is the identity, hence Proposition 3.3 gives an isomorphism of Galois modules

(24) $$\mathrm{Br}(\bar{A})_n = \mathrm{End}(E_n)/\mathbb{Z}/n,$$

where $\mathbb{Z}/n$ is the subring of scalars in $\mathrm{End}(E_n)$.

Remark. Let $A = E \times E$, where E is an elliptic curve without complex multiplication over $\bar{k}$, such that the image of Γ in $\mathrm{Aut}(E_2)$ is $\mathrm{GL}(2, \mathbb{F}_2)$. It is easy to check that $\mathrm{Br}(\bar{A})_2^\Gamma = \big(\mathrm{End}(E_2)/\mathbb{Z}/2\big)^\Gamma$ has order 2; in fact, the non-zero element of this group can be represented by a symmetric 2×2-matrix S over $\mathbb{F}_2$ such that $S^3 = \mathrm{Id}$. Thus the 2-primary component of $\mathrm{Br}(\bar{A})^\Gamma$ is finite cyclic. In this example the map $\mathrm{H}^2_{\mathrm{ét}}(\bar{A}, \mathbb{Z}/2)^\Gamma \to \mathrm{Br}(\bar{A})_2^\Gamma$ is zero. By the second formula of Proposition 3.3 the map $\mathrm{Br}(A)_2 \to \mathrm{Br}(\bar{A})_2^\Gamma$ is *not* surjective. The following proposition shows that the non-zero element of $\mathrm{Br}(\bar{A})_2^\Gamma$ does not belong to the image of the map $\mathrm{Br}(A) \to \mathrm{Br}(\bar{A})^\Gamma$.

Proposition 4.3.   *Let $A = E \times E$, where E is an elliptic curve such that for every prime ℓ the image of Γ in $\mathrm{Aut}(E_\ell)$ is $\mathrm{GL}(2, \mathbb{F}_\ell)$. Then we have*:

(i) $\mathrm{Br}(A) = \mathrm{Br}_1(A)$.

(ii) $\mathrm{Br}(\bar{A})^\Gamma \simeq \mathbb{Z}/2^m$ *for some $m \geqq 1$*.

Proof.   (i) We note that the argument in the proof of Proposition 4.1 shows that the curve E has no complex multiplication. In view of the second formula of Proposition 3.3 it is enough to prove the following lemma:

Lemma 4.4.   *Let $G \subset \mathrm{GL}(2, \mathbb{Z}_\ell)$ be a subgroup that maps surjectively onto $\mathrm{GL}(2, \mathbb{F}_\ell)$. Let $\mathrm{Mat}_2(\mathbb{Z}/\ell^n)$ be the abelian group of 2×2-matrices with entries in $\mathbb{Z}/\ell^n$, and let $\mathrm{Mat}_2(\mathbb{Z}/\ell^n)^G$ be the subgroup of matrices commuting with the image of G in $\mathrm{GL}(2, \mathbb{Z}/\ell^n)$. Then for any positive integer n we have $\mathrm{Mat}_2(\mathbb{Z}/\ell^n)^G = \mathbb{Z}/\ell^n \cdot \mathrm{Id}$.*

Proof.   We proceed by induction starting with the obvious case $n = 1$. Suppose we know the statement for n, and need to prove it for $n + 1$. Consider the exact sequence of G-modules

$$0 \to \mathrm{Mat}_2(\mathbb{Z}/\ell) \to \mathrm{Mat}_2(\mathbb{Z}/\ell^{n+1}) \to \mathrm{Mat}_2(\mathbb{Z}/\ell^n) \to 0,$$

where the second arrow comes from the injection $\mathbb{Z}/\ell = \ell^n \mathbb{Z}/\ell^{n+1} \hookrightarrow \mathbb{Z}/\ell^{n+1}$, and the third one is the reduction modulo ℓ^n. By induction assumption $\mathrm{Mat}_2(\mathbb{Z}/\ell^n)^G = \mathbb{Z}/\ell^n \cdot \mathrm{Id}$. Thus, the map $\mathrm{Mat}_2(\mathbb{Z}/\ell^{n+1})^G \to \mathrm{Mat}_2(\mathbb{Z}/\ell^n)^G$ is surjective, and every element in $\mathrm{Mat}_2(\mathbb{Z}/\ell^{n+1})^G$ is the sum of a scalar multiple of Id and an element of $\mathrm{Mat}_2(\mathbb{Z}/\ell)^G$. But $\mathrm{Mat}_2(\mathbb{Z}/\ell)^G = \mathbb{Z}/\ell \cdot \mathrm{Id}$, and so the lemma, and hence also part (i) of the proposition, are proved.

(ii) For an odd prime ℓ we have a direct sum decomposition of Γ-modules $\mathrm{End}(E_\ell) = \mathbb{Z}/\ell \oplus \mathrm{Br}(\bar{A})_\ell$, where $\mathrm{Br}(\bar{A})_\ell$ is identified with the group of endomorphisms of trace zero. Our assumption implies that $\mathrm{Br}(\bar{A})_\ell^\Gamma = 0$. The remark before the proposition shows that $\mathrm{Br}(\bar{A})^\Gamma$ is a finite cyclic 2-group. $\square$

Remark.   By a theorem of W. Duke [2] 'almost all' elliptic curves over $\mathbb{Q}$ satisfy the assumption of Proposition 4.3. More precisely, if $y^2 = x^3 + ax + b$ is the unique equation for E such that $a, b \in \mathbb{Z}$ and $\gcd(a^3, b^2)$ does not contain twelfth powers, the height $H(E)$ of E is defined to be $\max(|a|^3, |b|^2)$. For $x > 0$ write $\mathscr{C}(x)$ for the set of elliptic curves E over $\mathbb{Q}$ (up to isomorphism) such that $H(E) \leqq x^6$, and $\mathscr{E}(x)$ for the set of curves in $\mathscr{C}(x)$ for which there exists a prime ℓ such that the image of Γ in $\mathrm{Aut}(E_\ell)$ is not equal to $\mathrm{GL}(2, \mathbb{F}_\ell)$. Then $\lim_{x \to +\infty} |\mathscr{E}(x)|/|\mathscr{C}(x)| = 0$. By Proposition 1.4 and Theorem 2.4 this implies that for most Kummer surfaces

$$z^2 = (x^3 + ax + b)(y^3 + ay + b)$$

we have $\mathrm{Br}(X) = \mathrm{Br}(\mathbb{Q})$. In particular, there are infinitely many such surfaces.

Proposition 4.5.   *Let E be an elliptic curve over $\mathbb{Q}$ satisfying the assumptions of Proposition 4.2. Then E has no complex multiplication, and $\mathrm{End}_\Gamma(E_\ell)$ is the subring of*

scalars $\mathbb{F}_\ell \cdot \mathrm{Id} \subset \mathrm{End}(E_\ell)$ *for any prime* $\ell \neq 2$. *If* $A = E \times E$, *then* $\mathrm{Br}(\bar{A})^\Gamma$ *is a finite abelian 2-group.*

Proof.   The curve E has no complex multiplication because $j(E)$ is not an algebraic integer. Let ℓ be an odd prime, $\ell \neq p = 5$, and let ϕ be a non-zero endomorphism of the Γ-module E_ℓ such that $\mathrm{Tr}(\phi) = 0$. From the proof of Proposition 4.2 we know that there exists a nilpotent $N \neq 0$ in $\mathrm{End}(E_\ell)$ such that the image of $I(\mathfrak{p})$ in $\mathrm{Aut}(E_\ell)$ contains $\mathrm{Id} + N$. Since ϕ is an endomorphism of the $I(\mathfrak{p})$-module E_ℓ, it commutes with N, and it follows from $\mathrm{Tr}(\phi) = 0$ that ϕ is also nilpotent. As was explained in the proof of Proposition 4.2, the existence of such an endomorphism ϕ implies that $\mathrm{Gal}(\bar{\mathbb{Q}}_p/K)$-modules $\mathbb{Z}/\ell$ and μ_ℓ are isomorphic. However, K does not contain non-trivial roots of 1 of order ℓ when $p = 5$ and $\ell \neq 5$ is odd, because the residue field of K is $\mathbb{F}_5$. This contradiction shows that $\mathrm{End}_\Gamma(E_\ell)$ is the subring of scalars $\mathbb{Z}/\ell \subset \mathrm{End}(E_\ell)$. If $\ell = 5$ we repeat these arguments with $p = 7$ taking into account that $\mathbb{Q}_7$, and hence K, does not contain non-trivial 5-th roots of 1. $\square$

Example (B1).   Let $A = E \times E$, where $E/\mathbb{Q}$ is an elliptic curve such that the representation of Γ in E_ℓ is a surjection $\Gamma \to \mathrm{GL}(E_\ell)$ for every odd prime ℓ. Then E has no complex multiplication over $\bar{\mathbb{Q}}$ (see the proof of Proposition 4.1). Then $\mathrm{Br}(\bar{A})_\ell^\Gamma = 0$ if $\ell \neq 2$. This assumption holds in the following examples which have the additional property that the 2-torsion of E is rational, i.e., $E_2 \subset E(\mathbb{Q})$. For example, one can consider the curve $y^2 = (x - 1)(x - 2)(x + 2)$ of conductor 24, or the curve $y^2 = (x + 1)(x + 2)(x - 3)$ of conductor 40. The computation of residues shows that in each of these two cases we have $\mathrm{Br}(X) = \mathrm{Br}(\mathbb{Q})$ for $X = \mathrm{Kum}(E \times E)$ (see Example 4 in the next section).

Case (C).   Lastly, we would like to consider the case when $A = E \times E$, where E is an elliptic curve with complex multiplication.

Theorem 4.6.   *Let E be an elliptic curve over $\mathbb{Q}$ with complex multiplication, and let ℓ be an odd prime such that E has no rational isogeny of degree ℓ, i.e., E_ℓ does not contain a Galois-invariant subgroup of order ℓ. Let G_ℓ be the image of $\Gamma = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ in $\mathrm{Aut}(E_\ell)$. Then G_ℓ is nonabelian, the order $|G_\ell|$ is not divisible by ℓ, and the centralizer of G_ℓ in $\mathrm{End}(E_\ell)$ is $\mathbb{F}_\ell = \mathbb{Z}/\ell$.*

The theorem remains true if one replaces $\mathbb{Q}$ by any real number field (with the same proof).

Proof of Theorem 4.6.   Suppose that E has complex multiplication by an order $\mathcal{O}$ of an imaginary quadratic field K, that is, $\mathrm{End}(\bar{E}) = \mathcal{O}$. We start with the observation that ℓ is unramified in $\mathcal{O}$, or, equivalently, the 2-dimensional $\mathbb{F}_\ell$-algebra $\mathcal{O}/\ell \subset \mathrm{End}(E_\ell)$ has no nilpotents. Indeed, if the radical of $\mathcal{O}/\ell$ is non-zero, it is an $\mathbb{F}_\ell$-vector space of dimension 1, and so is spanned by one element. Its kernel in E_ℓ is a Galois-invariant cyclic subgroup of order ℓ. We assumed that such subgroups do not exist, so this is a contradiction.

Therefore, $\mathcal{O}/\ell$ is either $\mathbb{F}_\ell \oplus \mathbb{F}_\ell$ or the field $\mathbb{F}_{\ell^2}$. In the first case $(\mathcal{O}/\ell)^*$ is a split Cartan subgroup of order $(\ell - 1)^2$, whereas in the second case it is a non-split Cartan subgroup of order $\ell^2 - 1$, so that ℓ does not divide $|(\mathcal{O}/\ell)^*|$. On the other hand, the image of $\mathrm{Gal}(\bar{\mathbb{Q}}/K)$ in $\mathrm{Aut}(E_\ell)$ commutes with the Cartan subgroup $(\mathcal{O}/\ell)^*$, and so belongs to $(\mathcal{O}/\ell)^*$. Since $\mathrm{Gal}(\bar{\mathbb{Q}}/K)$ is a subgroup of Γ of index 2, we conclude that the order of G_ℓ divides $2|(\mathcal{O}/\ell)^*|$ and so is not divisible by ℓ.

The group G_ℓ contains an element c corresponding to the complex conjugation. Any $z \in \mathcal{O} \setminus \ell \mathcal{O}$ such that $\mathrm{Tr}(z) = 0$ anticommutes with complex conjugation. Since ℓ is odd, the non-zero image of z in $\mathcal{O}/\ell$ anticommutes with c. Thus c is not a scalar; in particular, c has exact order 2. If G_ℓ is abelian, then both eigenspaces of c in E_ℓ are Galois-invariant cyclic subgroups of order ℓ, but these do not exist. This implies that G_ℓ is nonabelian.

Finally, the absence of Galois-invariant order ℓ subgroups in E_ℓ implies that the G_ℓ-module E_ℓ is simple, so the centralizer of G_ℓ in $\mathrm{End}(E_\ell)$ is $\mathbb{F}_\ell$.   □

Corollary 4.7. *Let $A = E \times E$, where E is an elliptic curve over $\mathbb{Q}$ with complex multiplication, and let ℓ be an odd prime such that E has no rational isogeny of degree ℓ. Then $\mathrm{Br}(\bar{A})_\ell^\Gamma = 0$.*

Proof. It follows from Theorem 4.6 that the Γ-module $\mathrm{End}(E_\ell)$ is semisimple, hence $\mathrm{H}^2_{\mathrm{\acute{e}t}}(\bar{A}, \mu_\ell) = (\mathbb{Z}/\ell)^2 \oplus \mathrm{End}(E_\ell)$ is also semisimple. Thus $\mathrm{H}^2_{\mathrm{\acute{e}t}}(\bar{A}, \mu_\ell) = \mathrm{NS}(\bar{A})/\ell \oplus \mathrm{Br}(\bar{A})_\ell$ is a direct sum of Γ-modules. Since the identity in $\mathrm{End}(E_\ell)$ corresponds to the diagonal in $\bar{E} \times \bar{E}$, it is contained in $\mathrm{NS}(\bar{A})/\ell$. By Theorem 4.6 we have $\mathrm{H}^2_{\mathrm{\acute{e}t}}(\bar{A}, \mu_\ell)^\Gamma \subset \mathrm{NS}(\bar{A})/\ell$, so that $\mathrm{Br}(\bar{A})_\ell^\Gamma = 0$.   □

Example (C1). Let $A = E \times E$, where E is the curve $y^2 = x^3 - x$ with complex multiplication by $\mathbb{Z}[\sqrt{-1}]$. An application of sage [23] gives that every isogeny of prime degree $E \to E'$ defined over $\mathbb{Q}$ is the factorization by a subgroup of $E(\mathbb{Q})_{\mathrm{tors}} = E_2$. Hence $\mathrm{Br}(\bar{A})_\ell^\Gamma = 0$ for every odd prime ℓ.

Example (C2). Let $A = E \times E$, where E is the curve $y^2 = x^3 - 1$ with complex multiplication by $\mathbb{Z}\left[\dfrac{1 + \sqrt{-3}}{2}\right]$. An application of sage gives that every isogeny of prime degree $E \to E'$ over $\mathbb{Q}$ is the factorization by a subgroup of $E(\mathbb{Q})_{\mathrm{tors}} \simeq \mathbb{Z}/6$. Hence $\mathrm{Br}(\bar{A})_\ell^\Gamma = 0$ for every prime $\ell \geqq 5$.

5. Brauer groups of Kummer surfaces

Example 1. Let $k = \mathbb{Q}$. Examples (A1) and (A2) show that the Kummer surfaces X given by the following affine equations have trivial Brauer group $\mathrm{Br}(X) = \mathrm{Br}(\mathbb{Q})$:

$$(25) \qquad z^2 = (x^3 + 6x - 2)(y^3 + 1),$$

$$(26) \qquad z^2 = (4x^3 - 4x + 1)(4y^3 + 4y^2 + 1).$$

In both examples we have $\mathrm{Br}(\bar{X})^\Gamma = 0$.

Example 2. Other examples can be obtained using Proposition 4.3 in conjunction with Theorem 2.4. For example, for the following Kummer surface X we also have $\mathrm{Br}(X) = \mathrm{Br}(\mathbb{Q})$, whereas $\mathrm{Br}(\bar{X})^\Gamma \simeq \mathbb{Z}/2^m$ for some $m \geqq 1$:

$$(27) \qquad z^2 = (x^3 + 6x - 2)(y^3 + 6y - 2).$$

The interest of the following series of examples is that for them the image of $\mathrm{Br}(A)$ in $\mathrm{Br}(\bar{A})$ contains $\mathrm{Br}(\bar{A})_2$, so in order to prove the triviality of $\mathrm{Br}(X)$ we need to compute the residues at the nine lines in $X \backslash W$.

Example 3.   Let X be the Kummer surface over $\mathbb{Q}$ with affine equation

$$z^2 = x(x-a)(x-b)y(y-a')(y-b'),$$

such that $a = 5 + 35m$, $b = 7 + 35n$, where $m, n \in \mathbb{Z}$, m is not congruent to 2 modulo 5, n is not congruent to 4 modulo 7, and $a' = 35m' + 1$, $b' = 35n' + 2$ for any $m', n' \in \mathbb{Z}$. We have $X = \mathrm{Kum}(E \times E')$, where the elliptic curves E and E' are as in Example A3. Since $X(\mathbb{Q}) \neq \emptyset$ we see that $\mathrm{Br}(\mathbb{Q})$ is a direct factor of $\mathrm{Br}(X)$. By Propositions 4.2 and 1.4 (ii) to show that $\mathrm{Br}(X) = \mathrm{Br}(\mathbb{Q})$ it is enough to prove that every element of $\mathrm{Br}(X)$ of order 2 is algebraic. By Proposition 3.7 we need to compute the dimension d of the kernel of the matrix (22). Considering the first two entries in each row, and taking their valuations at 5 and 7 immediately shows that no product of some of the rows of (22) is trivial. Thus $d = 0$, hence $\mathrm{Br}(X) = \mathrm{Br}(\mathbb{Q})$.

Example 4.   Let $X = \mathrm{Kum}(E \times E)$, where E is as in Example 3, or the elliptic curve with conductor 24 or 40 mentioned in Example B1. In the latter case X is given by one of the following equations:

$$(28) \qquad z^2 = (x-1)(x-2)(x+2)(y-1)(y-2)(y+2),$$

$$(29) \qquad z^2 = (x+1)(x+2)(x-3)(y+1)(y+2)(y-3).$$

One checks that the dimension of the kernel of (22) is 1, so that $\mathrm{Br}(X) = \mathrm{Br}(\mathbb{Q})$ by Proposition 3.7.

Kummer surfaces without rational points.   There is a more general construction of Kummer surfaces than the one previously considered. Let c be a 1-cocycle of Γ with coefficients in A_2 so that $[c] \in \mathrm{H}^1(k, A_2)$. All quasi-projective varieties and Galois modules acted on by the k-group scheme A_2 can be twisted by c. The twist of A is a principal homogeneous space A^c, also called a 2-covering of A. The action of A_2 on A by translations descends to an action of A_2 on $X = \mathrm{Kum}(A)$, so we obtain a twisted Kummer surface X^c together with morphisms $A^c \leftarrow A'^c \to X^c$. For example, if $A = E \times E'$, a 2-covering C of E is given by $y^2 = f(x)$, where $f(x)$ is a separable polynomial of degree 4, and a 2-covering C' of E' is given by a similar equation $y^2 = g(x)$, then the twisted Kummer surface X^c is given by the affine equation

$$z^2 = f(x)g(y).$$

The Hasse principle on such surfaces over number fields was studied in [21].

Proposition 5.1.   *Suppose that for every integer $n > 1$ we have*

$$\mathrm{H}^2_{\text{ét}}(\bar{A}, \mu_n)^\Gamma = \left(\mathrm{NS}(\bar{A})/n\right)^\Gamma$$

(this condition is satisfied when $\mathrm{Br}(\bar{A})^\Gamma = 0$). Then $\mathrm{Br}(X^c) = \mathrm{Br}_1(X^c)$ for any $[c] \in \mathrm{H}^1(k, A_2)$.

Proof. By Remark 3 after Proposition 1.3, $A_2(\overline{k})$ acts trivially on $\mathrm{Br}(\overline{A})$ and on $\mathrm{Br}(\overline{X})$, thus we have the following isomorphisms of Γ-modules:

$$\mathrm{Br}(\overline{X^c}) \simeq \mathrm{Br}(\overline{X}) \simeq \mathrm{Br}(\overline{A^c}) \simeq \mathrm{Br}(\overline{A}).$$

Translations act trivially on étale cohomology groups of $\overline{A}$, hence we have a canonical isomorphism of Γ-modules $\mathrm{H}^2_{\text{ét}}(\overline{A^c}, \mu_n) = \mathrm{H}^2_{\text{ét}}(\overline{A}, \mu_n)$. In the commutative diagram

$$
\begin{array}{ccc}
\mathrm{H}^2_{\text{ét}}(\overline{A^c}, \mu_n)^{\Gamma} & \longrightarrow & \mathrm{Br}(\overline{A^c})_n^{\Gamma} \\
\uparrow & & \uparrow \\
\mathrm{H}^2_{\text{ét}}(A^c, \mu_n) & \longrightarrow & \mathrm{Br}(A^c)_n
\end{array}
$$

the bottom arrow is surjective, and the top arrow is zero by assumption. It follows that $\mathrm{Br}(A^c) = \mathrm{Br}_1(A^c)$. We conclude by Theorem 2.4. $\square$

This proposition in conjunction with Proposition 4.1 gives many examples of twisted Kummer surfaces X^c such that $\mathrm{Br}(X^c) = \mathrm{Br}_1(X^c)$.

Kummer surfaces with non-trivial transcendental Brauer group. Let E be an elliptic curve over $\mathbb{Q}$. As pointed out by Mazur [10], p. 133, the elliptic curves E' such that the Galois modules E_ℓ and E'_ℓ are symplectically isomorphic correspond to $\mathbb{Q}$-points on the modular curve $X(\ell)$ twisted by E_ℓ. Thus for $\ell \leqq 5$ there are infinitely many possibilities for E' due to the fact that the genus of $X(\ell)$ is zero, see [19], [16]. Now let $\ell = 7$, 11 or 13. Examples of pairs of non-isogenous elliptic curves with isomorphic Galois modules $E_\ell \simeq E'_\ell$ for these values of ℓ can be found in [4] and [1]. Our results imply that $\mathrm{Br}(X)$, where $X = \mathrm{Kum}(E \times E')$, contains an element of order ℓ whose image in $\mathrm{Br}(\overline{X})$ is non-zero.

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