Cohomology and the Brauer group of double covers

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1 Introduction

Let k be a field of characteristic different from 2. Let $\pi : X \to S$ be a finite surjective morphism of degree 2, where X and S are smooth, projective and geometrically integral varieties over k. Let $i : C \to S$ and $j : C \to X$ be the natural closed embeddings of the branch locus of π, so that $i = \pi j$. Assume that C is non-empty and smooth, but not necessarily connected. In this note we are interested in the cokernel of the natural map between the 2-torsion subgroups of the Brauer groups of S and X:

$$\pi^* : \mathrm{Br}(S)[2] \longrightarrow \mathrm{Br}(X)[2].$$

Consider the Gysin sequence in étale cohomology (or Betti cohomology if $k = \mathbb{C}$):

$$\mathrm{H}^0(C, \mathbb{Z}/2) \xrightarrow{\theta} \mathrm{H}^2(S, \mathbb{Z}/2) \longrightarrow \mathrm{H}^2(S \setminus C, \mathbb{Z}/2) \longrightarrow \mathrm{H}^1(C, \mathbb{Z}/2) \xrightarrow{\theta} \mathrm{H}^3(S, \mathbb{Z}/2);$$

the construction of this sequence is recalled in the remark before Lemma 2.4 below. Let us write $\mathrm{H}^1(C, \mathbb{Z}/2)[\theta]$ for the kernel of the Gysin map θ. We define a canonical map

$$\Phi : \mathrm{H}^1(C, \mathbb{Z}/2)[\theta] \longrightarrow \mathrm{Br}(X)[2]/\pi^*(\mathrm{Br}(S)[2]),$$

and obtain results about the kernel and the cokernel of Φ. To construct Φ we note that the Gysin sequence for the smooth pair $(S, i(C))$ is linked with a similar sequence for $(X, j(C))$ by the maps induced by π. Since π^* induces the zero map on $\mathrm{H}^1(C, \mathbb{Z}/2)$, we obtain a map from $\mathrm{H}^1(C, \mathbb{Z}/2)[\theta]$ to the quotient of $\mathrm{H}^2(X, \mathbb{Z}/2)$ by the sum of $\pi^*\mathrm{H}^2(S, \mathbb{Z}/2)$ and $\theta\mathrm{H}^0(C, \mathbb{Z}/2)$. The latter subgroup consists of algebraic classes, so the Kummer sequence gives us our map Φ.

We denote the kernel of $\pi_* : \mathrm{Pic}(X) \to \mathrm{Pic}(S)$ by $\mathrm{Pic}(X)[\pi_*]$. Using diagram (4) below, it is not hard to see that j^* maps $\mathrm{Pic}(X)[\pi_*]$ to the 2-torsion subgroup $\mathrm{Pic}(C)[2]$. A description of Φ in the general case can be read off from the exact sequence (16) in §2.

Let $\bar{k}$ be a separable closure of k, and let $\Gamma = \mathrm{Gal}(\bar{k}/k)$. Write $\overline{S} = S \times_k \bar{k}$. In Proposition 2.5 we show that if C is geometrically connected and $\mathrm{H}^1(\overline{S}, \mathbb{Z}/2) = 0$

(for example, S is geometrically simply connected), then Φ gives rise to an exact sequence

$$0 \longrightarrow j^*(\mathrm{Pic}(X)[\pi_*]) \longrightarrow \mathrm{Pic}(C)[2][\theta] \longrightarrow \mathrm{Br}(X)[2]/\pi^*(\mathrm{Br}(S)[2]).$$

We also give a description of the cokernel of the last map in this sequence. Note that $\mathrm{Pic}(X)[\pi_*] = 0$ when $\mathrm{Pic}(X) \cong \mathbb{Z}$. In the case $S = \mathbb{P}^2_k$ a description of the map $\theta : \mathrm{H}^1(C, \mathbb{Z}/2) \to \mathrm{H}^3(\mathbb{P}^2_k, \mathbb{Z}/2)$ in terms of corestrictions can be found in [1, Prop. 5.3].

When S is a surface, we obtain an explicit description of the kernel and the cokernel of Φ over $\bar{k}$ under certain simplifying assumptions. Over $\bar{k}$ the Kummer sequence gives an isomorphism $\mathrm{H}^1(\overline{C}, \mathbb{Z}/2) = \mathrm{Pic}(\overline{C})[2]$.

Theorem 1.1 *Let k be a field of characteristic different from 2 with a separable closure $\bar{k}$. Let S be a smooth, projective and geometrically integral surface over k with $\mathrm{Pic}(\overline{S})[2] = \mathrm{Br}(\overline{S})[2] = 0$, for example, a geometrically rational surface. For any finite surjective morphism $\pi : X \to S$ of degree 2 ramified in a non-empty smooth curve $j : C \hookrightarrow X$, the map $\Phi : \mathrm{Pic}(\overline{C})[2] \to \mathrm{Br}(\overline{X})[2]$ gives rise to an exact sequence of Γ-modules*

$$0 \longrightarrow \mathrm{Pic}(\overline{C})[2]/j^*(\mathrm{Pic}(\overline{X})[\pi_*]) \longrightarrow \mathrm{Br}(\overline{X})[2] \longrightarrow \mathrm{Pic}(\overline{S})^{\mathrm{even}}/\pi_*\mathrm{Pic}(\overline{X}) \longrightarrow 0, \quad (1)$$

where $\mathrm{Pic}(\overline{S})^{\mathrm{even}}$ is the subgroup of $\mathrm{Pic}(\overline{S})$ consisting of the classes that have even intersection with each connected component of $\overline{C}$.

Remarks 1. The class of C in $\mathrm{Pic}(S)$ is divisible by 2. Thus if C is geometrically connected we have $\mathrm{Pic}(\overline{S})^{\mathrm{even}} = \mathrm{Pic}(\overline{S})$.
2. When k has characteristic zero, the condition $\mathrm{Br}(\overline{S})[2] = 0$ follows from $\mathrm{Pic}(\overline{S})[2] = 0$ and $\mathrm{H}^2(\overline{S}, \mathcal{O}_S) = 0$. Indeed, by Hodge theory the latter condition implies the triviality of the divisible subgroup of $\mathrm{Br}(\overline{S})$, see formula (8.7) in [8, §III.8]. Then $\mathrm{Br}(\overline{S})[2] = 0$ follows from $\mathrm{Pic}(\overline{S})[2] = 0$ by formula (8.12) of *loc. cit.*

When $k = \bar{k}$ the 2-torsion subgroup $\mathrm{Br}(X)[2]$ was studied by T.J. Ford in [5]. He used the results of Knus, Parimala and Srinivas [13], in particular, the observation that for an *unramified* double cover $\pi : V \to U$ the cokernel of the canonical adjunction map $(\mathbb{Z}/2)_U \to \pi_*(\mathbb{Z}/2)_V$ is isomorphic to $(\mathbb{Z}/2)_U$. Although the methods used by Ford appear to be rather general, the results of [5] are proved under the assumption that S has Picard rank 1, so they apply to double covers of the projective plane but not to double covers of more general rational surfaces.

In the last few years there was a renewed interest in constructing elements of $\mathrm{Br}(X)$ motivated in part by the desire to compute the Brauer–Manin obstruction on X. Van Geemen's explicit geometric construction [20] of elements of $\mathrm{Br}(X)[2]$ as Azumaya algebras was recently extended in [12] to arbitrary double covers $X \to \mathbb{P}^2$

ramified in a smooth curve C. In particular, the exact sequence of [12, Thm. 1.1] represents $\mathrm{Br}(X)[2]$ as the quotient of the 2-torsion subgroup of $\mathrm{Pic}(C)/K_C$ by $\mathrm{Pic}(X)/(\mathbb{Z}\pi^*\mathcal{O}(1) + 2\mathrm{Pic}(X))$, where K_C is the canonical class of C. See [6] for another recent work on the subject. When S is a rational surface, [20, Thm. 6.2] says that under some additional assumptions there is an injective map $\mathrm{Pic}(C)[2] \to \mathrm{Br}(X)[2]$ and calculates the size of its cokernel. Finally, in their recent paper [2] B. Creutz and B. Viray give a presentation of the elements of $\mathrm{Br}(X)[2]$ by central simple algebras when S is a ruled surface.

The main point of this article is to show that if one is only interested in Φ as a homomorphism of Galois modules $\mathrm{Pic}(\overline{C})[2] \to \mathrm{Br}(\overline{X})[2]$, and not in a geometric construction of resulting Azumaya algebras on X, then fairly general results can be obtained using only standard cohomological methods without recourse to the geometry of underlying varieties. Our approach was influenced by the calculations of Colliot-Thélène and Wittenberg in [1, §5.1], which seems to be the only place in the literature where the ground field is not assumed to be separably closed.

We set up our cohomological machinery in §2 where we work with a double cover of a smooth, projective and geometrically integral variety S over an arbitrary field k, $\mathrm{char}(k) \neq 2$, ramified in a smooth subvariety C of codimension 1. In §3 we assume that S is a surface such that $\mathrm{H}^1(\overline{S}, \mathbb{Z}/2) = 0$. In §3.1 we spell out some useful exact sequences for the cohomology groups of double covers that are most likely well known but do not seem to be readily available in the literature. In Proposition 3.1 we obtain the exact sequence

$$0 \longrightarrow \mathrm{H}^1(\overline{C}, \mathbb{Z}/2) \longrightarrow \mathrm{H}^2(\overline{X}, \mathbb{Z}/2)/\pi^*\mathrm{H}^2(\overline{S}, \mathbb{Z}/2) \xrightarrow{\pi_*} \mathrm{H}^2(\overline{S}, \mathbb{Z}/2)^{\perp C} \longrightarrow 0,$$

where $\mathrm{H}^2(\overline{S}, \mathbb{Z}/2)^{\perp C}$ is the subgroup consisting of the elements orthogonal to the connected components of $\overline{C}$ with the respect to the cup-product pairing. In Corollary 3.2 we establish the following amusing fact: if C is geometrically connected, then for any $i \geq 1$ we have a canonical isomorphism of Γ-modules

$$\mathrm{H}^i(\mathbb{Z}/2, \mathrm{H}^2(\overline{X}, \mathbb{Z}/2)) = \mathrm{H}^1(\overline{C}, \mathbb{Z}/2),$$

where $\mathbb{Z}/2$ is the automorphism group of the covering $\pi : X \to S$.

Theorem 1.1 is proved in §3.2. For concrete calculations of the Galois module structure of $\mathrm{Br}(\overline{X})[2]$ using (1) we need to have enough information about $\mathrm{Pic}(\overline{S})$, $\mathrm{Pic}(\overline{X})$, $\mathrm{Pic}(\overline{C})[2]$ and the natural maps between these Galois modules. We sketch a few applications in §3.3. Corollary 3.5 concerns arbitrary K3 surfaces with a non-symplectic involution having a non-empty set of fixed points. We also show that the geometric Brauer group of a diagonal quartic surface defined over any field of characteristic not equal to 2 contains a Galois-invariant element of order 2, see Proposition 3.6.

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2 Cohomology of double covers

Let k be a field of characteristic not equal to 2. In this section we do not assume k to be algebraically closed.

Let $\pi : X \to S$, $i : C \hookrightarrow S$ and $j : C \hookrightarrow X$ be as in the first paragraph of the introduction. Since π and i are finite morphisms, π_* and i_* are exact functors between respective categories of étale sheaves [14, Cor. II.3.6]. In particular, for any étale sheaf $\mathcal{E}$ on X we have $\mathrm{R}^n\pi_*\mathcal{E} = 0$ for all $n > 0$, and similarly for i_*. Thus the spectral sequence $\mathrm{H}^p(S, \mathrm{R}^q\pi_*\mathcal{E}) \Rightarrow \mathrm{H}^{p+q}(X, \mathcal{E})$ degenerates, so that for $n \geq 1$ we have canonical isomorphisms

$$\mathrm{H}^n(S, \pi_*\mathcal{E}) = \mathrm{H}^n(X, \mathcal{E}), \quad \mathrm{H}^n(S, i_*\mathcal{F}) = \mathrm{H}^n(C, \mathcal{F}),$$

for any étale sheaf $\mathcal{E}$ on X and any étale sheaf $\mathcal{F}$ on C. We shall use these isomorphisms without further comment.

Let $\mathcal{O}_X$ and $\mathcal{O}_S$ be the structure sheaves. There is a natural map of coherent sheaves $\pi_*\mathcal{O}_X \to \mathcal{O}_S$ that induces norm on the function fields, see [15, Lecture 10]. The composition of the canonical map $\mathcal{O}_S \to \pi_*\mathcal{O}_X$ with $\pi_*\mathcal{O}_X \to \mathcal{O}_S$ sends a function to its square. This gives natural morphisms of étale sheaves

$$\mathbb{G}_{m,S} \longrightarrow \pi_*\mathbb{G}_{m,X}, \qquad \pi_*\mathbb{G}_{m,X} \longrightarrow \mathbb{G}_{m,S},$$

whose composition is $[2] : \mathbb{G}_{m,S} \to \mathbb{G}_{m,S}$. The first of these morphisms is injective and the second one is surjective.

The canonical morphism $\mathbb{G}_{m,X} \to j_*\mathbb{G}_{m,C}$ gives rise to the morphism

$$\pi_*\mathbb{G}_{m,X} \longrightarrow \pi_*j_*\mathbb{G}_{m,C} = i_*\mathbb{G}_{m,C},$$

where the last equality follows from $i = \pi j$. An immediate verification on stalks shows the commutativity of the following diagram of étale sheaves on S:

$$\begin{array}{ccc} \pi_*\mathbb{G}_{m,X} & \longrightarrow & \mathbb{G}_{m,S} \\ \downarrow & & \downarrow \\ i_*\mathbb{G}_{m,C} & \xrightarrow{[2]} & i_*\mathbb{G}_{m,C} \end{array} \tag{2}$$

Let T be the kernel of $\pi_*\mathbb{G}_{m,X} \to \mathbb{G}_{m,S}$. We can extend (2) to a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T & \longrightarrow & \pi_*\mathbb{G}_{m,X} & \longrightarrow & \mathbb{G}_{m,S} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & i_*(\mathbb{Z}/2)_C & \longrightarrow & i_*\mathbb{G}_{m,C} & \xrightarrow{[2]} & i_*\mathbb{G}_{m,C} & \longrightarrow & 0 \end{array} \tag{3}$$

The map $T \to i_*(\mathbb{Z}/2)_C$ is adjoint to the isomorphism $i^*T \tilde{\longrightarrow} (\mathbb{Z}/2)_C$.

The map of sheaves $\pi_*\mathbb{G}_{m,X} \to \mathbb{G}_{m,S}$ induces a homomorphism of abelian groups $\pi_* : \mathrm{Pic}(X) \to \mathrm{Pic}(S)$, whose kernel we denoted by $\mathrm{Pic}(X)[\pi_*]$. We deduce from (2) a commutative diagram with exact rows, which shows that j_* maps $\mathrm{Pic}(X)[\pi_*]$ to $\mathrm{Pic}(C)[2]$:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Pic}(X)[\pi_*] & \longrightarrow & \mathrm{Pic}(X) & \xrightarrow{\pi_*} & \mathrm{Pic}(S) \\ & & \downarrow{\scriptstyle j^*} & & \downarrow{\scriptstyle j^*} & & \downarrow{\scriptstyle i^*} \\ 0 & \longrightarrow & \mathrm{Pic}(C)[2] & \longrightarrow & \mathrm{Pic}(C) & \xrightarrow{[2]} & \mathrm{Pic}(C) \end{array} \quad (4)$$

Let U be the complement to $i(C)$ in S, and V be the complement to $j(C)$ in X. We write $\rho : U \to S$ and $\sigma : V \to X$ for the corresponding open embeddings, so that there is a commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{\sigma} & X \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ U & \xrightarrow{\rho} & S \end{array}$$

By restricting the map of étale sheaves $\pi_*\mathbb{G}_{m,X} \to \mathbb{G}_{m,S}$ to the kernels of [2] we obtain a map $f : \pi_*(\mathbb{Z}/2)_X \to (\mathbb{Z}/2)_S$.

The following lemma is a variation on the Smith exact sequence in topology.

Lemma 2.1 *In the bounded derived category $D(S)$ of étale sheaves on S the cone of $f : \pi_*(\mathbb{Z}/2)_X \to (\mathbb{Z}/2)_S$ is canonically quasi-isomorphic to $\mathrm{R}\rho_*(\mathbb{Z}/2)_U[1]$. Thus in $D(S)$ there is a canonical exact triangle*

$$\mathrm{R}\rho_*(\mathbb{Z}/2)_U \longrightarrow \pi_*(\mathbb{Z}/2)_X \longrightarrow (\mathbb{Z}/2)_S. \quad (5)$$

The associated exact sequence of cohomology is

$$0 \longrightarrow (\mathbb{Z}/2)_S \longrightarrow \pi_*(\mathbb{Z}/2)_X \xrightarrow{f} (\mathbb{Z}/2)_S \longrightarrow i_*(\mathbb{Z}/2)_C \longrightarrow 0. \quad (6)$$

Proof. We have an exact sequence of Knus, Parimala and Srinivas

$$0 \longrightarrow (\mathbb{Z}/2)_U \longrightarrow \pi_*(\mathbb{Z}/2)_V \longrightarrow (\mathbb{Z}/2)_U \longrightarrow 0. \quad (7)$$

Indeed, for any geometric point of U the sequence of stalks of (7) is

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow (\mathbb{Z}/2)^2 \longrightarrow \mathbb{Z}/2 \longrightarrow 0,$$

where the second arrow is the diagonal embedding and the third arrow is the sum. This sequence is visibly exact, so (7) is exact, too.

The cohomological purity theorem for the smooth pair $(S, i(C))$ gives the following canonical isomorphisms:

$$\rho_*(\mathbb{Z}/2)_U = (\mathbb{Z}/2)_S, \quad \mathrm{R}^1\rho_*(\mathbb{Z}/2)_U = i_*(\mathbb{Z}/2)_C, \quad \mathrm{R}^n\rho_*(\mathbb{Z}/2)_U = 0 \text{ for } n > 1, \quad (8)$$

and similarly for $\sigma : V \to X$, see [3, Thm. 3.4, pp. 63–64]. For any étale sheaf $\mathcal{F}$ on S we have a canonical truncation morphism in the derived category $D(S)$

$$\rho_*\mathcal{F} = \tau_{\leq 0}\mathrm{R}\rho_*\mathcal{F} \longrightarrow \mathrm{R}\rho_*\mathcal{F}.$$

Since $\rho_*\pi_*(\mathbb{Z}/2)_V = \pi_*\sigma_*(\mathbb{Z}/2)_V = \pi_*(\mathbb{Z}/2)_X$, we get a commutative diagram of exact triangles in $D(S)$

$$\begin{array}{ccccc} \pi_*(\mathbb{Z}/2)_X & \xrightarrow{f} & (\mathbb{Z}/2)_S & \longrightarrow & \mathrm{Cone}(f) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{R}\rho_*(\pi_*(\mathbb{Z}/2)_V) & \longrightarrow & \mathrm{R}\rho_*(\mathbb{Z}/2)_U & \longrightarrow & \mathrm{R}\rho_*(\mathbb{Z}/2)_U[1] \end{array} \quad (9)$$

where the bottom triangle is obtained by applying the derived functor $\mathrm{R}\rho_*$ to (7) and shifting by 1.

We claim that the morphism $\mathrm{Cone}(f) \to \mathrm{R}\rho_*(\mathbb{Z}/2)_U[1]$ defined by diagram (9), is a quasi-isomorphism. For this we note that the long exact sequence of cohomology attached to the exact triangle $\mathrm{R}\rho_*(\mathbb{Z}/2)_U \to \mathrm{R}\rho_*(\pi_*(\mathbb{Z}/2)_V) \to \mathrm{R}\rho_*(\mathbb{Z}/2)_U$ gives the exact sequence (6), provided we can justify the surjectivity of the fourth arrow in (6). This can be checked on stalks. The stalk of $i_*(\mathbb{Z}/2)_C$ at $x \notin i(C)$ is zero, so we can assume $x \in i(C)$. Then the sequence of stalks of (6) is

$$0 \longrightarrow \mathbb{Z}/2 \xrightarrow{\sim} \mathbb{Z}/2 \longrightarrow \mathbb{Z}/2 \xrightarrow{\sim} \mathbb{Z}/2 \longrightarrow 0,$$

which is exact, because the middle arrow here is the zero map. Hence (6) is exact.

By (8) we now conclude from (9) that $\mathrm{Cone}(f) \to \mathrm{R}\rho_*(\mathbb{Z}/2)_U[1]$ induces an isomorphism on all the cohomology groups, so is indeed a quasi-isomorphism. □

The long exact sequence of cohomology associated to (5) has the following form:

$$\longrightarrow \mathrm{H}^{n-1}(S, \mathbb{Z}/2) \xrightarrow{\delta} \mathrm{H}^n(U, \mathbb{Z}/2) \xrightarrow{\alpha} \mathrm{H}^n(X, \mathbb{Z}/2) \xrightarrow{\pi_*} \mathrm{H}^n(S, \mathbb{Z}/2) \longrightarrow \quad (10)$$

which is the definition of the maps α and δ.

Corollary 2.2 *For any $n \geq 0$ the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{H}^n(X, \mathbb{Z}/2) & \xrightarrow{\sigma^*} & \mathrm{H}^n(V, \mathbb{Z}/2) \\ {\scriptstyle \pi^*}\uparrow & \nwarrow^{\alpha} & \uparrow{\scriptstyle \pi^*} \\ \mathrm{H}^n(S, \mathbb{Z}/2) & \xrightarrow{\rho^*} & \mathrm{H}^n(U, \mathbb{Z}/2) \end{array} \quad (11)$$

Proof. By Lemma 2.1, after extending (9) to the left we obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{R}\rho_*(\mathbb{Z}/2)_U & \longrightarrow & \pi_*(\mathbb{Z}/2)_X \\ {\scriptstyle\cong}\downarrow & & \downarrow \\ \mathrm{R}\rho_*(\mathbb{Z}/2)_U & \longrightarrow & \mathrm{R}\rho_*(\pi_*(\mathbb{Z}/2)_V) \end{array}$$

which implies the commutativity of the upper triangle of (11). For the lower triangle it is enough to note that the truncation map

$$(\mathbb{Z}/2)_S = \tau_{\leq 0}\mathrm{R}\rho_*(\mathbb{Z}/2)_U \longrightarrow \mathrm{R}\rho_*(\mathbb{Z}/2)_U$$

composed with the first arrow in (5) is the natural map $(\mathbb{Z}/2)_S \to \pi_*(\mathbb{Z}/2)_X$. □

Lemma 2.1 implies that the obvious commutative diagram of exact triangles

$$\begin{array}{ccccc} T & \longrightarrow & \pi_*\mathbb{G}_{m,X} & \longrightarrow & \mathbb{G}_{m,S} \\ \downarrow{\scriptstyle [2]} & & \downarrow{\scriptstyle [2]} & & \downarrow{\scriptstyle [2]} \\ T & \longrightarrow & \pi_*\mathbb{G}_{m,X} & \longrightarrow & \mathbb{G}_{m,S} \end{array}$$

gives rise to a commutative diagram in $D(S)$ with exact rows and columns:

$$\begin{array}{ccccc} \mathrm{R}\rho_*(\mathbb{Z}/2)_U & \longrightarrow & \pi_*(\mathbb{Z}/2)_X & \longrightarrow & (\mathbb{Z}/2)_S \\ \downarrow & & \downarrow & & \downarrow \\ T & \longrightarrow & \pi_*\mathbb{G}_{m,X} & \longrightarrow & \mathbb{G}_{m,S} \\ \downarrow{\scriptstyle [2]} & & \downarrow{\scriptstyle [2]} & & \downarrow{\scriptstyle [2]} \\ T & \longrightarrow & \pi_*\mathbb{G}_{m,X} & \longrightarrow & \mathbb{G}_{m,S} \end{array} \qquad (12)$$

Lemma 2.3 *The map $T \to i_*(\mathbb{Z}/2)_C$ from* (3) *coincides with the composition of the differential $T \to \mathrm{R}\rho_*(\mathbb{Z}/2)_U[1]$ attached to the left hand column of* (12) *and the truncation map $\mathrm{R}\rho_*(\mathbb{Z}/2)_U[1] \to \big(\tau_{\geq 1}\mathrm{R}\rho_*(\mathbb{Z}/2)_U\big)[1] = i_*(\mathbb{Z}/2)_C$.*

Proof. This can be checked by a direct calculation on stalks. For this we note that the composed map $T \to \mathrm{R}\rho_*(\mathbb{Z}/2)_U[1] \to i_*(\mathbb{Z}/2)_C$ can be viewed as provided by the snake lemma applied to the middle and right hand columns of (12). If a geometric point $x \in S$ is not in $i(C)$, then the stalk of $i_*(\mathbb{Z}/2)_C$ at x is zero, so there is nothing to check. The stalk of T at $x \in i(C)$ is an invertible element of the strictly local ring $\mathcal{O}_{x,X}$ with norm 1. We can write $\mathcal{O}_{x,X} = \mathcal{O}_{x,S}[\sqrt{t}]$, where $t \in \mathcal{O}_{x,S}$ is a local equation of C. Extracting a square root we write our element as $(a+b\sqrt{t})^2$ for some $a, b \in \mathcal{O}_{x,S}$ such that $(a^2 - b^2t)^2 = 1$, that is, $a^2 - b^2t = s$ where $s = \pm 1$. We need to check that $(a + b\sqrt{t})^2$ is congruent to s modulo the maximal ideal $(\sqrt{t})$, which is immediate. □

From diagram (12) we obtain a commutative diagram of abelian groups with exact rows and columns

$$\begin{array}{ccccccc}
k^* & \longrightarrow & \mathrm{H}^1(S,T) & \longrightarrow & \mathrm{Pic}(X) & \xrightarrow{\pi_*} & \mathrm{Pic}(S) \\
\downarrow{\scriptstyle [2]} & & \downarrow{\scriptstyle [2]} & & \downarrow{\scriptstyle [2]} & & \downarrow{\scriptstyle [2]} \\
k^* & \longrightarrow & \mathrm{H}^1(S,T) & \longrightarrow & \mathrm{Pic}(X) & \xrightarrow{\pi_*} & \mathrm{Pic}(S) \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{cl}} & & \downarrow{\scriptstyle \mathrm{cl}} \\
\mathrm{H}^1(S,\mathbb{Z}/2) & \xrightarrow{\delta} & \mathrm{H}^2(U,\mathbb{Z}/2) & \xrightarrow{\alpha} & \mathrm{H}^2(X,\mathbb{Z}/2) & \xrightarrow{\pi_*} & \mathrm{H}^2(S,\mathbb{Z}/2) \\
 & & & & \downarrow & & \downarrow \\
 & & & & \mathrm{Br}(X)[2] & \xrightarrow{\pi_*} & \mathrm{Br}(S)[2] \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
\end{array} \tag{13}$$

where cl is the cycle class map. From (13) we cut out the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Pic}(X)[\pi_*] & \longrightarrow & \mathrm{Pic}(X) & \xrightarrow{\pi_*} & \pi_*\mathrm{Pic}(X) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \mathrm{cl}} & & \downarrow{\scriptstyle \mathrm{cl}} & & \\
0 & \longrightarrow & \mathrm{H}^2(U,\mathbb{Z}/2)/\delta\mathrm{H}^1(S,\mathbb{Z}/2) & \longrightarrow & \mathrm{H}^2(X,\mathbb{Z}/2) & \xrightarrow{\pi_*} & \mathrm{H}^2(S,\mathbb{Z}/2)[\delta] & \longrightarrow & 0
\end{array} \tag{14}$$

where $\mathrm{H}^2(S,\mathbb{Z}/2)[\delta]$ is the kernel of δ, and the map β is defined by the diagram.

Remark. The spectral sequence $\mathrm{H}^p(S,\mathrm{R}^q\rho_*(\mathbb{Z}/2)_U) \Rightarrow \mathrm{H}^{p+q}(U,\mathbb{Z}/2)$ combined with the purity theorem (8) gives rise to the Gysin exact sequence:

$$0 \longrightarrow \mathrm{H}^1(S,\mathbb{Z}/2) \xrightarrow{\rho^*} \mathrm{H}^1(U,\mathbb{Z}/2) \longrightarrow \mathrm{H}^0(C,\mathbb{Z}/2) \xrightarrow{\theta} \mathrm{H}^2(S,\mathbb{Z}/2) \xrightarrow{\rho^*} \mathrm{H}^2(U,\mathbb{Z}/2) \longrightarrow$$
$$\mathrm{H}^1(C,\mathbb{Z}/2) \xrightarrow{\theta} \mathrm{H}^3(S,\mathbb{Z}/2) \xrightarrow{\rho^*} \mathrm{H}^3(U,\mathbb{Z}/2) \longrightarrow \mathrm{H}^2(C,\mathbb{Z}/2) \xrightarrow{\theta} \mathrm{H}^4(S,\mathbb{Z}/2)$$

The maps marked with θ are called the Gysin maps; we denote by $\mathrm{H}^n(C,\mathbb{Z}/2)[\theta]$ the kernel of the corresponding Gysin map.

Lemma 2.4 *The composition of* $\delta : \mathrm{H}^{n-1}(S,\mathbb{Z}/2) \to \mathrm{H}^n(U,\mathbb{Z}/2)$ *with the map* $\mathrm{H}^n(U,\mathbb{Z}/2) \to \mathrm{H}^{n-1}(C,\mathbb{Z}/2)$ *from the Gysin sequence is the restriction map* i^*.

Proof. The composition of the differential $(\mathbb{Z}/2)_S \to \mathrm{R}\rho_*(\mathbb{Z}/2)_U[1]$ defined by (5) with the truncation map $\mathrm{R}\rho_*(\mathbb{Z}/2)_U[1] \to i^*(\mathbb{Z}/2)_C$ is the map $(\mathbb{Z}/2)_S \to i^*(\mathbb{Z}/2)_C$ given by the restriction to C. $\Box$

We would like to trim (14) a bit more. By Corollary 2.2 the map $\pi^* : \mathrm{H}^2(S,\mathbb{Z}/2) \to \mathrm{H}^2(X,\mathbb{Z}/2)$ factors through $\mathrm{H}^2(U,\mathbb{Z}/2)$. Since $\pi_*\pi^* = [2]$, the subgroup $\pi^*\mathrm{H}^2(S,\mathbb{Z}/2)$

of $\mathrm{H}^2(X,\mathbb{Z}/2)$ is in the kernel of the map π_* to $\mathrm{H}^2(S,\mathbb{Z}/2)$. The Gysin sequence shows that the quotient of $\mathrm{H}^2(U,\mathbb{Z}/2)$ by $\rho^*\mathrm{H}^2(S,\mathbb{Z}/2)$ is $\mathrm{H}^1(C,\mathbb{Z}/2)[\theta]$. Taking quotients by the images of $\mathrm{H}^2(S,\mathbb{Z}/2)$ we obtain from (14) the following commutative diagram with exact rows, where we have also used Lemma 2.4:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Pic}(X)[\pi_*] & \longrightarrow & \mathrm{Pic}(X) & \xrightarrow{\pi_*} & \pi_*\mathrm{Pic}(X) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\mathrm{cl}} & & \downarrow{\scriptstyle\mathrm{cl}} & & \\ 0 & \longrightarrow & \frac{\mathrm{H}^1(C,\mathbb{Z}/2)[\theta]}{i^*\mathrm{H}^1(S,\mathbb{Z}/2)} & \longrightarrow & \frac{\mathrm{H}^2(X,\mathbb{Z}/2)}{\pi^*\mathrm{H}^2(S,\mathbb{Z}/2)} & \xrightarrow{\pi_*} & \mathrm{H}^2(S,\mathbb{Z}/2)[\delta] & \longrightarrow & 0 \end{array} \quad (15)$$

The composition $\pi_*\pi^* : \mathrm{Pic}(S) \to \mathrm{Pic}(S)$ is [2], hence $2\mathrm{Pic}(S) \subset \pi_*\mathrm{Pic}(X)$. In other words, $\pi_*\mathrm{Pic}(X)$ contains $2\mathrm{Pic}(S)$, which is the kernel of the cycle class map $\mathrm{Pic}(S) \to \mathrm{H}^2(S,\mathbb{Z}/2)$. Since $2\mathrm{Pic}(S)$ is the surjective image of $\pi^*\mathrm{Pic}(S) \subset \mathrm{Pic}(X)$, which is in the kernel of the middle vertical map in (15), an application of the snake lemma to (15) gives rise to the following exact sequence

$$0 \longrightarrow \frac{\mathrm{H}^1(C,\mathbb{Z}/2)[\theta]/i^*\mathrm{H}^1(S,\mathbb{Z}/2)}{\beta(\mathrm{Pic}(X)[\pi_*])} \longrightarrow \frac{\mathrm{Br}(X)[2]}{\pi^*(\mathrm{Br}(S)[2])} \longrightarrow \frac{\mathrm{H}^2(S,\mathbb{Z}/2)[\delta]}{\mathrm{cl}(\pi_*\mathrm{Pic}(X))} \longrightarrow 0 \quad (16)$$

From construction it is clear that the map $\mathrm{H}^1(C,\mathbb{Z}/2)[\theta] \longrightarrow \mathrm{Br}(X)[2]/\pi^*(\mathrm{Br}(S)[2])$ given by (16) is the map Φ defined in the introduction. To say more about Φ we need to make some simplifying assumptions. The case when the ground field k is algebraically closed will be considered in the next section.

Proposition 2.5 *If C is geometrically connected and $\mathrm{H}^1(\overline{S},\mathbb{Z}/2) = 0$, then the Gysin map $\theta : \mathrm{H}^1(C,\mathbb{Z}/2) \to \mathrm{H}^3(S,\mathbb{Z}/2)$ factors through $\mathrm{Pic}(C)[2]$. Let us denote by $\mathrm{Pic}(C)[2][\theta]$ the kernel of the resulting map $\mathrm{Pic}(C)[2] \to \mathrm{H}^3(S,\mathbb{Z}/2)$. Then Φ gives rise to the exact sequence*

$$0 \longrightarrow \frac{\mathrm{Pic}(C)[2][\theta]}{j^*(\mathrm{Pic}(X)[\pi_*])} \longrightarrow \frac{\mathrm{Br}(X)[2]}{\pi^*(\mathrm{Br}(S)[2])} \longrightarrow \frac{\mathrm{H}^2(S,\mathbb{Z}/2)[\delta]}{\mathrm{cl}(\pi_*\mathrm{Pic}(X))} \longrightarrow 0.$$

Proof. The spectral sequence $\mathrm{H}^p(k,\mathrm{H}^q(\overline{S},\mathbb{Z}/2)) \Rightarrow \mathrm{H}^{p+q}(S,\mathbb{Z}/2)$ shows that in our assumptions the structure morphism $S \to \mathrm{Spec}(k)$ induces an isomorphism $\mathrm{H}^1(k,\mathbb{Z}/2)\tilde{\longrightarrow}\mathrm{H}^1(S,\mathbb{Z}/2)$. Since C is geometrically connected, we have $\mathrm{H}^0(C,\mathbb{G}_m) = k^*$. The Kummer sequence then gives an exact sequence

$$0 \longrightarrow k^*/k^{*2} \longrightarrow \mathrm{H}^1(C,\mathbb{Z}/2) \longrightarrow \mathrm{Pic}(C)[2] \longrightarrow 0$$

The second arrow here is the map $\mathrm{H}^1(k,\mathbb{Z}/2) \to \mathrm{H}^1(C,\mathbb{Z}/2)$ induced by the structure morphism $C \to \mathrm{Spec}(k)$, hence its image is $i^*\mathrm{H}^1(S,\mathbb{Z}/2)$.

The composed map $\mathrm{H}^1(S,T) \to \mathrm{H}^2(U,\mathbb{Z}/2) \to \mathrm{H}^1(C,\mathbb{Z}/2)$, after passing to the quotients by the image of $\mathrm{H}^1(k,\mathbb{Z}/2)$, becomes

$$\mathrm{Pic}(X)[\pi_*] \xrightarrow{\beta} \mathrm{H}^2(U,\mathbb{Z}/2)/\delta\mathrm{H}^1(S,\mathbb{Z}/2) \longrightarrow \mathrm{Pic}(C)[2].$$

By Lemma 2.3 this composition is the restriction map j^*. □

Remarks. **1**. If $S = \mathbb{P}^2_k$, then C is a geometrically irreducible curve. We have $\mathrm{Pic}(S) = \mathbb{Z}$ and $\mathrm{Br}(S) = \mathrm{Br}(k)$.
2. If $\mathrm{Pic}(X) = \mathbb{Z}$, then $\mathrm{Pic}(S) = \mathbb{Z}$ and $\mathrm{Pic}(X)[\pi_*] = 0$.
3. If we assume that $S = \mathbb{P}^2_k$ and $\mathrm{Pic}(X) = \mathbb{Z}$, then Φ is an injective map from $\mathrm{Pic}(C)[2][\theta]$ to the cokernel of the natural map $\mathrm{Br}(k)[2] \to \mathrm{Br}(X)[2]$.

3 Surfaces

3.1 Cohomology of double covers of surfaces

In this section we describe $\mathrm{H}^2(\overline{X}, \mathbb{Z}/2)$, where $X \to S$ is a double covering of a geometrically simply connected surface. This material is related to the classical Smith theory and is probably well known to the experts. We spell out these descriptions here as they do not seem to be readily available in this form in the literature.

Proposition 3.1 *Let S be a smooth, projective and geometrically integral surface over k with $\mathrm{H}^1(\overline{S}, \mathbb{Z}/2) = 0$, for example, a geometrically simply connected surface. For any finite surjective morphism $\pi : X \to C$ of degree 2 ramified in a non-empty smooth curve C we have an exact sequence of Γ-modules*

$$0 \longrightarrow \mathrm{H}^1(\overline{C}, \mathbb{Z}/2) \longrightarrow \mathrm{H}^2(\overline{X}, \mathbb{Z}/2)/\pi^*\mathrm{H}^2(\overline{S}, \mathbb{Z}/2) \xrightarrow{\pi_*} \mathrm{H}^2(\overline{S}, \mathbb{Z}/2)^{\perp C} \longrightarrow 0, \quad (17)$$

where $\mathrm{H}^2(\overline{S}, \mathbb{Z}/2)^{\perp C}$ is the subgroup consisting of the elements orthogonal to the connected components of $\overline{C}$ with the respect to the cup-product pairing.

Proof. We obtain (17) from the bottom exact sequence of (15) considered over $\bar{k}$. By the Poincaré duality we have $\mathrm{H}^3(\overline{S}, \mathbb{Z}/2) = 0$. This implies that $\mathrm{H}^1(\overline{C}, \mathbb{Z}/2)[\theta]$ is all of $\mathrm{H}^1(\overline{C}, \mathbb{Z}/2)$. It remains to identify $\mathrm{H}^2(\overline{S}, \mathbb{Z}/2)[\delta]$ with $\mathrm{H}^2(\overline{S}, \mathbb{Z}/2)^{\perp C}$. As $\mathrm{H}^3(\overline{S}, \mathbb{Z}/2) = 0$ the Gysin sequence gives the injectivity of the map $\mathrm{H}^3(\overline{U}, \mathbb{Z}/2) \to \mathrm{H}^2(\overline{C}, \mathbb{Z}/2)$. By Lemma 2.4 we now obtain that $\mathrm{H}^2(\overline{S}, \mathbb{Z}/2)[\delta]$ is the kernel of the restriction map

$$i^* : \mathrm{H}^2(\overline{S}, \mathbb{Z}/2) \longrightarrow \mathrm{H}^2(\overline{C}, \mathbb{Z}/2) = \bigoplus_{i=1}^{n} \mathrm{H}^2(\overline{C}_i, \mathbb{Z}/2) = (\mathbb{Z}/2)^n,$$

where $\overline{C}_1, \ldots, \overline{C}_n$ are the connected components of $\overline{C}$. Each restriction map

$$\mathrm{H}^2(\overline{S}, \mathbb{Z}/2) \longrightarrow \mathrm{H}^2(\overline{C}_i, \mathbb{Z}/2) = \mathbb{Z}/2$$

coincides with the cup-product with the class of $\overline{C}_i$ in $\mathrm{H}^2(\overline{S}, \mathbb{Z}/2)$, and the proposition follows. □

Corollary 3.2 *In the assumptions of Proposition 3.1 assume further that C is geometrically connected. Then we have the following properties:*

(i) *the map* $\pi^* : \mathrm{H}^2(\overline{S}, \mathbb{Z}/2) \to \mathrm{H}^2(\overline{X}, \mathbb{Z}/2)$ *is injective;*

(ii) *the map* $\pi_* : \mathrm{H}^2(\overline{X}, \mathbb{Z}/2) \to \mathrm{H}^2(\overline{S}, \mathbb{Z}/2)$ *is surjective;*

(iii) *there is an exact sequence of* Γ*-modules*

$$0 \longrightarrow \mathrm{H}^1(\overline{C}, \mathbb{Z}/2) \longrightarrow \mathrm{H}^2(\overline{X}, \mathbb{Z}/2)/\pi^*\mathrm{H}^2(\overline{S}, \mathbb{Z}/2) \xrightarrow{\pi_*} \mathrm{H}^2(\overline{S}, \mathbb{Z}/2) \longrightarrow 0; \qquad (18)$$

(iv) *The automorphism group of the double covering* $\pi : X \to S$ *defines an action of* $\mathbb{Z}/2$ *on* $\mathrm{H}^2(\overline{X}, \mathbb{Z}/2)$*. For each* $i \geq 1$ *there is a canonical isomorphism of* Γ*-modules*

$$\mathrm{H}^i(\mathbb{Z}/2, \mathrm{H}^2(\overline{X}, \mathbb{Z}/2)) = \mathrm{H}^1(\overline{C}, \mathbb{Z}/2).$$

Proof. (i) In the commutative diagram (11) the map $\alpha : \mathrm{H}^2(\overline{U}, \mathbb{Z}/2) \to \mathrm{H}^2(\overline{X}, \mathbb{Z}/2)$ is injective because its kernel is the image of $\mathrm{H}^1(\overline{S}, \mathbb{Z}/2) = 0$. Since $\overline{C}$ is connected we have $\mathrm{H}^0(\overline{C}, \mathbb{Z}/2) = \mathbb{Z}/2$. The class of the connected unramified double covering $\pi : \overline{V} \to \overline{U}$ is a non-zero element of $\mathrm{H}^1(\overline{U}, \mathbb{Z}/2)$. As $\mathrm{H}^1(\overline{S}, \mathbb{Z}/2) = 0$ the Gysin sequence shows that $\rho^* : \mathrm{H}^2(\overline{S}, \mathbb{Z}/2) \to \mathrm{H}^2(\overline{U}, \mathbb{Z}/2)$ is injective. By the commutativity of the diagram (11) we conclude that $\pi^* : \mathrm{H}^2(\overline{S}, \mathbb{Z}/2) \to \mathrm{H}^2(\overline{X}, \mathbb{Z}/2)$ is injective.

(ii) By Proposition 3.1 we need to show that the class of $\overline{C}$ in $\mathrm{H}^2(\overline{S}, \mathbb{Z}/2)$ is zero. This class comes from the class $[\overline{C}] \in \mathrm{Pic}(\overline{S})$ under the cycle map. But $\overline{C}$ is the ramification divisor of $\pi : \overline{X} \to \overline{S}$, so $[\overline{C}]$ is divisible by 2 in $\mathrm{Pic}(\overline{S})$.

(iii) This follows from (ii) and the exact sequence (17).

(iv) Let $\iota : X \to X$ be the involution defined by the double covering $\pi : X \to S$. It is well known that for even $i \geq 2$ the group $\mathrm{H}^i(\mathbb{Z}/2, \mathrm{H}^2(\overline{X}, \mathbb{Z}/2))$ is canonically isomorphic to $\mathrm{Ker}(\mathrm{Id} - \iota^*)/\mathrm{Im}(\mathrm{Id} + \iota^*)$. For odd $i \geq 1$ the same is true once we replace ι^* by $-\iota^*$. Since the coefficient group is $\mathbb{Z}/2$, the sign plays no role and hence the answer is the same for even and odd i. The map $\mathrm{Id} + \iota^*$ can be written as the composition

$$\mathrm{H}^2(\overline{X}, \mathbb{Z}/2) \xrightarrow{\pi_*} \mathrm{H}^2(\overline{S}, \mathbb{Z}/2) \xrightarrow{\pi^*} \mathrm{H}^2(\overline{X}, \mathbb{Z}/2).$$

By (i) the second map here is injective, hence $\mathrm{Ker}(\mathrm{Id} + \iota^*) = \mathrm{H}^2(\overline{X}, \mathbb{Z}/2)[\pi_*]$. By (ii) the first map here is surjective, hence $\mathrm{Im}(\mathrm{Id} + \iota^*) = \pi^*\mathrm{H}^2(\overline{S}, \mathbb{Z}/2)$. □

The case when C is a curve of degree 4 in $S = \mathbb{P}^2_k$, so that X is a del Pezzo surface of degree 2, was discussed in [4, Section IX.1], see also [21, Lemma 1.1].

The exact sequence (18) is the analogue of a similar sequence for double coverings of curves. Let $\pi : C \to D$ be a double covering of smooth, projective and geometrically integral curves ramified in a non-empty 0-dimensional closed subscheme $B \subset D$. Let us denote by $(\mathbb{Z}/2)^B$ the permutation Γ-module whose generators bijectively correspond to the $\bar{k}$-points of B. Let $(\mathbb{Z}/2)^B_0 \subset (\mathbb{Z}/2)^B$ be the submodule of vectors with the zero sum of coordinates. Since $|B(\bar{k})|$ is even, the diagonal

$\mathbb{Z}/2 \subset (\mathbb{Z}/2)^B$ is contained in $(\mathbb{Z}/2)^B_0$. Using the same methods as above, that is, the Gysin sequence and (10), one obtains the well known exact sequence of Γ-modules

$$0 \longrightarrow (\mathbb{Z}/2)^B_0/\mathbb{Z}/2 \longrightarrow \mathrm{H}^1(\overline{C},\mathbb{Z}/2)/\pi^*\mathrm{H}^1(\overline{D},\mathbb{Z}/2) \xrightarrow{\pi_*} \mathrm{H}^1(\overline{D},\mathbb{Z}/2) \longrightarrow 0 \qquad (19)$$

and the injectivity of $\pi^* : \mathrm{H}^1(\overline{D},\mathbb{Z}/2) \to \mathrm{H}^1(\overline{C},\mathbb{Z}/2)$. The proof is left to the reader.

Remarks. **1**. Let us return to the situation of Proposition 3.1. In this case the proof of Corollary 3.2 shows that the kernel of the map $\pi^* : \mathrm{H}^2(\overline{S},\mathbb{Z}/2) \to \mathrm{H}^2(\overline{X},\mathbb{Z}/2)$ is the subgroup L_C generated by the classes of the connected curves $\overline{C}_i$, for $i = 1,\ldots,n$. Thus the non-degenerate cup-product pairing

$$\mathrm{H}^2(\overline{S},\mathbb{Z}/2) \ \times \ \mathrm{H}^2(\overline{S},\mathbb{Z}/2) \ \longrightarrow \mathbb{Z}/2$$

induces an isomorphism of Γ-modules

$$\pi^*\mathrm{H}^2(\overline{S},\mathbb{Z}/2) = \mathrm{H}^2(\overline{S},\mathbb{Z}/2)/L_C = \mathrm{Hom}(\mathrm{H}^2(\overline{S},\mathbb{Z}/2)^{\perp C},\mathbb{Z}/2).$$

2. The cup-product pairing satisfies the property $(\pi^*(x),\pi^*(y)) = 2(x,y)$ for any $x,y \in \mathrm{H}^2(\overline{S},\mathbb{Z}/2)$. Therefore, $\pi^*\mathrm{H}^2(\overline{S},\mathbb{Z}/2)$ is a hyperbolic subspace of $\mathrm{H}^2(\overline{X},\mathbb{Z}/2)$ with respect to the intersection pairing. In the particular case when $\overline{C}$ is a disjoint union of projective lines, so that $\mathrm{H}^1(\overline{C},\mathbb{Z}/2) = 0$, we obtain that $\pi^*\mathrm{H}^2(\overline{S},\mathbb{Z}/2)$ is a maximal hyperbolic subspace of $\mathrm{H}^2(\overline{X},\mathbb{Z}/2)$. This situation arises when X is the blowing-up of the eight fixed points of a symplectic involution on a K3 surface, and S is the K3 surface which is the quotient of X by this involution. Likewise, it arises when X is the blowing-up of the sixteen fixed points of the antipodal involution of an abelian surface, and S is the associated Kummer surface.

3.2 Proof of Theorem 1.1

Let k be an algebraically closed field of characteristic not equal to 2. We now assume that S is a surface such that $\mathrm{Pic}(S)[2] = \mathrm{Br}(S)[2] = 0$. Since S is projective, the Kummer sequence gives $\mathrm{H}^1(S,\mathbb{Z}/2) = 0$. The vanishing of $\mathrm{Br}(S)[2]$ implies the surjectivity of the class map $\mathrm{Pic}(S)/2 \tilde{\longrightarrow} \mathrm{H}^2(S,\mathbb{Z}/2)$. The Kummer sequence for C gives an isomorphism $\mathrm{H}^1(C,\mathbb{Z}/2) = \mathrm{Pic}(C)[2]$. By Poincaré duality $\mathrm{H}^1(S,\mathbb{Z}/2) = 0$ implies $\mathrm{H}^3(S,\mathbb{Z}/2) = 0$.

Lemma 3.3 *The composition* $\mathrm{Pic}(X)[\pi_*] \xrightarrow{\beta} \mathrm{H}^2(U,\mathbb{Z}/2) \to \mathrm{Pic}(C)[2]$ *is the restriction map* j^*.

Proof. Since k is algebraically closed, and both S and X are connected varieties, from the definition of T we obtain $\mathrm{H}^1(S,T) = \mathrm{Pic}(X)[\pi_*]$. By Lemma 2.3 the map $j^* : \mathrm{H}^1(S,T) = \mathrm{Pic}(X)[\pi_*] \to \mathrm{Pic}(C)[2]$ factors through $\mathrm{H}^2(U,\mathbb{Z}/2)$, as required. □

Therefore, in our situation, (16) takes the form

$$0 \longrightarrow \frac{\mathrm{Pic}(C)[2]}{j^*\mathrm{Pic}(X)[\pi_*]} \longrightarrow \mathrm{Br}(X)[2] \longrightarrow \frac{\mathrm{H}^2(S,\mathbb{Z}/2)[\delta]}{\mathrm{cl}(\pi_*\mathrm{Pic}(X))} \longrightarrow 0. \tag{20}$$

To complete the proof of Theorem 1.1 we use the following lemma.

Lemma 3.4 *The kernel of* $\delta : \mathrm{H}^2(S,\mathbb{Z}/2) \to \mathrm{H}^3(U,\mathbb{Z}/2)$ *is* $\mathrm{Pic}(S)^{\mathrm{even}}/2\mathrm{Pic}(S)$.

Proof. By Lemma 2.4 the composition of $\delta : \mathrm{H}^2(S,\mathbb{Z}/2) \to \mathrm{H}^3(U,\mathbb{Z}/2)$ with the map $\mathrm{H}^3(U,\mathbb{Z}/2) \to \mathrm{H}^2(C,\mathbb{Z}/2)$ from the Gysin sequence is the restriction map i^*. Since $\mathrm{H}^3(S,\mathbb{Z}/2) = 0$, we obtain from the Gysin sequence that $\mathrm{H}^2(S,\mathbb{Z}/2)[\delta]$ is the kernel of the restriction map

$$\mathrm{H}^2(S,\mathbb{Z}/2) \to \mathrm{H}^2(C,\mathbb{Z}/2) = (\mathbb{Z}/2)^{\pi_0(C)}.$$

If C' is a connected component of C, the corresponding restriction map $\mathrm{Pic}(S)/2 \to \mathbb{Z}/2$ is the cup-product modulo 2 with the class of C' in $\mathrm{Pic}(S)$. □

Remark. If U is affine, we have $\mathrm{H}^3(U,\mathbb{Z}/2) = 0$ by the affine Lefschetz theorem.

3.3 Surfaces to which the theorem can be applied

A K3 surface X that is a double cover of a rational surface S has an involution σ such that $S = X/\sigma$. This involution is *non-symplectic*, that is, it acts on $\mathrm{H}^0(X,\Omega^2)$ by -1. Furthermore, the set of fixed points X^σ is a non-empty smooth curve which is not necessarily connected [17, Section 4].

Corollary 3.5 *Let X be a smooth K3 surface over $\mathbb{C}$ with a non-symplectic involution σ such that $X^\sigma \neq \emptyset$. Let $\pi : X \to X/\sigma$ be the quotient map, and let $j : X^\sigma \to X$ be the natural closed embedding. Then there is an exact sequence*

$$0 \longrightarrow \mathrm{Pic}(X^\sigma)[2]/j^*(\mathrm{Pic}(X)[\pi_*]) \longrightarrow \mathrm{Br}(X)[2] \to \mathrm{Pic}(X/\sigma)^{\mathrm{even}}/\pi_*\mathrm{Pic}(X) \to 0.$$

Proof. By [17, Section 4] $S = X/\sigma$ is a smooth rational surface and $C = X^\sigma$ is a non-empty smooth curve in S. Thus the corollary follows from Theorem 1.1. □

Below we list examples of del Pezzo surfaces S doubly covered by K3 (or more general) surfaces. Here k is a field of characteristic different from 2.

(1) Let $S = \mathbb{P}^2_k$ and let $\pi : X \to \mathbb{P}^2_k$ be a double covering ramified in a smooth curve of even degree C. If $\mathrm{Pic}(\overline{X})$ is generated by $\pi^*\mathcal{O}(1)$, the exact sequence (1) takes the form

$$0 \longrightarrow \mathrm{Pic}(\overline{C})[2] \xrightarrow{\Phi} \mathrm{Br}(\overline{X})[2] \longrightarrow \mathbb{Z}/2 \longrightarrow 0.$$

This is the exact sequence of [5, Thm. 2.4] or [20, Thm. 6.2]. Without assuming that the Picard rank of $\overline{X}$ is 1, with some extra work one can deduce from (1) the exact sequence of [12, Thm. 1.1] mentioned in the introduction.

(2) See [20, Examples 5.6, 6.3] for K3 surfaces that are double covers of the quadric $\mathbb{P}^1_k \times \mathbb{P}^1_k$ or the ruled surface $\mathbb{F}_1$.

(3) A del Pezzo surface S of degree 4 is a smooth complete intersection of two quadrics in $\mathbb{P}^4_k$. A smooth complete intersection of the projective cone over S and a quadric in $\mathbb{P}^5_k$ that does not pass through its vertex is a K3 surface doubly covering S. According to [19, Cor. 3.3] for any del Pezzo surface S of degree 4 over k there exists a curve D of genus 2 and a 2-covering J_λ of the Jacobian J of D with the following property. Let K_λ be the Kummer surface associated to J_λ. Then $J[2]$ acts on K_λ and on S and there is a $J[2]$-equivariant finite morphism $\pi : K_\lambda \to S$ of degree 2 ramified in a canonical curve C of genus 5. This morphism maps the sixteen rational curves of K_λ (corresponding to the $J[2]$-torsor in J_λ which is the inverse image of 0 under the canonical map $J_\lambda \to J$) to the sixteen lines in S. See [19, §3.2] for details; see also [16] and [7]. Since the classes of lines on $\overline{S}$ generate $\mathrm{Pic}(\overline{S})$, we see that in this case we have $\pi_*\mathrm{Pic}(\overline{X}) = \mathrm{Pic}(\overline{S})$.

(4) A del Pezzo surface S of degree 3 is a smooth cubic surface in $\mathbb{P}^3_k$. A smooth complete intersection of the projective cone over S and a quadric in $\mathbb{P}^4_k$ not passing through the vertex of the cone is a K3 surface that is a double cover of S.

(5) Finally, a del Pezzo surface S of degree 2 is a double cover of $\mathbb{P}^2_k$ ramified in a smooth quartic curve C. The class of C in $\mathrm{Pic}(S)$ is divisible by 2, so there is a double cover $X \to S$ ramified exactly in C. If C is given by a quartic form $f(x, y, z) = 0$, then X is a quartic K3 surface with equation $t^4 = cf(x, y, z)$ for some $c \in k^*$. In the particular case when $f(x, y, z)$ is a diagonal quartic form we obtain the following statement.

Proposition 3.6 *Let k be a field of characteristic different from 2, and let $X \subset \mathbb{P}^3_k$ be the surface given by*

$$ax^4 + by^4 + cz^4 + dw^4 = 0,$$

where $a, b, c, d \in k^*$. The group $\mathrm{Br}(\overline{X})^\Gamma$ contains an element of order 2.

Proof. The branch curve $\overline{C}$ is isomorphic to the Fermat quartic curve, so we may assume $f(x, y, z) = x^4 + y^4 + z^4$. It is well known that $\mathrm{Pic}(\overline{X}) \simeq \mathbb{Z}^{20}$, so that $\mathrm{Br}(\overline{X})[2] \simeq (\mathbb{Z}/2)^2$. The group $\mathrm{Pic}(\overline{S})$ is generated by the (-1)-curves in $\overline{S}$ that map to the 28 bitangents to $\overline{C}$ in $\mathbb{P}^2_{\bar{k}}$. On the other hand, $\mathrm{Pic}(\overline{X})$ is generated by the obvious 48 lines on $\overline{X}$, see [18, Lemma 1]. The morphism $\pi : \overline{X} \to \overline{S}$ maps these 48 lines to those (-1)-curves in $\overline{S}$ that lie above the lines in $\mathbb{P}^2_{\bar{k}}$ which meet $\overline{C}$ in exactly one point with multiplicity 4. Hence $\pi_*\mathrm{Pic}(\overline{X})$ is the subgroup of $\mathrm{Pic}(\overline{S})$ generated by the (-1)-curves above such lines in $\mathbb{P}^2_{\bar{k}}$. A calculation performed by Martin Bright shows that $\pi_*\mathrm{Pic}(\overline{X})$ is a subgroup of $\mathrm{Pic}(\overline{S})$ of index 2. Now (1) shows that $j^*(\mathrm{Pic}(\overline{X})[\pi_*])$ has index 2 in $\mathrm{Pic}(\overline{C})[2]$. We conclude from Theorem 1.1 that $\Phi : \mathrm{Pic}(\overline{C})[2] \to \mathrm{Br}(\overline{X})[2]$ factors through an injective map of Galois modules $\mathbb{Z}/2 \to \mathrm{Br}(\overline{X})[2]$. $\Box$

Questions. **1**. When is the non-zero element of $\mathrm{Br}(\overline{X})[2]^\Gamma$ from Proposition 3.6 contained in the image of the natural map $\mathrm{Br}(X) \to \mathrm{Br}(\overline{X})$?
2. How does the Galois group Γ act on $\mathrm{Br}(\overline{X})[2]$?

For $k = \mathbb{Q}$ a theorem of Evis Ieronymou [9, Thm. 5.2] says that if 2 is not in the subgroup of $\mathbb{Q}^*$ generated by -1, 4, b/a, c/a, d/a and $\mathbb{Q}^{*4}$, then the map $\mathrm{Br}(X)\{2\} \to \mathrm{Br}(\overline{X})$ is zero, where $\mathrm{Br}(X)\{2\}$ is the 2-primary subgroup of $\mathrm{Br}(X)$. Thus in this case no non-zero element of $\mathrm{Br}(\overline{X})[2]^\Gamma$ comes from $\mathrm{Br}(X)$. I do not know what happens if this condition is not fulfilled. See [9, 10, 11] and references in these papers for known facts about the Brauer group of diagonal quartic surfaces.

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