Gödel's Proof

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24.4.06-14.1.78

ON FORMALLY UNDECIDABLE PROPOSITIONS OF PRINCIPIA MATHEMATICA AND RELATED SYSTEMS 11

by Kurt Gödel, Vienna

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The development of mathematics in the direction of greater exactness has–as is well known–led to large tracts of it becoming formalized, so that proofs can be carried out according to a few mechanical rules. The most comprehensive formal systems yet set up are, on the one hand, the system of Principia Mathematica (**PM**)2 and, on the other, the axiom system for set theory of Zermelo-Fraenkel (later extended by J. v. Neumann).3 These two systems are so extensive that all methods of proof used in

27.

Su x(n|y) ° ez {z £ [Pr(l(x)+l(y))]x+y & [($u,v)u,v £ x & x = u * R(b Gl x) * v & z = u * y * v & n = l(u)+1]}

Su x(n|y) derives from **x** on substituting **y** in place of the **n**-th term of **x** (it being assumed that **0 < n £ l(x)**).

28.

0 St v,x ° en {n £ l(x) & v Fr n,x & not ($p)[n < p £ l(x) & v Fr p,x]}

(k+1) St v,x ° en {n < k St v,x & v Fr n,x & ($p)[n < p < k St v,x & v Fr p,x]}

systems, including, in particular, all those arising from

The essence

- ☹ First theorem of undecidability:

If axiomatic theory is consistent,
there exist theorems which can
neither be proved or disproved

The essence

- ☺ Second theorem of undecidability:

There is no constructive procedure
which will prove
axiomatic theory to be consistent.

Euclid's *Elements*

lived circa 300 BC

23 definitions

5 postulates

465 propositions

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Euclid’s *Elements*

The axioms

It is possible

- ► to draw a straight line from any point to any point.
- ► to produce a finite straight line continuously in a straight line.
- ► to describe a circle with any centre and radius.

That all right angles equal one another.

Parallel lines don’t cross

Euclid’s *Elements*

Consistency

- Can mutually inconsistent statements be derive from a set of axioms.

Say in Euclid’s geometry

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Euclid’s *Elements*

In other words

- Can we be sure no one some day derives a proposition which contradicts another proposition.

PRINCIPIA MATHEMATICA

BY

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VOLUME I

Cambridge
at the University Press
1910

PM

∗1. PRIMITIVE IDEAS AND PROPOSITIONS

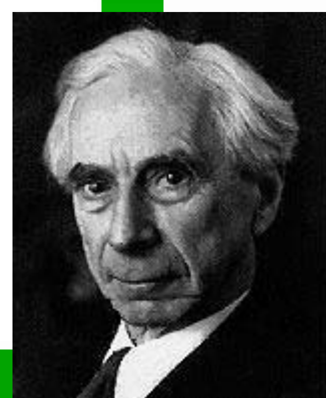
Since all definitions of terms are effected by means of other terms, every system of definitions which is not circular must start from a certain apparatus of undefined terms. It is to some extent optional what ideas we take as undefined in mathematics; the motives guiding our choice will be (1) to make the number of undefined ideas as small as possible, (2) as between two systems in which the number is equal, to choose the one which seems the simpler and easier. We know no way of proving that such and such a system of undefined ideas contains as few as will give such and such results*. Hence we can only say that such and such ideas are undefined in such and such ..., not that they are indefinable. Following Peano, we shall call the ...ed ideas and the undemonstrated propositions *primitive* ideas and ...tive propositions respectively. The primitive ideas are *explained* by means ... descriptions intended to point out to the reader what is meant; but the explanations do not constitute definitions, because they really involve the ideas they explain.

In the present number, we shall first enumerate the primitive ideas required in this section; then we shall define *implication*; and then we shall enunciate the primitive propositions required in this section. Every definition or proposition in the work has a number, for purposes of reference. Following Peano, we use numbers having a decimal as well as an integral part, in order to be able to insert new propositions between ... in the integral part of the number will be used to ... chapter. Definitions will generally have numbers whose ... than ·1, and will be usually put at the beginning of cha... the integral parts of the numbers of propositions will ... being preceded by a star; thus "∗1·01" will mean the defi... so numbered, and "∗1" will mean the chapter in whic... numbers whose integral part is 1, i.e. the present cha... generally be called "numbers."

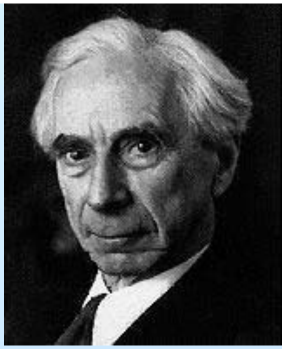
PRIMITIVE IDEAS.

...lementary propositions. By an "elementary" p... ...h does not involve any variables, or, in other ... ...nvolve such words as "all," "some," "the" or ... proposition such as "this is red," where "this" ... ...ion, will be elementary. Any combination o... ...ns by means of negation, disjunction or conjunc...

* ...ecognized methods of proving independence are not applic... ...ls. Cf. *Principles of Mathematics*, § 17. What is there s... ... applies with even greater force to primitive ideas.



Undecidable



- Russell's paradox:

Two types of sets:

Normal

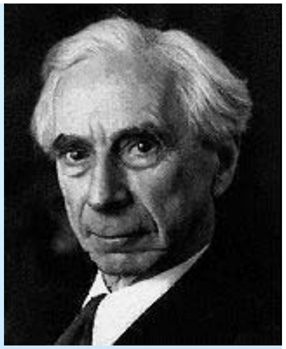
▶ those who don't contain themselves: $A \not\subseteq A$

&

Non-normal

▶ those who do contain themselves $B \subseteq B$

Undecidable



Examples:

Normal set:

$A =$ the set of MPhys 2 student

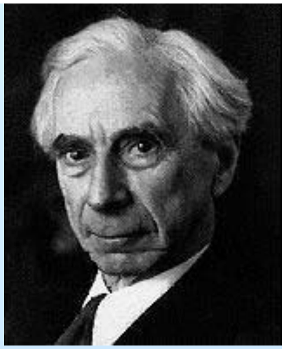
$$\Rightarrow A \not\subseteq A$$

Non-normal:

$B =$ the set of all thinkable things

$$\Rightarrow B \subseteq B$$

Undecidable



Define: N = Set of all Normal sets

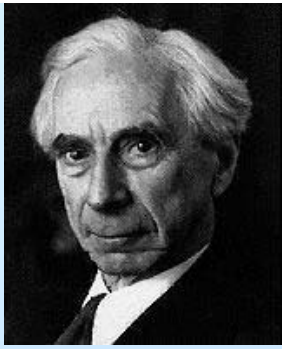
Question: Is N normal?

☻ Assume N is Normal ⇨ then N is member of itself, since N contains all Normal Sets per its definition i.e., N $\subseteq$ N.

☹ But if N $\subseteq$ N then N is non-Normal

So N being Normal implies
N being non-Normal !

Undecidable



Define: N = Set of all Normal sets

☻ Assume N is non-Normal ⇨ then N is member of itself per definition of non-Normal.

☹ But if N is non-Normal it is a member of itself, and N contains per definition Normal sets, i.e., N is Normal .

So N being non-Normal implies N being Normal !

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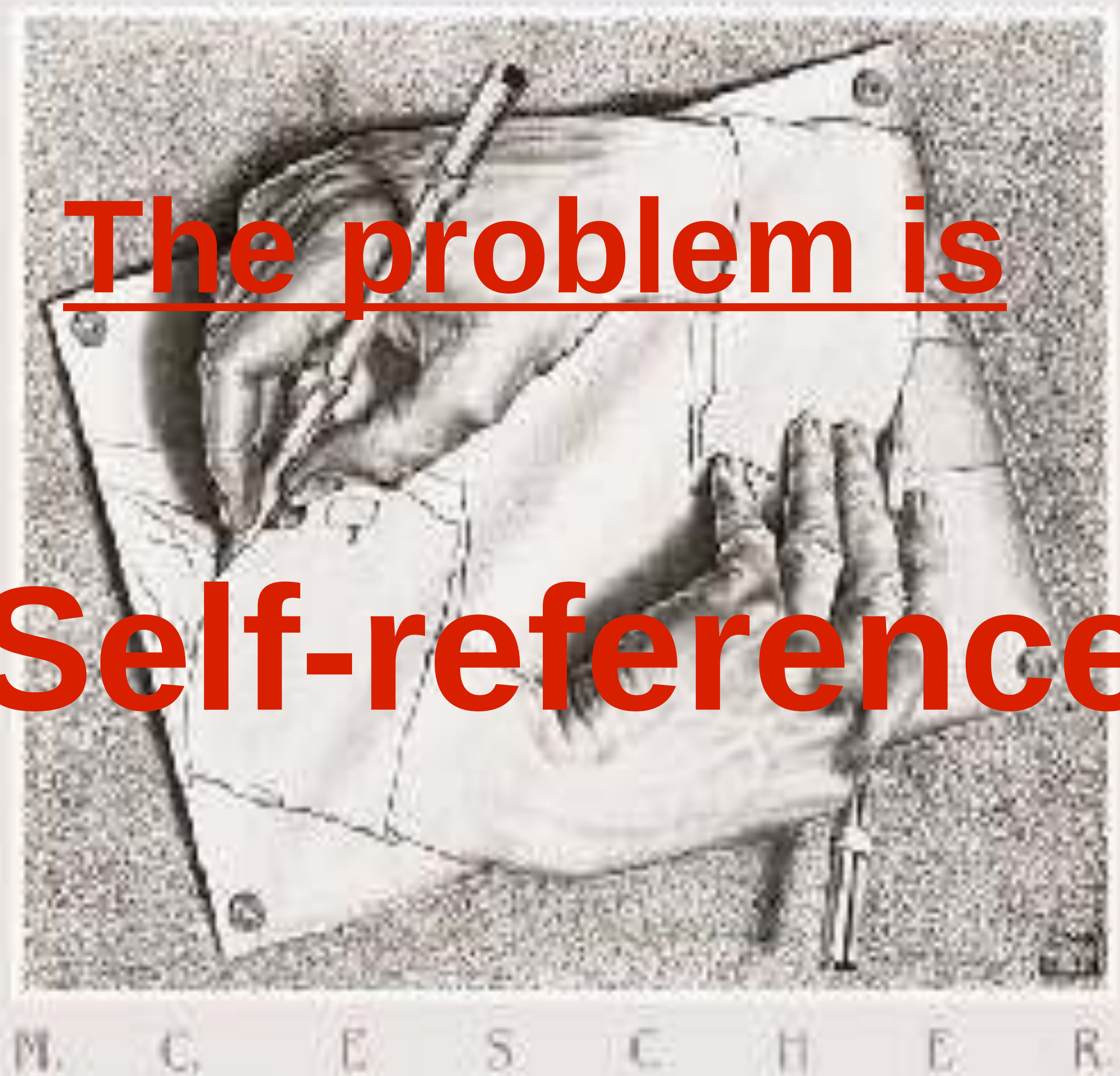
The problem is

Self-reference



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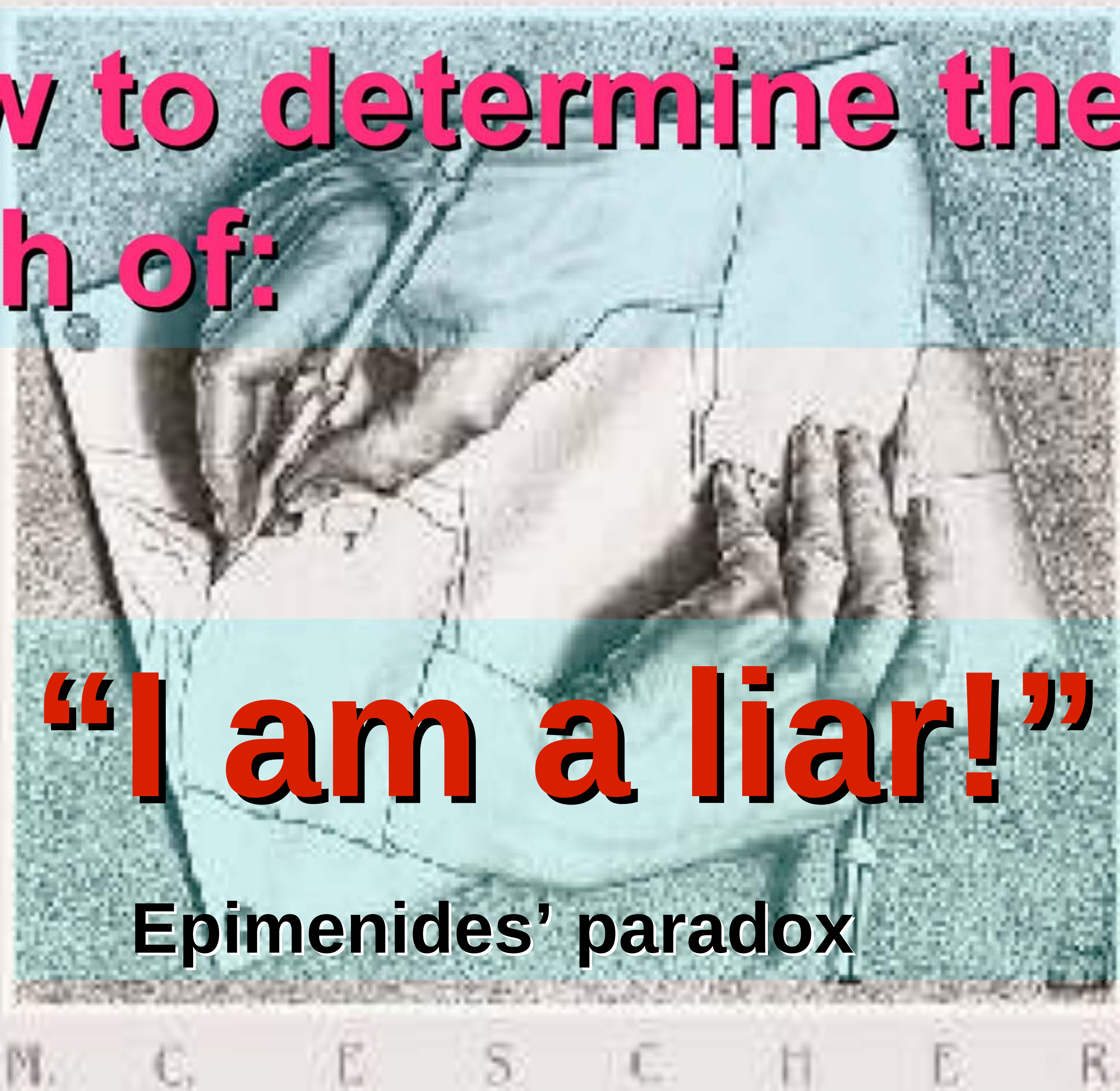
How to determine the truth of:



“I am a liar!”

Epimenides’ paradox

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The strategy of Gödel's proof

Distinguish between:

mathematics → $x = x,$ $x^2 = 9$

&

meta-mathematics → *x=4* is not a solution of x+2=3,

PM is consistent

The strategy of Gödel's proof

Enumeration of formalised system:

Signs:

Sign	Gödel number	
$\neg$	1	not
$\vee$	2	or
$\Rightarrow$	3	If … then
$\exists$	4	there is an
$=$	5	equals
0	6	zero
s	7	immediate successor

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The strategy of Gödel's proof

Enumeration of formalised system:

Math formulas:

☻

$(\exists x)(x = sy)$

☻

☻

Gödel number

$2^8 \times 3^4 \times \cdots$

☻

There is a number *x* following right after *y*

☻

☻

The strategy of Gödel's proof

Enumeration of formalised system:

Meta-maths:

The formula G is not demonstrable using the rules of PM

Gödel number

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The crunch

Gödel constructed a formula G for which he showed that:

G is demonstrable $\Leftrightarrow$ non G is demonstrable

More crunch

The meta-mathematical content of G is:

G= ‘The formula G is not demonstrable’

Or formally within PM:

$$\neg(\exists x)\mathrm{Dem}(x,\mathrm{Sub}(\mathrm{n},17,\mathrm{n}))$$

And more crunch

G= 'The formula G is not demonstrable'

So since G cannot be demonstrated, it is, per definition, TRUE, though its TRUTH cannot be proved within PM

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The Conclusion

All axiomatic systems will contain true propositions which cannot be proved <u>within</u> the system

And contain propositions which cannot be determined whether true of false

Some consequences:

Cantor

The continuum hypothesis:

No set
can have a number of elements between

the cardinality of the natural numbers
and
the cardinality of the real numbers

Cardinality:

Cantor

The real numbers cannot be countered

Proof: assume the opposite

$$r_1 = 0.x_{11}x_{12}x_{13}\cdots$$

$$r_2 = 0.x_{21}x_{22}x_{23}\cdots$$

$$r_3 = 0.x_{31}x_{32}x_{33}\cdots$$

$$\vdots$$

$$r = 0.x_{11}x_{22}x_{33}$$

$$r = 0.\tilde{x}_{11}\tilde{x}_{22}\tilde{x}_{33}$$

Cardinality:

Cantor

So clearly:

reals > # integers

But:

is there a set with a number of elements in between?

Cantor said: “No” - but could not prove it.

Consequences continued:

The continuum hypothesis:

Was the first problem in Hilbert's list of 23 unsolved important problems.

Paul Cohen showed in 1963 that the continuum hypothesis is undecidable

More consequences:

☺ Truth cannot be identified with provability

☺ Roger Penrose: Creative mathematicians do not think in a mechanistic way. They often have a kind of insight into the Platonic realm which exists independently of us.

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More consequences:

- ☺ We cannot build one all embracing explanation of everything based on one finite set of axioms.
- ☺ There exist more true statements than the countable number of truths that can be recursively deduced from a finite set of axioms.

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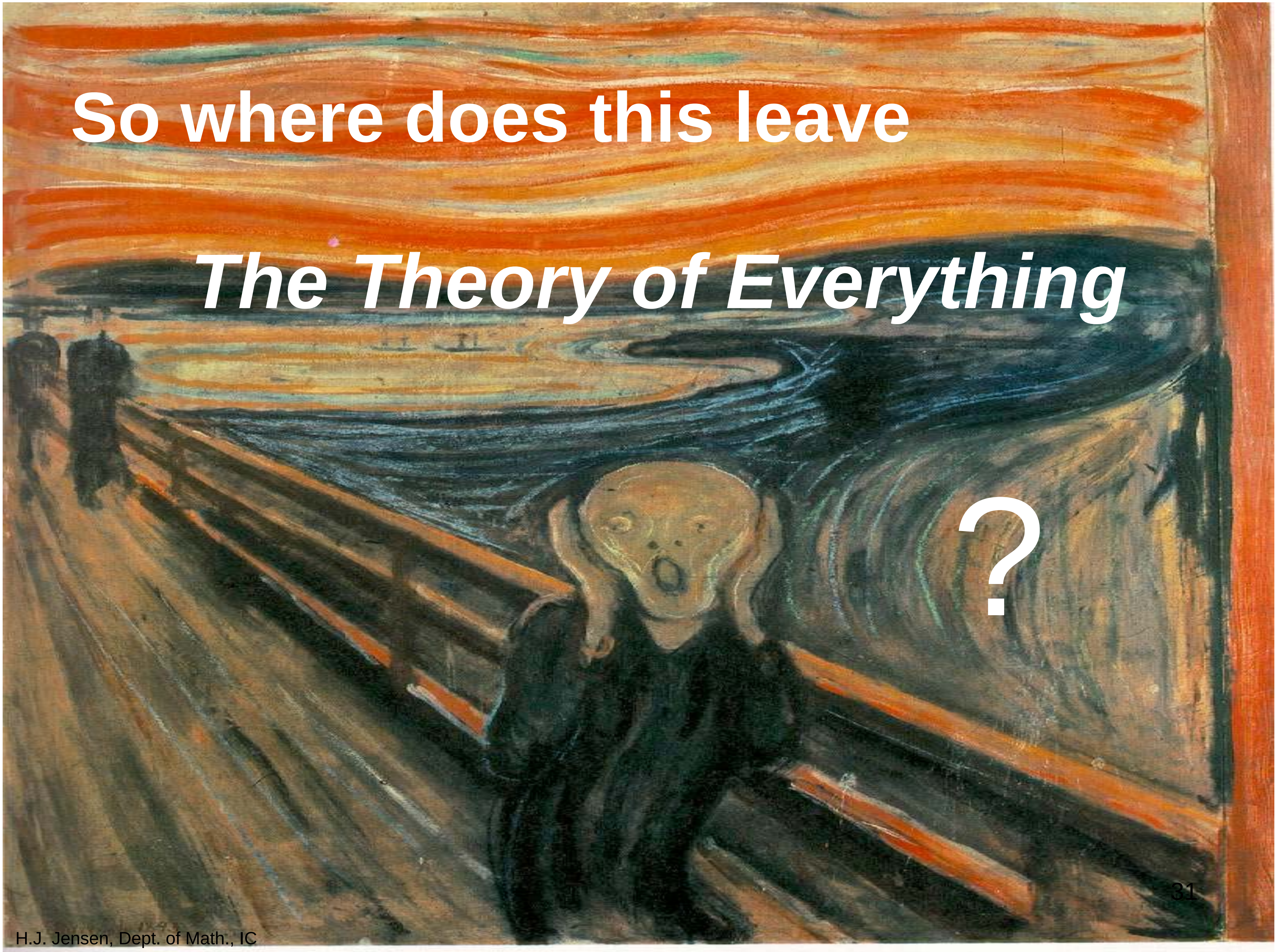
The world is too complex for a `finitistic' axiomatic approach to suffice.
Creativity is needed at all levels of description.



So where does this leave

The Theory of Everything

?