Self-Organised Criticality:

What does it mean and is it important?

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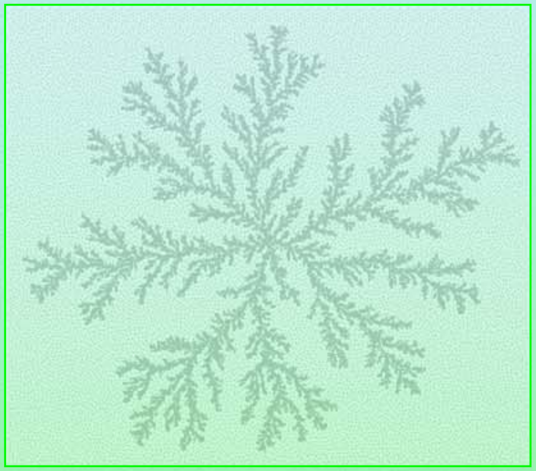
List of Content

- What is criticality
- What is "self-organised"
- Why is it interesting:
  - Examples from the real world
- Models
- Mathematical formalism

What is criticality

Lack of characteristic scale

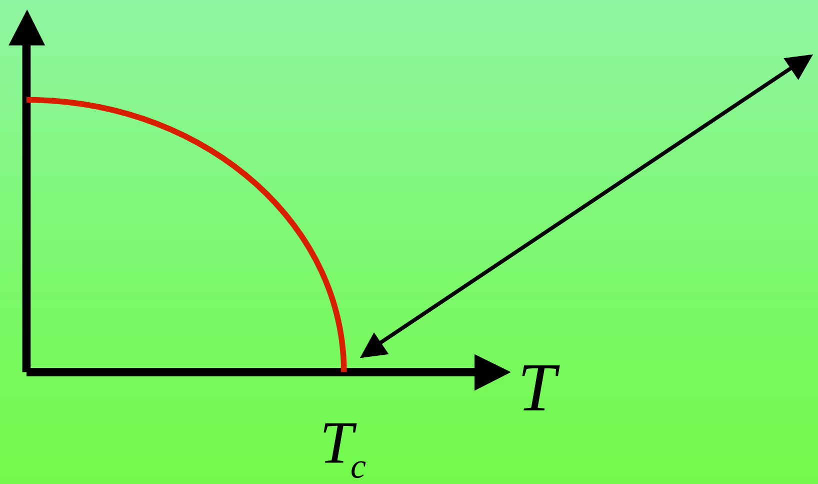


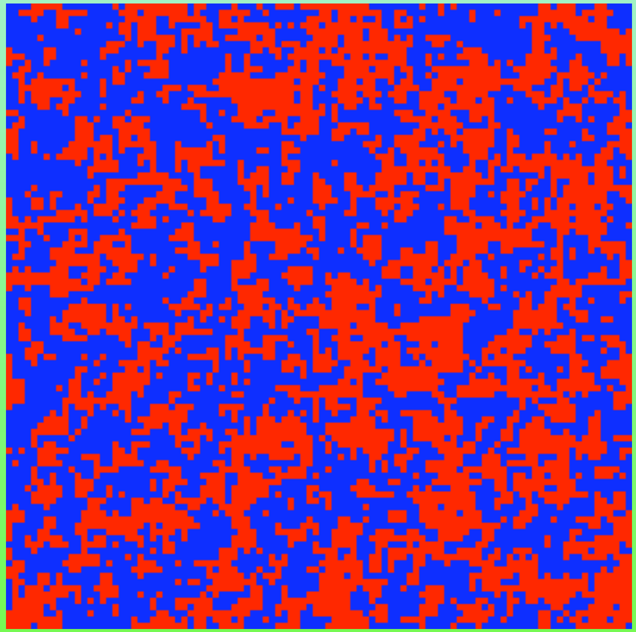
Typical scale

And when does it occur

In thermodynamic systems at the critical temperature

$$M = \langle S(r) \rangle$$





The correlation function

The critical behaviour is identified from the functional form of the correlation function

$$C(r,0) = \left\langle [S(0,0) - \langle S \rangle][S(r,0) - \langle S \rangle] \right\rangle = r^{-\eta} \exp\{- r/\xi\}$$

Where the correlation length ξ diverges as the critical temperature is approached

$$\xi(T) \propto (T - T_c)^{-\nu} \longrightarrow C(r) = r^{-\eta}$$

Temporal behaviour

$$C(0,t) = \left\langle [S(0,0) - \langle S \rangle][S(0,t) - S] \right\rangle = t^{-\alpha} \exp\{-t/\tau\}$$

As critical behaviour is approach: $\tau \to \infty$

The power spectrum

$$\left|\hat{S}(\omega)\right|^2 \propto \omega^{-\beta}$$

where $\alpha = 1 - \beta$

Hence $\beta \approx 1$ is very interesting

Scale free behaviour out of equilibrium

Spatial fractals

- Clouds
- Mountains
- Cauliflower
-

Snow on ground

Canopy

Temporal

O. Moriya et al, *Phys. Rev. Lett*. 80, 2833 (1998)

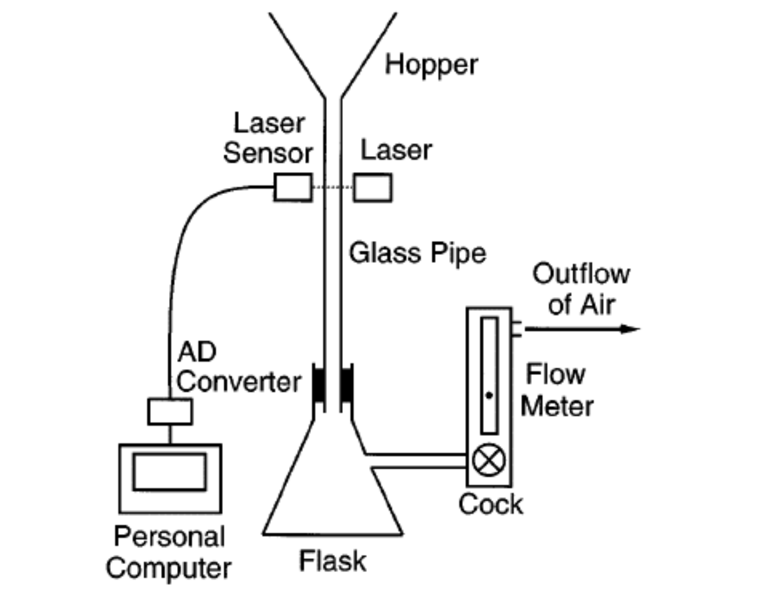


FIG. 1. Schematic illustration of the experimental setup.

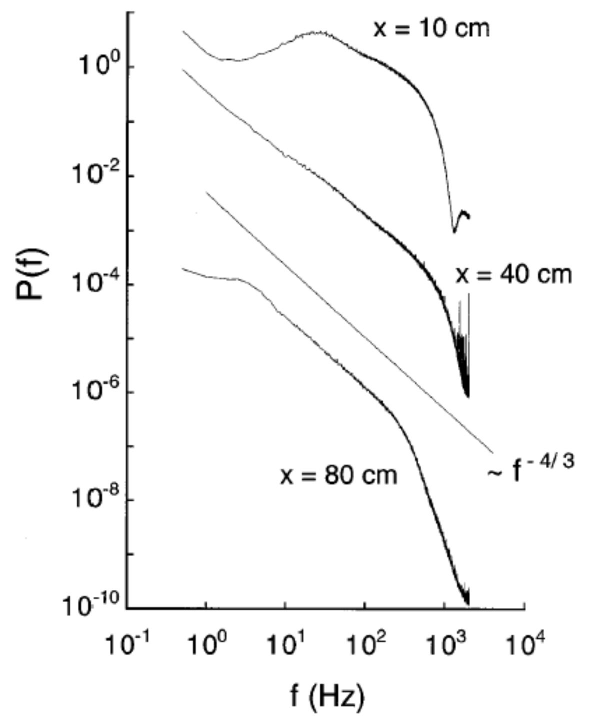


FIG. 3. Log-log plot of power spectra $P(f)$ of time series signals in *fully closed*. The straight line with the slope of $-4/3$ in the figure is a guide for the eyes.

- Quasars
- Ocean current
- Pressure variation in speech
-

An explanation needed!

If fractals and 1/f spectra are so common there must surely be one universal mechanism behind

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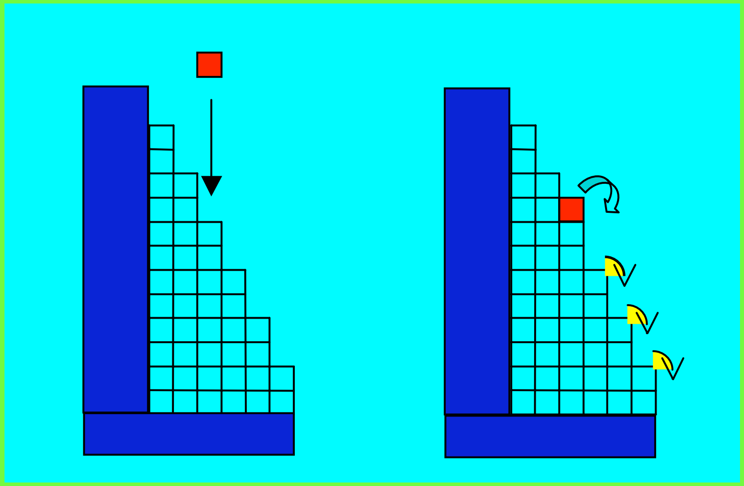
Self-Organized Criticality: An Explanation of $1/f$ Noise

Per Bak, Chao Tang, and Kurt Wiesenfeld
Physics Department, Brookhaven National Laboratory, Upton, New York 11973
(Received 13 March 1987)

We show that dynamical systems with spatial degrees of freedom naturally evolve into a self-organized critical point. Flicker noise, or $1/f$ noise, can be identified with the dynamics of the critical state. This picture also yields insight into the origin of fractal objects.

The sandpile model by BTW

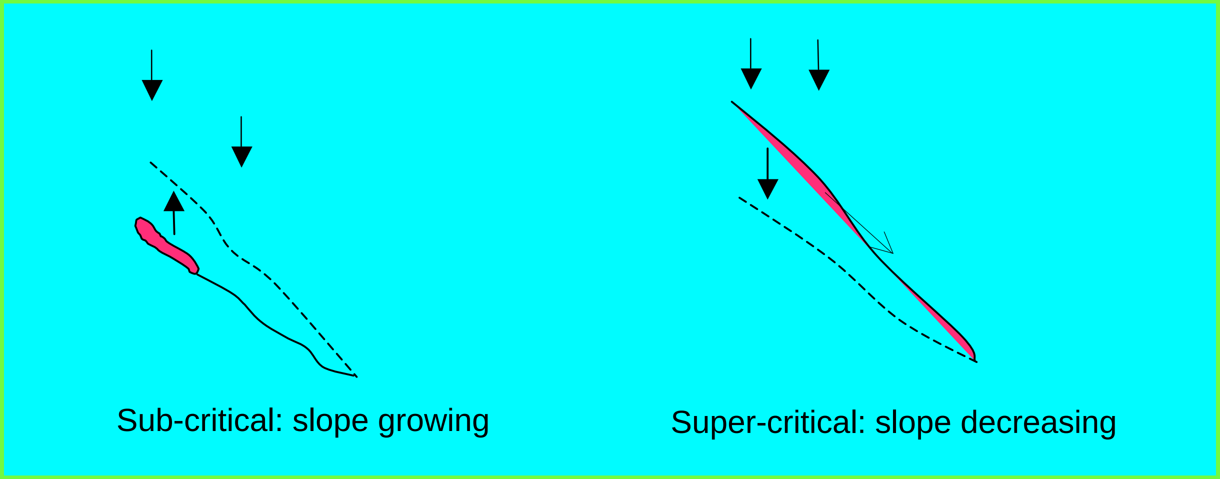
- Add sand grain by grain
- If local slope $z > z_c$ then relax



Induce avalanches of different sizes.

Self organisation:

Non tuning beside slow driving



Properties of the sandpile model

Power law distribution of avalanches

From BTW's PRL

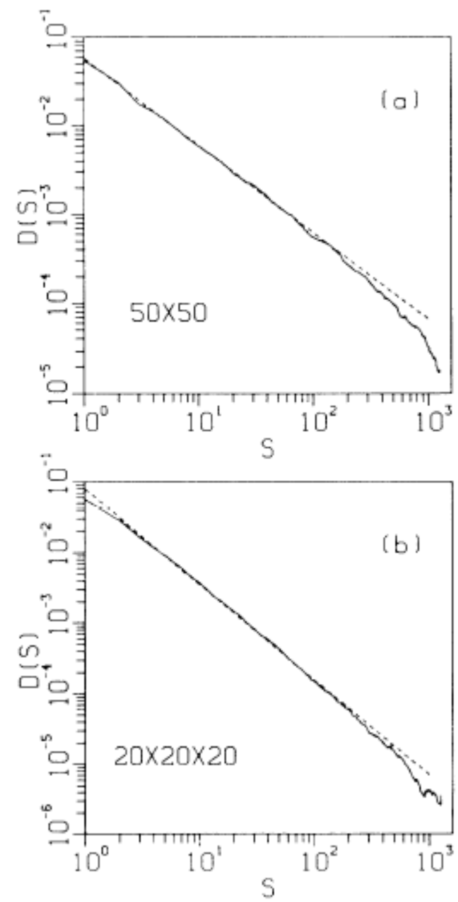


FIG. 2. Distribution of cluster sizes at criticality in two and three dimensions, computed dynamically as described in the text. (a) 50×50 array, averaged over 200 samples; (b) 20×20×20 array, averaged over 200 samples. The data have been coarse grained.

But not so fractal

Spatial extent of avalanches

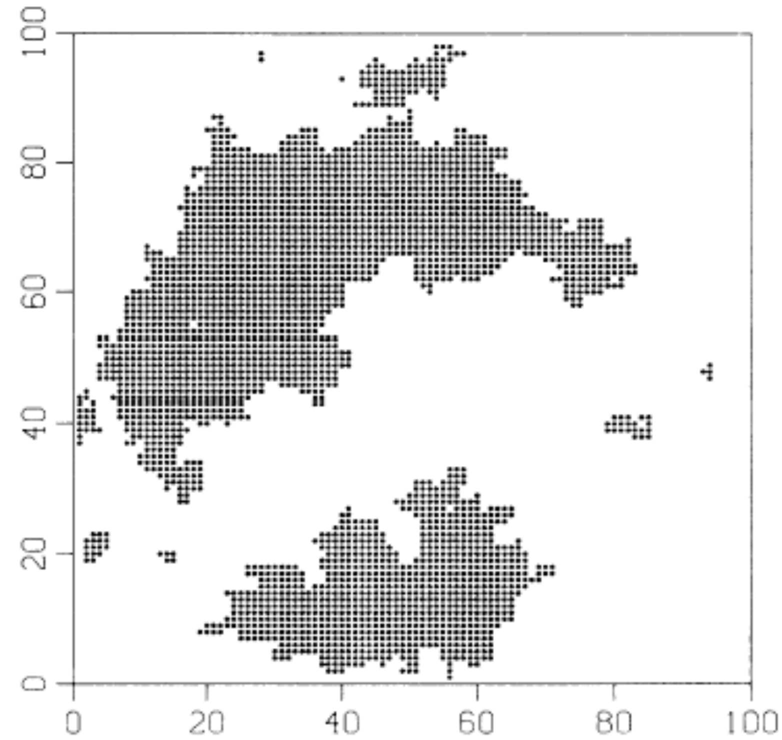


FIG. 1. Self-organized critical state of minimally stable clusters, for a 100×100 array.

And the power spectrum turned out to be $1/f^2$, except when driven at edge only.

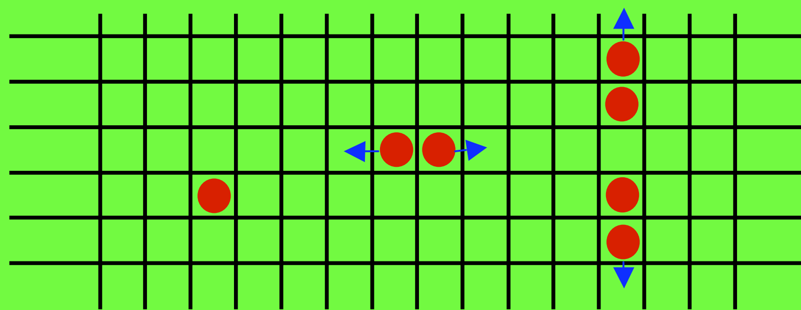
But other models does exhibit fractals and $\frac{1}{f}$

Density fluctuations in a Lattice Gas Model

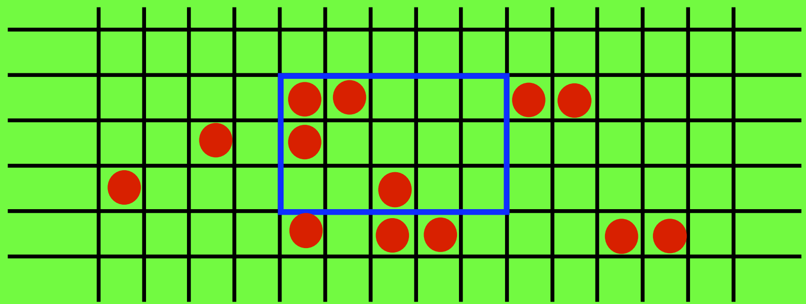
HJJ, *Phys. Rev. Lett.* 64, 3103 (1990)

Repulsive particles on a lattice.

- Deterministic motion.

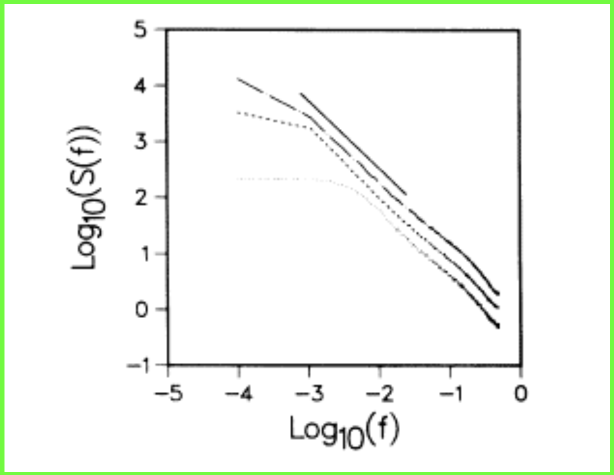


Monitor number of particles in a sub-section:

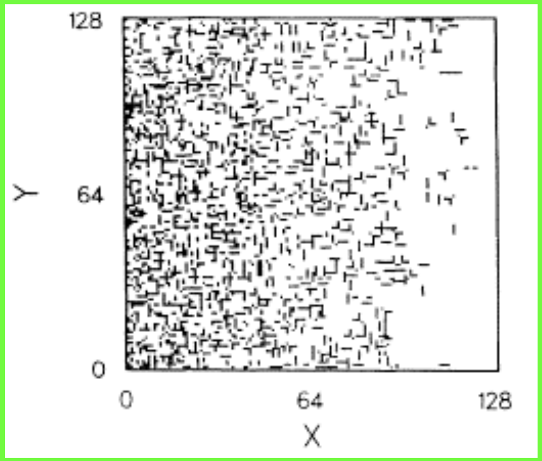


$N(t)$ = # particles in blue box

Power spectrum of *N(t)*



Instantaneous dissipation



Many more models:

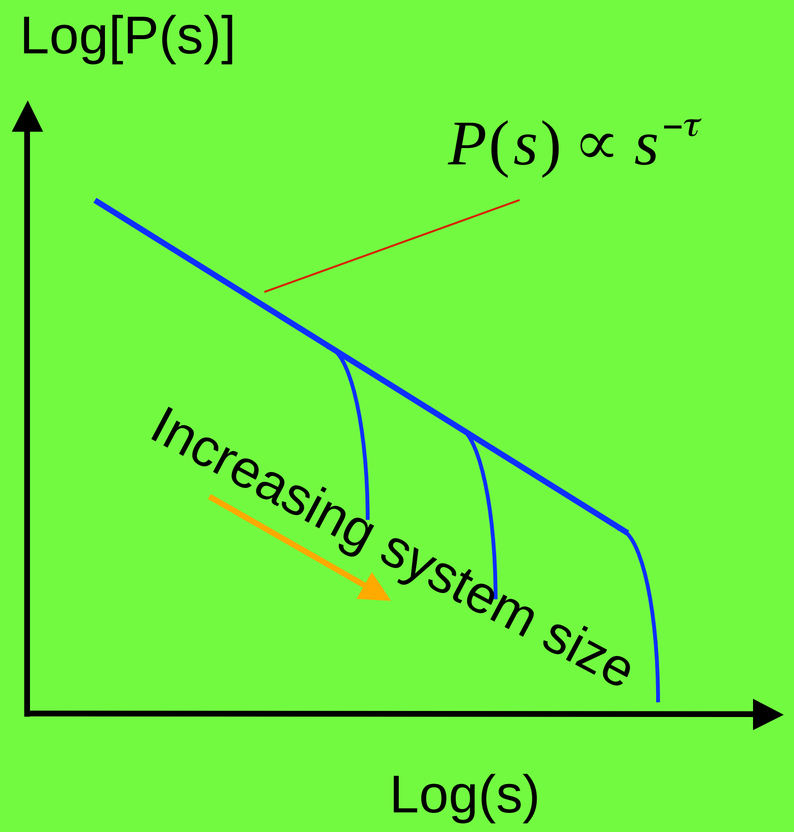
- Earth quake model (Olami-Feder-Christensen)
- Forest fires or epidemics (Drossel-Schwabl)
-

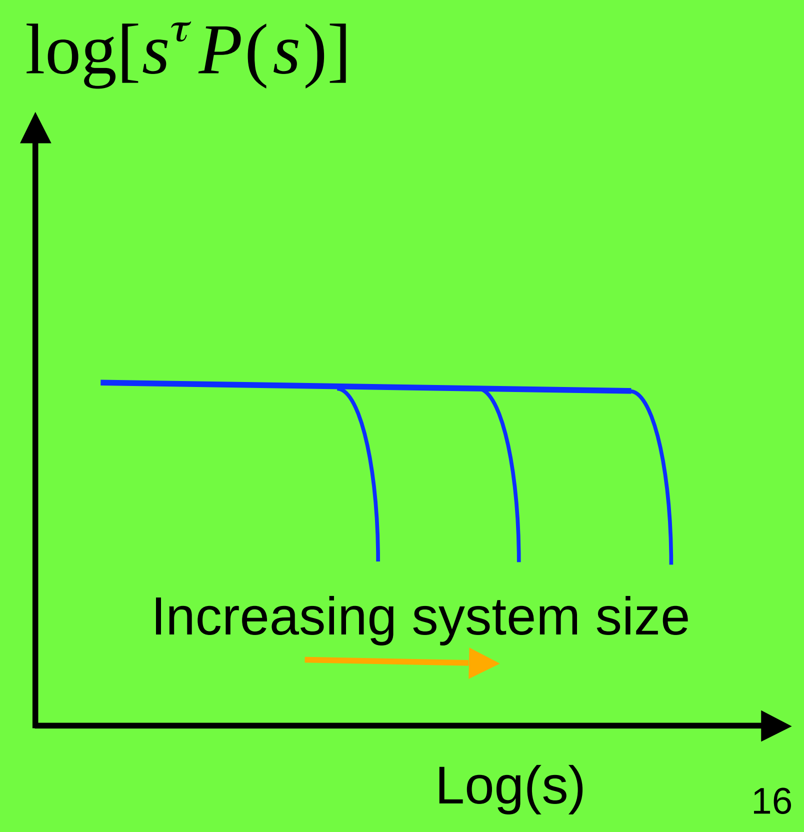
All exhibit scale invariance in the form of power laws for the distributions of events or avalanches.

Well, at least to some degree

Broken scaling

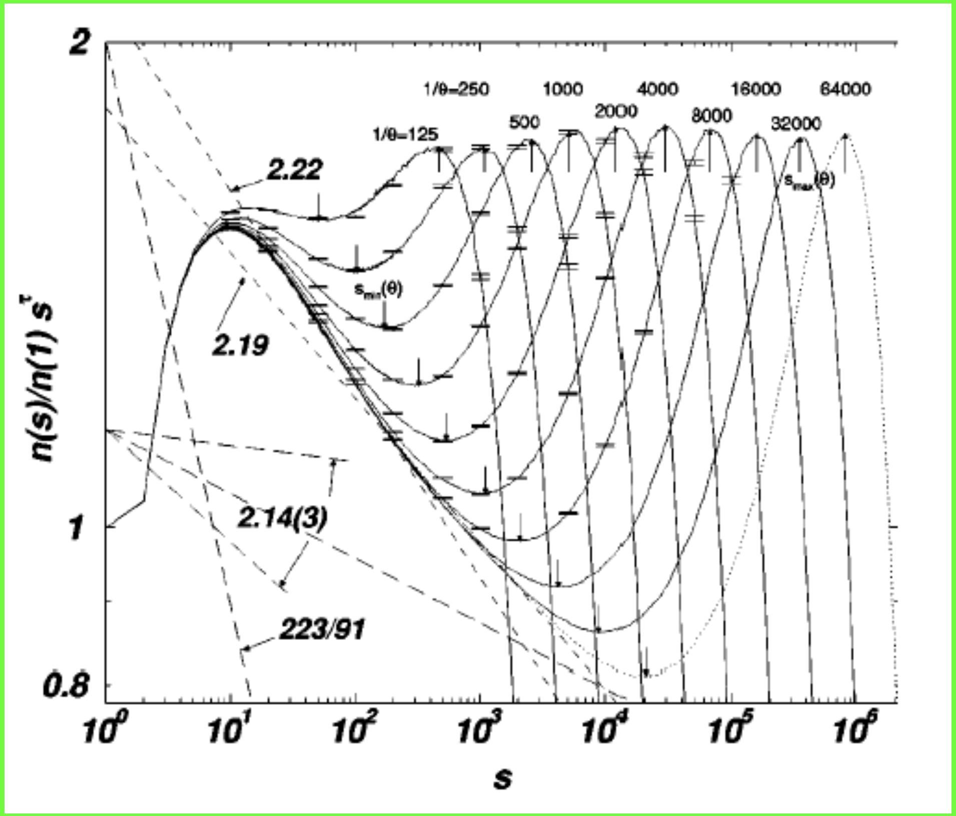
The ideal situation





The real situation

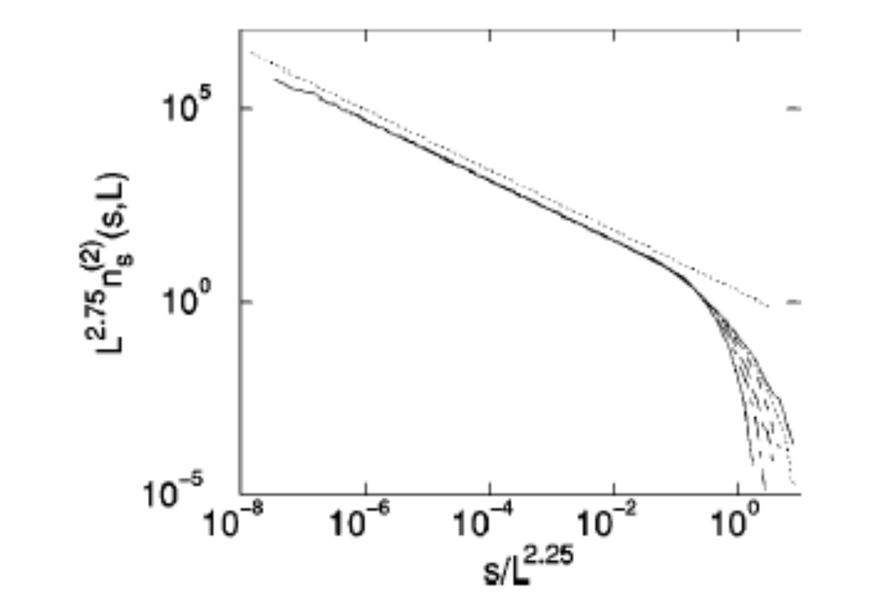
The Drossel-Schwabl forest fire model



From G. Prussner & HJJ, Phys. Rev. E. **65**, 056707 (2002).

See also Grassberger.

Scaling in the BTW sandpile model

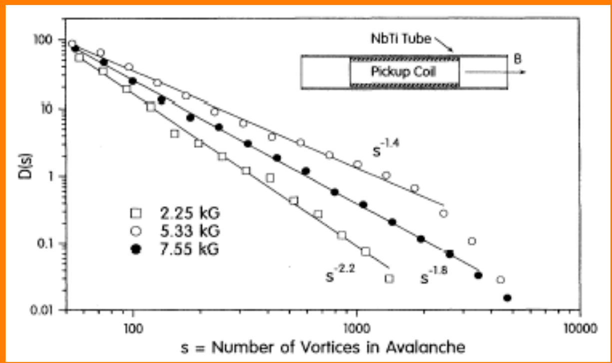


Only avalanches reaching the system edge. Still cut-off does not collapse. (slope = -7/9)

From B. Drossel, *Phys. Rev. E*, **61**, R2168 (2000)

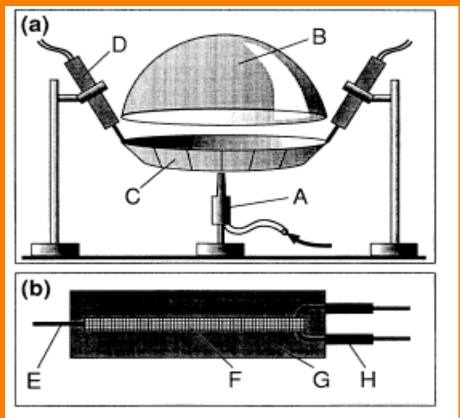
Experimental evidence:

Superconductors



Field et al. PRL 74, 1206 (1995)

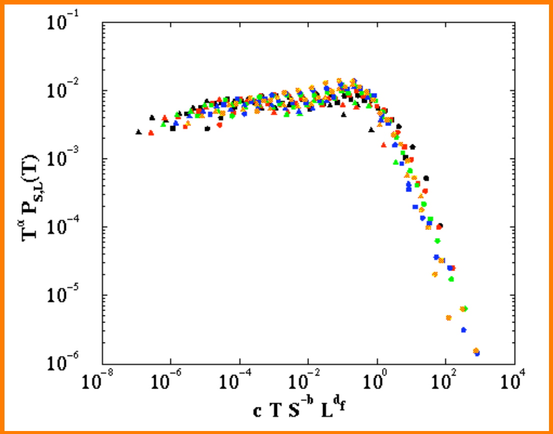
Droplet formation



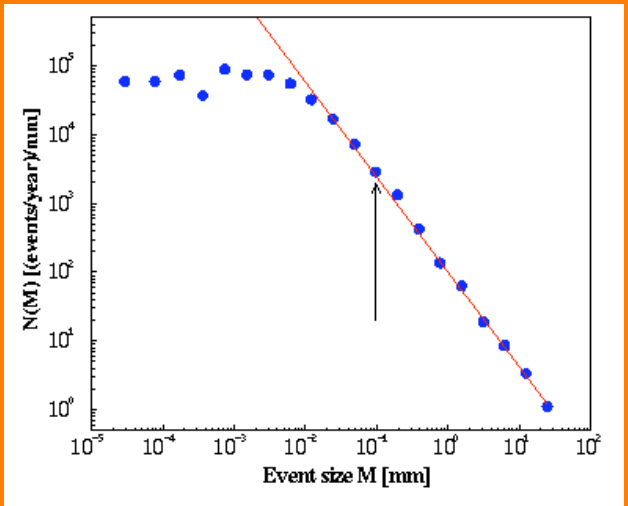
Plourde et al. PRL **71**, 2749 (1993)

Rice pile

Earthquakes



Rain



K. Christensen et al. Physics, Imperial

http://www.cmth.ph.ic.ac.uk/~kimchris

Biological evolution

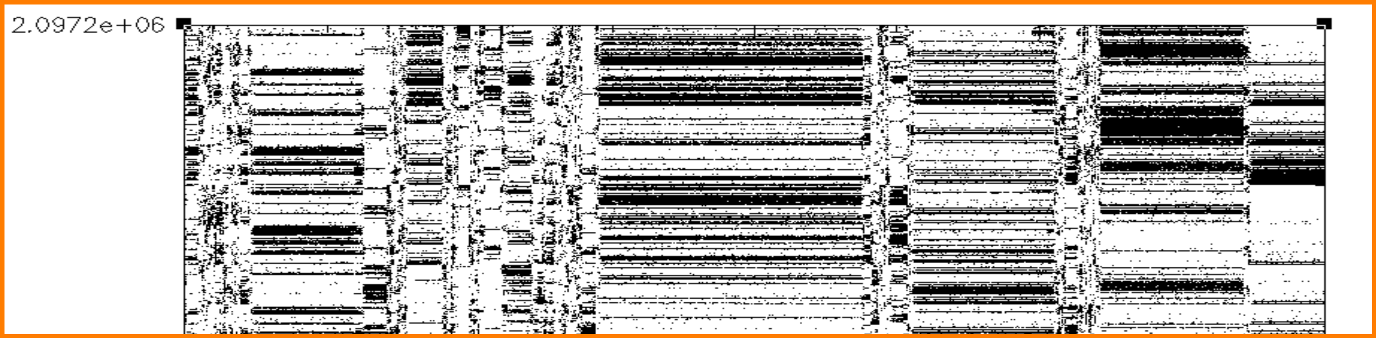
Gould, Eldredge

Punctuated equilibrium intermittent dynamics

and Raup:

Extinctions power law distributed

Something like:



Tangled Nature model of evolution

see http://www.ma.ic.ac.uk/~hjjens

So what is essential?

Back to Bak, Tang and Wiesenfeld

- Slow driving
- Threshold $\longleftrightarrow$ local rigidity

$\downarrow$

System keeps getting stuck in one of many meta-stable states

Formalisms

Eq. of motion for the BTW model:

The algorithm:

- Let $E_{\mathbf{r}}$ denote the height of site no. $\boldsymbol{r}$
- Add one grain at a random site $\mathbf{r}_0 \Rightarrow E_{\mathbf{r}_0} \mapsto E_{\mathbf{r}_0} + 1$
- If $E_{\mathbf{r}_0} > E_c$ then $E_{\mathbf{r}_0} \mapsto E_{\mathbf{r}_0} - E_c$

$$E_{\mathbf{r}_n} \mapsto E_{\mathbf{r}_n} + E_c / q$$

Or in equation form:

$$E(\mathbf{r},t+1) = E(\mathbf{r},t) - E_c\Theta(E(\mathbf{r},t) - E_c) + \sum_{\mathbf{r}_n} \frac{E_c}{q}\Theta(E(\mathbf{r}_n,t) - E_c)$$

$$E(\mathbf{r},t+1) - E(\mathbf{r},t) = \frac{E_c}{q}\sum_{\mathbf{r}_n}\Theta(E(\mathbf{r}_n,t) - E_c) - \Theta(E(\mathbf{r},t) - E_c)$$

Continuum limit

$$\boxed{\frac{\partial}{\partial t}E(\mathbf{r},t) = D\nabla^2\Theta(E(\mathbf{r},t) - E_c) + \eta(\mathbf{r},t)}$$

$$\frac{\partial}{\partial t} E(\mathbf{r},t) = D\nabla^2 \Theta(E(\mathbf{r},t) - E_c) + \eta(\mathbf{r},t)$$

Need to regularise the θ-function. Consider e.g.

Albert Diaz-Guilera

Europhys. Lett. 26, 177 (1994)

$$\Theta(x) = \lim_{\beta \to \infty} f(\beta x)$$

Where *f(x)* is some nice function with

$$\lim_{x \to \infty} f(x) = 1$$
$$\lim_{x \to -\infty} f(x) = 0$$

Then expand $f(x) = \sum_n a_n x^n$

Include more and more non-linearities and study – using Renormalisation Group – how the correlator

$\langle E(\mathbf{0},t) E(\mathbf{r},t) \rangle$ behaves.

Result of analysis

☺ That the model may be critical.

(at least as judged from continuum equation)

☺ That criticality in the non-conservative case is only possible if uniformly driven.

(Consistent with the numerics of the OFC model)

☹ But procedure is non-rigorous, not clear if results can be trusted, and very heavy.

☹ Nor can one calculate the exponent of the avalanche distribution

Another approach

Absorbing state phase transitions

(See e.g. R. Dickman, *Physica* A **306**, 90-97 (2002))

Consider two fields:

the density of active sites ρ_a

the “particle” density ς

Elimination of ς leads to the following eq.

Langevin eq. with memory term

$$\partial_t \rho_a(\mathbf{x},t) = D\nabla^2 \rho_a - r(\mathbf{x})\rho_a - b\rho_a^2 + w\rho_a \int_0^t d\tilde{t}\, \nabla^2 \rho_a(\mathbf{x},\tilde{t}) + \sqrt{\rho_a}\,\eta(\mathbf{x},t)$$

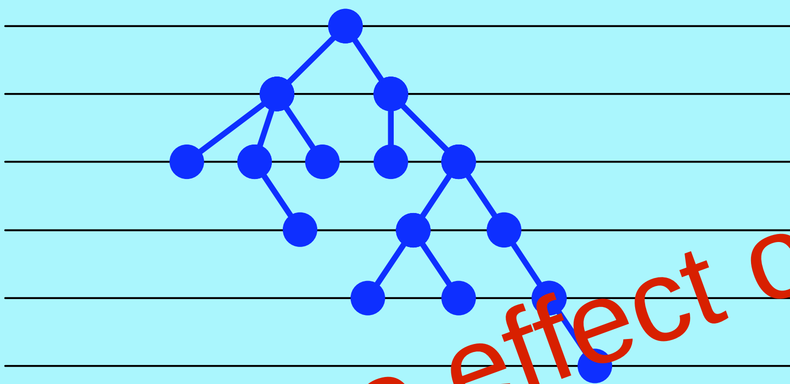
Where the growth rate $r(\mathbf{x})$ is given by the initial condition and the noise is correlated according to

$$\langle \eta(\mathbf{x},t)\eta(\mathbf{x}',t')\rangle = \Gamma\delta(\mathbf{x}-\mathbf{x}')\delta(t-t')$$

Related to Directed Percolation.

Nevertheless, difficult to handle.
Renormalisation group not yet applied with success.

Relation to branching processes



The effect of correlations?

Uncorrelated process

Distribution of tree sizes $p(s) \propto s^{-3/2} \exp\left(-\frac{s}{s_0(\sigma)}\right)$

The branching ratio $\sigma = \sum_{n=1}^{\infty} n p_n$

Characteristic size $s_0(\sigma) \propto (1-\sigma)^{-2}$

Exact results

D. Dhar's Δ-matrix formalism for Abelian sandpiles

see e.g. Dhar, Phys. Rev. Lett 64, 1613, (1990)

Consider a system consisting of N sites *1,2,3,…,* N.

Dynamics:

(1) *Addition rule.* $z_i \mapsto z_i + 1$

(2) *Toppling rule.* $z_i > z_{ic} \Rightarrow z_j \mapsto z_j - \Delta_{ij}$ for $j = 1,2,\ldots,N$

The Δ-matrix: $N \times N$

(a) $\Delta_{ii} > 0 \quad \forall i;$

Overcritical sites decrease height

(b) $\Delta_{ij} \leq 0 \quad \forall i \neq j;$

Neighbour sites receive height

(c) $\sum_{i=1}^{N} \Delta_{ij} \geq 0 \quad \forall i.$

Particles are not created during relaxation

Definitions

Set of stable configurations

$$S = \left\{ C = \{z_i\} \mid 1 \leq z_i \leq z_{ic} \quad \forall i \right\}$$

Operators on S

$$a_i : S \rightarrow S$$

Take configuration C.

Add one particle to site i according to rule (1) and relax according to rule (2).

Result = $a_i C$

Operators commute: $a_i a_j = a_j a_i$

Set of recurrent configurations

$$R = \left\{ C \in S \mid \exists m_i : \mathrm{a}_\mathrm{i}^{\mathrm{m}_\mathrm{i}} \mathrm{C} = \mathrm{C} \;\; \forall i \right\}$$

Inverse a_i^{-1} exists on ***R***

$$P(C,t) = const. \text{ for } \forall C \in R$$

$$\boxed{P(C,t) = \frac{1}{|R|} = \frac{1}{\det \Delta}}$$

The avalanche exponent τ

Attempts have been made using the Δ-matrix formalism.

But no exact result obtained so far.

Summary and conclusion

What does it mean?

Marginal stability and response of all sizes

And is it important?

Yes, but exactly how, we don't know yet.

References:

P. Bak

How Nature Works. The since of Self-organized criticality

Oxford Univ. Press 1997

H.J. Jensen

Organized Criticality. Emergent Complex Behavior in Physical and Biological Systems

Cambridge Univ. Press. 1998