

Anisotropy and Universality: The Oslo Model, the Rice Pile Experiment, and the Quenched Edwards-Wilkinson Equation

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(Received 17 July 2003; published 11 December 2003)

We show that any amount of anisotropy moves the Oslo model to another *known* universality class, the exponents of which can be derived exactly. This amounts to an exact solution of the quenched Edwards-Wilkinson equation with a drift term. We argue that anisotropy is likely to be experimentally relevant and may explain why consistent exponents have not been extracted in the rice pile experiments.

DOI: 10.1103/PhysRevLett.91.244303

PACS numbers: 45.70.Ht, 05.65.+b, 64.60.Ht, 68.35.Fx

The suggestion that power-law-like distributions frequently reported in experiments may be the effect of threshold dynamics and metastability was made by Bak, Tang, and Wiesenfeld in 1987 [1] and was called self-organized criticality (SOC). Avalanche dynamics in granular piles was from the onset used as a metaphor and laboratory for SOC behavior and has inspired many models and experiments [2]. One of the most celebrated of these efforts is the experimental study of avalanches in one dimensional rice piles by Frette *et al.* [3] and the theoretical Oslo model [4] inspired by the rice pile experiment. The general interest and relevance of such studies rely on the assumption, guided by equilibrium critical phenomena, that the critical behavior of scale invariant systems falls into universality classes determined solely by a few general characteristics of the system, such as symmetry and dimension. So-called “relevant” parameters can decide which of the symmetries the system is asymptotically dominated by.

The role of anisotropy in SOC has been highlighted very early by Hwa and Kardar [5] and Grinstein *et al.* [6], who used anisotropic Langevin equations to describe sandpiles. On the cellular automata level, Kadanoff *et al.* [7] have conjectured that the net flux of particles is a relevant parameter. In this Letter we confirm this conjecture for the Oslo model, which shows a clear-cut and consistent relevant dependence on anisotropy. This is of great importance for the interpretation of experimental results [3] and more generally for the much studied quenched Edwards-Wilkinson (EW) equation [8,9]. Moreover, contrary to suggestions in former studies, the switch between different universality classes (crossover) is *not* triggered by the introduction of stochasticity [10–13] nor by multiple topplings [12,14–16].

Similar to [17], the system size at crossover, L_X , depends on the strength of the anisotropy v . We exemplify two possible mechanisms causing anisotropy in experiments, one of which vanishes with system size L fast enough to keep vL constant. This represents a marginal case and consequently makes a unique identification of the critical exponents impossible.

The original Oslo model (OOM) consists, in one dimension, of a lattice of sites $i = 1, \dots, L$. Two coupled dynamical variables are associated with each lattice site: the primary variable $z_i \in \{0, 1, 2, \dots\}$ and the threshold variable $z_i^c \in \{1, 2\}$. The initial configuration consists of $z_i = 0 \forall i$ and a random configuration of the z_i^c . The system is driven by increasing z_1 by one (a “grain”) followed by a relaxation of all sites $1 \leq i \leq L$ for which $z_i > z_i^c$ (“overcritical” sites). In case a site i is overcritical, the following updates are performed (“toppling” or “relaxation”): $z_i \rightarrow z_i - 2$ and $z_{i\pm 1} \rightarrow z_{i\pm 1} + 1$ and, importantly, the existing value of the threshold z_i^c is afterwards replaced by 1 with probability p and by 2 with probability $1 - p$. The boundaries are updated the same way except that for $i = 1$ ($i = L$) addition on site 0 ($L + 1$) is omitted.

We now introduce a tunable degree of anisotropy into the dynamics. An overcritical site i is relaxed in the following way. Only left movement: with probability $p_l(1 - p_r)$ perform the updates: $z_i \rightarrow z_i - 1$ and $z_{i-1} \rightarrow z_{i-1} + 1$. Only right movement: with probability $p_r(1 - p_l)$ perform the updates $z_i \rightarrow z_i - 1$ and $z_{i+1} \rightarrow z_{i+1} + 1$. Both left and right movement: with probability $p_l p_r$ perform the updates $z_i \rightarrow z_i - 2$ and $z_{i\pm 1} \rightarrow z_{i\pm 1} + 1$. A new z_i^c is chosen, at random, after every successful update, i.e., when at least one grain has been redistributed. We call this version of the model the anisotropic Oslo model (AOM). The strength of the anisotropy is described by the drift velocity $v = (p_r - p_l) / (p_r + p_l)$ which is the net flux of grains through the system. Clearly it is only sensible to study $p_r + p_l > 0$. The case $p_r = p_l = 1$ corresponds exactly to the OOM, while $p_r = p_l \neq 1$ represents a stochastic variant of the OOM. The avalanche exponents for the extreme, totally asymmetric case $p_l = 0$ and $p_r = 1$ can be obtained exactly [18] and describe, as we shall see below, the scaling behavior for all $v > 0$. We are interested in the statistics of the sizes, s , of the avalanches of relaxation induced by the driving $z_1 \rightarrow z_1 + 1$. The size of an avalanche is in both versions of the model defined as the number of times the relaxation rule was successfully

applied after the drive $z_1 \rightarrow z_1 + 1$ in order to make $z_i < z_i^c \forall_i$ yet again. Thus $s \geq 0$. The model remains essentially unchanged if the case of no grain redistribution is omitted, as was done in the numerical results presented below.

In the totally anisotropic or asymmetric limit [16] the model resembles some features of other exactly solved, *directed* models [10,13,15,16,19]. In two dimensions a very similar model has been studied numerically [14]. However, we stress that contrary to some other “exact” solutions, the model is solvable directly on the lattice and without assuming any scaling behavior [18]. Also the amplitudes of the moments can be calculated exactly.

It is a tedious, but straightforward task to show that the AOM is “Abelian”; i.e., the order of updates is irrelevant for its statistical properties. Since the microdynamics which prescribes the order of updates is irrelevant, there is no unique way to define a microscopic time scale. Presumably universal exponents of the duration of avalanches are therefore mainly a property of the arbitrary choice of the microdynamics. According to Hughes and Paczuski [20], a non-Abelian variant of an Abelian model may or may not remain in the same universality class.

Moreover, one notes that the OOM as well as the AOM contains multiple topplings, i.e., a single site can relax several times during a single avalanche. We now describe the avalanche statistics of the AOM. In Fig. 1 we demonstrate that the avalanche size distribution $\mathcal{P}(s; L, p_r, p_l)$ follows simple (finite size) scaling

$$\mathcal{P}(s; L, p_r, p_l) = a(p_r, p_l) s^{-\tau} \mathcal{G}\left[\frac{s}{b(p_r, p_l) L^D}\right] \quad \text{for } s > s_l, \quad (1)$$

where \mathcal{G} is the universal scaling function, $a(p_r, p_l)$ and $b(p_r, p_l)$ are two anisotropy and system dependent parameters, and s_l is the lower cutoff independent of L . The values for the scaling exponents are $\tau_a = 4/3$ and $D_a = 3/2$, which can be derived exactly in the asymmetric limit $\nu = 1$ [18] and represent the known universality class of the stochastic, directed sandpile in *two* dimensions [12,13,15], which is in turn closely related to the directed sandpile [10]. Numerically, these exponents have been found for all $\nu > 0$ studied at sufficiently large system sizes $L \gg L_X$. The crossover that occurs around L_X is discussed in detail below. The two exponents of the AOM are to be compared with the exponents for the OOM, of $\tau_O = 1.556(4)$ and $D_O = 2.25(2)$. Since the average avalanche size scales linearly with the system size, the exponents are related by $D(2 - \tau) = 1$ [4,21].

The easiest way to derive the exponents from numerical data is by analysis of the moments [22], which scale according to (1) for $n > \tau - 1$ in leading order like

$$\langle s^n \rangle = \int_0^\infty ds s^n \mathcal{P}(s; L, p_r, p_l) = a(bL^D)^{1+n-\tau} g_n + \dots, \quad (2)$$

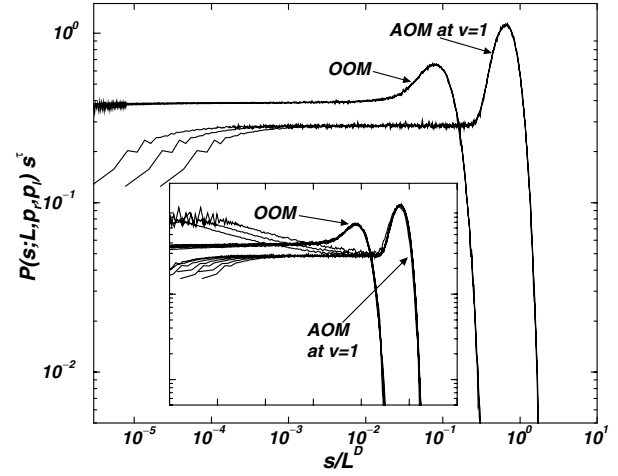


FIG. 1. Main panel: Data collapse [$\mathcal{P}(s; L, p_r, p_l) s^\tau$ vs s/L^D] of the normalized and binned distributions for the two extreme cases ($p_r = 1$ with $p_l = 0$ and $p_l = 1$) for $L = 1280, 2560, 5120$. The rescaling was done using $\tau = 1.333\dots$ and $D = 1.5$ for the AOM and $\tau = 1.555\dots$ and $D = 2.25$ for the OOM. Inset: Distributions for three choices of $p_r > p_l$, namely $(p_r, p_l) = (1.0, 0.95), (1.0, 0.25), (0.75, 0.25)$, and two choices of $p_r = p_l$ (0.75 and 0.25), for $L = 640, 1280, 2560$. The data collapse of each tuple (p_r, p_l) would form a single line, thereby fixing τ and D . By tuning a and b in Eq. (1), as done in the inset, the result collapses with one of the extreme cases shown in both panels.

where g_n is discussed below and \dots denotes subleading terms, especially Wegner’s corrections to scaling [23]. In the following, the crossover is studied by means of the rescaled second moment, $\langle s^2 \rangle / L^{5/2}$, which is shown in Fig. 2. For nonvanishing anisotropy, $\nu > 0$, it approaches a constant as $L \rightarrow \infty$. For very small but finite values of ν and L the rescaled moment increases with L like $L^{0.75}$, corresponding to the OOM behavior, but at $L \approx L_X(\nu)$ it crosses over and eventually converges to a finite constant. Below, we shall relate the behavior of $L_X(\nu)$ to the effective anisotropy relevant to an experiment of a given size. Here we emphasize that Fig. 2 clearly demonstrates that the universality class of the extremely anisotropic case, $p_r = 1, p_l = 0$, contains all systems with nonvanishing anisotropy $\nu > 0$. That renders the OOM with $\nu = 0$ a special case; remarkably, even for $p_r = p_l \neq 1$ the model still shows OOM behavior. Thus, it is not the stochasticity itself [10–12] which induces the change in critical behavior.

Equation (1) allows the definition of universal amplitude ratios

$$g_n = \langle s^n \rangle \langle s \rangle^{(n-2)} / \langle s^2 \rangle^{(n-1)} \quad (3)$$

which can easily be proven to be asymptotically independent of a, b , and L . The two constraints on \mathcal{G} , which fix a and b in (1), can be chosen such that g_n are the moments of $x^{-\tau} \mathcal{G}(x)$ as used in (2), namely, by imposing that

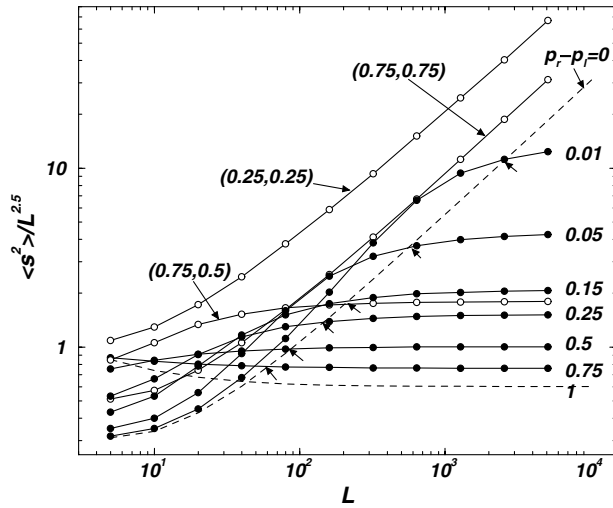


FIG. 2. Scaling of $\langle s^2 \rangle / L^{2.5}$ for different anisotropies. The solid circles are results for $p_r - p_l$ as indicated and $p_r = 1$. Open circles show other parameters (p_r, p_l) . The dashed lines are the two extreme cases OOM ($p_r = p_l = 1$) and the solvable model ($p_r = 1, p_l = 0$). The short arrows mark the approximate crossover.

$\int_0^\infty x^{1-\tau} \mathcal{G}(x) = \int_0^\infty x^{2-\tau} \mathcal{G}(x) = 1$. The universal amplitude ratio g_3 as shown in Fig. 3 indicates not only the same crossover behavior as observed in Fig. 2, but also the universality of \mathcal{G} .

The importance of the above result is highlighted when we recall that the OOM in the continuum limit is described by the quenched EW equation [9]. A similar derivation shows that the AOM is a quenched EW equa-

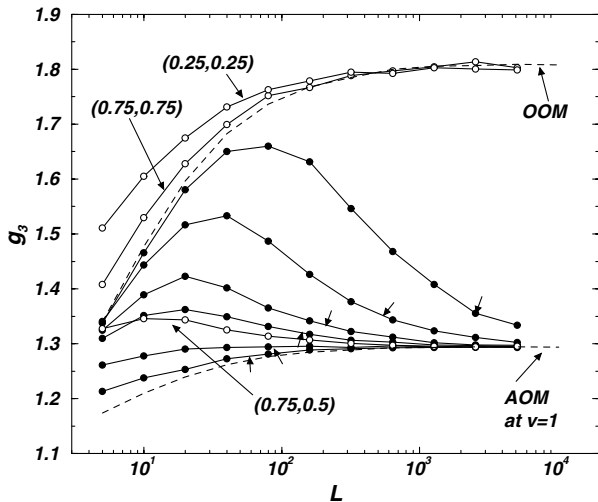


FIG. 3. Behavior of g_3 [see Eq. (3)] for different anisotropies. The solid circles are results for $p_r - p_l$ in the same vertical order as shown in Fig. 2 and $p_r = 1$. Open circles show other parameters (p_r, p_l) as indicated. The dashed lines are the two extreme cases OOM ($p_r = p_l = 1$) and the solvable model ($p_r = 1, p_l = 0$). The short arrows mark the crossover points in Fig. 2.

tion with an additional drift term

$$\partial_t h(x, t) = \gamma \partial_x^2 h(x, t) - v \partial_x h(x, t) + \eta(x, h(x, t)), \quad (4)$$

where γ is the diffusion constant and v the anisotropy or drift velocity as defined above. In the AOM, the quenched noise $\eta(x, h(x, t))$ represents the randomly chosen z_i^c and $h(x, t)$ is the number of charges received by site x at time t . The quenched nature of the noise makes it difficult to solve (4) directly. Together with the boundary conditions [9], it prevents the drift term $v \partial_x h$ from being absorbed by a Galilean transformation. However, the above results determine the roughness exponent via $D = 1 + \chi$ [21] to be $\chi = 1/2$ for $v > 0$, as already suggested in another case of anisotropic depinning [24]. For $v = 0$ numerics for the OOM suggest correspondingly that $\chi = 1.25(2)$.

The crossover for small $v > 0$ with increasing L can also be illuminated by a study of the individual grains in the system, which behave like biased random walkers. This leads again to a diffusion equation with drift term and two absorbing boundaries [25]. The average avalanche size is the average time the particles spend in the system divided by the average number of grains redistributed per toppling. The crossover is expected as soon as the ballistic motion dominates over the diffusion, $L^2/\gamma > L/v$, thus $L_\chi(v) = \gamma/v$. This has been confirmed numerically, based on heuristic estimation of L_χ , as shown by the marks in Fig. 2.

We now discuss the relevance of anisotropy to real experimental granular systems. We stress that the anisotropy is in the amount of slope transported between sites involved in a relaxation event. Any net flux of the slope is eventually compensated by the toppling of the last site and the slope is therefore asymptotically stationary.

One process leading to an anisotropic redistribution of slope arises when the toppling grain is elongated; see Fig. 4. Most remarkably, in the original experiment [3] it was noted that only the elongated rice samples showed scale invariant behavior. A reorientation of a single grain, as shown in Fig. 4, leads to a net flux of slope to the right. It can happen only once, and in fact it depends on how

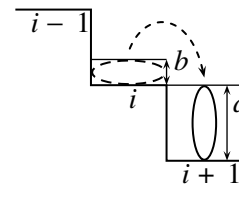


FIG. 4. A stylized toppling of a single grain: If an elongated grain of width b and height a topples from site i to site $i + 1$, it reduces the height at i by b and increases the height at $i + 1$ by a , thereby increasing the slope at site $i - 1$ by b , decreasing the slope at i by $a + b$ and increasing the slope at $i + 1$ by a . The net flux of slope is $a - b$ to the right.

and whether the rice enters and/or leaves the system with a typical orientation. If that is the case, then an average reorientation is distributed among all topplings on its way through the system, i.e., $v \propto 1/L$. Because $L_X \propto 1/v$ and $v \propto 1/L$, this represents a marginal case, because it is impossible to decide whether $L \gg L_X(v)$ or not. The dependence of, say, the ratio $\langle s^2 \rangle / L^{2.5}$ on L would in this case be given by an (unknown) trajectory through the diagram in Fig. 2 since a change in L also leads to a change in v , which allows nonuniversal quantities to enter.

We do not know if the experiment by Frette *et al.* [3] involves this complication, but the exponents extracted are not consistent [26].

Local rearrangements, such as expansions (on the site losing a grain) and compressions (on the site receiving the grain) lead to an anisotropy not vanishing with L : Say a grain moves from i to $i + 1$, then column i expands by $\epsilon(h_i)$ and column $i + 1$ is compressed by $\epsilon(h_{i+1})$. Then the changes of the slopes during toppling are $\Delta z_{i-1} = 1 - \epsilon(h_i)$, $\Delta z_i = -2 + \epsilon(h_i) + \epsilon(h_{i+1})$, and $\Delta z_{i+1} = 1 - \epsilon(h_{i+1})$. Assuming that the columns behave elastically, ϵ would be an increasing function of h , resulting in a net flow to the right, i.e., $v > 0$. However, it remains unclear whether any of these effects can be seen in experimental systems, because, for example, the boundary conditions might be different [4]. Moreover, having shown that anisotropy is a relevant field, it would not be surprising to find other relevant fields which lead to yet another universality class.

We have demonstrated that for any amount of anisotropy the exponents of the original Oslo model change and are given by simple rational numbers which can all be obtained exactly [18]. The crossover has been studied numerically using a moment analysis, Eq. (2), and universal amplitude ratios, Eq. (3), and the crossover length has been determined. The generalized model described above continuously connects the established original Oslo model and an exactly solvable, directed variant. This variant has, compared to the original Oslo model, an enormous basin of attraction, so that the latter may be regarded a special case of the former.

Moreover, we find a change in critical behavior of an SOC model, genuinely due to anisotropy, rather than stochasticity or the presence of multiple topplings.

The results are theoretically interesting especially because of their relation to the EW equation, the roughness exponent of which has been obtained in case of the presence of a drift term in one dimension to be $\chi = 1/2$. Moreover, according to our study, experiments are seriously complicated due to a coupling between system size and effective anisotropy. That might provide a clue as to the apparent difficulties to find theoretically predicted exponents in the real world.

It is a pleasure to thank Kim Christensen for very helpful discussions and Paul Anderson for proofreading. The authors gratefully acknowledge the support of

EPSRC. G. P. would like to thank Quincy Thoeren for hospitality.

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- [26] There are some problems in their derivation: The scaling ansatz is unphysical, as higher moments diverge even in finite systems. The result $\beta = \nu$ is due to a neglect of the lower cutoff. Moreover, the possible interpretation of $\alpha \approx 2.02$ as τ leads to a system size independent finite first moment, which contradicts $\langle s \rangle \propto L$. In this light it would be very interesting to repeat the data analysis.