1st progress test 2/11/2011

1. Let $f : S^1 \to S^1$ defined by

$$f(x) = (ax + b\sin(2\pi x) + c) \bmod 1$$

with $x \in [0, 1) \cong S^1$ and parameters $a, b, c \in \mathbb{R}$.

(a) Determine what conditions on the parameters a, b, and c must be satisfied for f to be an orientation preserving circle homeomorphism.
Answer: For f to be a circle map we need $a \in \mathbb{Z}$ so that $f(x + 1) = f(x)$. For f to be invertible it is necessary that its degree has absolute value 1. As $\deg(f) = a$, this implied that $|a| = 1$. Furthermore, in order to ensure monotinicity that is necessary for invertibility, we need $\frac{d}{dx}f(x) = a + 2\pi b \cos(2\pi x) \neq 0$ for all x, which implies that $|2\pi b| < 1$. Preservation of orientation implies that $a = 1$ (and not $a = -1$).

(b) Let $F : \mathbb{R} \to \mathbb{R}$ be a lift of f. Give all possible choices for this lift F.
Answer: $F(x) = ax + b\sin(2\pi x) + c + m$ with $m \in \mathbb{Z}$.

(c) Show that the rotation number of f is equal to $\frac{1}{2}$ if $a = 1$, $b = \frac{1}{8}$, $c = \frac{1}{2}$.
[Hint: use a property that f^2 must satisfy if $\rho(f) = \frac{1}{2}$.]
Answer: It suffices to show that f has a periodic orbit with period 2. We consider the lift $F(x) = x + \frac{1}{8}\sin(2\pi x) + \frac{1}{2}$. First we note that f has no fixed point since $\frac{1}{8}\sin(2\pi x) + \frac{1}{2} \neq 0 \bmod \mathbb{Z}$ for all x. We compute that $F^2(x) = x + \frac{1}{8}\sin(2\pi x) + 1 + \frac{1}{8}\sin(2\pi(x + \frac{1}{8}\sin(2\pi x) + \frac{1}{2}))$ so that $F^2(0) = 1$ and hence that 0 is a fixed point of f^2. It then immediately follows that $\rho(f) = 1/2$.

(d) Show that the circle map in part (c) is NOT topologically conjugate to the rigid rotation $R_{\frac{1}{2}} : x \to x + \frac{1}{2} \bmod 1$.
Answer: All $x \in S^1$ are periodic points of $R_{1/2}$ of period 2. It thus suffices to show that f has a point that does not have period 2. For instance $F^2(\frac{1}{4}) = \frac{5}{4} + \frac{1}{8}\frac{\sqrt{2}}{2} + \frac{1}{8}\sin(2\pi(\frac{3}{4} + \frac{1}{8}\frac{\sqrt{2}}{2}))$. Since $\frac{3}{4} > \frac{1}{8}\frac{\sqrt{2}}{2} + \frac{1}{8}\sin(2\pi(\frac{3}{4} + \frac{1}{8}\frac{\sqrt{2}}{2})) > \frac{1}{4}$. Alternatively, one could also verify that 0 is a hyperbolic fixed point (and hence isolated): $Df^2(0) > 1$.

2. (a) Show that if the rotation number of an orientation preserving circle homeomorphism is equal to zero, then f has a fixed point.

Answer: Without loss assume $F(0) \in (0,1)$. Suppose f has no fixed point then $\delta \leq F(x)-x \leq 1-\delta$ for some $\delta > 0$ and all x (since F is continuous). Some simple manipulation yields $F^n(0) = \sum_{k=1}^n F^k(0) - F^{k-1}(0)$ implying that for all n we have $\delta \leq \frac{F^n(0)}{n} \leq 1-\delta$ so that $\rho(f) \neq 0$. Hence if $\rho(f) = 0$, f must have a fixed point.

(b) Show that if F is the lift of an orientation preserving circle homeomorphism, then for all $x, y \in \mathbb{R}$ the following implication holds

$$|x - y| < 1 \quad \Rightarrow \quad |F(x) - F(y)| < 1.$$

Answer: First we note that if f is invertible, then $|x-y| = 1$ implies $|F(x)-F(y)| = 1$. Also if $y = x$ then the latter expression is trivially equal to 0. By strict monotinicity of F (implied by invertibility of f), $|F(x) - F(y)|$ is also a strictly monotonous function of y and when y increases from x to $x+1$, $|F(x) - F(y)|$ increases (monotonically) from 0 to 1, and thus if $|x - y| < 1$ then $|F(x) - F(y)| < 1$. The argument when $y \in x + (-1, 0)$ is similar.

(c) Let f be a continuous circle map and suppose that the limit $\lim_{n\to\infty} f^n(x)$ exists for some point $x \in S^1$. Show then that this limit point $z = \lim_{n\to\infty} f^n(x)$ is a fixed point of f.

Answer: $f(z) = f(\lim_{n\to\infty} f^n(x)) = \lim_{n\to\infty} f^{n+1}(x) = z$.