2nd progress test 24/11/2011

1. Let $f : S^1 \to S^1$ defined by

$$f(x) = (-2x + a\cos^2(bx)) \bmod 1$$

with $x \in [0, 1) \cong S^1$ and parameters $a, b \in \mathbb{R}$.

(a) Determine what conditions on the parameters a and b must be satisfied for f to be an expanding circle map.

Answer: First of all $b = k\pi$ for some $k \in \mathbb{Z}$ so that $f(x+1) = f(x)$. f is expanding is $|f'(x)| > 1$ for all $x \in S^1$. $f'(x) = -2 - ak\pi\sin(2k\pi x)$, so we need $|ak|\pi < 1$, ie $|ak| < 1/\pi$ (ie $|ab| < 1$).

(b) Show that if f is of the above form, and an expanding circle map, it has no periodic orbits of period 2. [Hint: derive (or recall without proof) a formula for the number of fixed points of an expanding circle map as a function of its degree.]

Answer: $\deg(f^n) = (\deg f)^n$ and $\deg f = -2$, and the number of fixed points $P(f)$ of f is equal to $|\deg f - 1|$. Hence we here have $P_1(f) = |-2-1| = 3$ and $P_2(f) = |4-1| = 3$, so the number of periodic orbits of period 2 is $P_2(f) - P_1(f) = 0$.

2. Let Ω_3 denote the set of bi-infinite sequences $\{\omega_i\}_{i\in\mathbb{Z}}$ whose entries ω_i are taken from a set of three symbols, for instance $\{0, 1, 2\}$.

(i) Consider the cylinder

$$C_{\alpha_{1-n},\ldots,\alpha_{n-1}} := \{\omega \in \Omega_3 \mid \omega_i = \alpha_i,\ |i| < n\}.$$

Let

$$d(\omega, \omega') := \sum_{m\in\mathbb{Z}} \frac{\delta(\omega_m, \omega'_m)}{4^m},$$

where $\delta(a, b) = 0$ if $a = b$ and $\delta(a, b) = 1$ if $a \neq b$.

(a) Show that d is a metric on Ω_3.

Answer: All properties follow by comparing components in the sum: (i) $d(x, y) = d(y, x)$ follows from the fact that $\delta(a, b) = \delta(b, a)$. (ii) $d(x, y) = 0 \Leftrightarrow x = y$ follows again from the definition of δ: as soon as two sequences have one different symbol, the distance is positive, and the distance between two equal sequences is equal to zero. (iii) $d(x, y) + d(y, z) \geq d(x, z)$ follows from the fact that $\delta(a, b) + \delta(b, c) \geq \delta(a, c)$. This is obviously satisfied if $a = c$. If $a \neq c$ then $b \neq c$ or $a \neq b$ so that the inequality is also satsified.

(b) Consider Ω_3 as a metric space with metric d. Show that the cylinder $C_{\alpha_{-1}\alpha_0\alpha_1}$ is a ball in Ω_3 around any point α of the form $\alpha = \ldots\alpha_{-1}\alpha_0\alpha_1\ldots$ and determine its radius.

Answer: Let $\alpha \in C_{\alpha_{1-n},\ldots,\alpha_{n-1}}$. If $\omega \in C_{\alpha_{1-n},\ldots,\alpha_{n-1}}$ then

$$d(\omega,\omega') := \sum_{|m|\geq\ n} \frac{\delta(\omega_m,\omega'_m)}{4^m} \leq \sum_{|m|\geq\ n} \frac{1}{4^m} = \frac{1}{4^{n-1}}\frac{2}{3} < \frac{1}{4^{n-1}}.$$

On the other hand if $\omega \notin C_{\alpha_{1-n},\ldots,\alpha_{n-1}}$,

$$d(\omega,\omega') \geq \frac{1}{4^{n-1}}.$$

Thus $C_{\alpha_{1-n},\ldots,\alpha_{n-1}}$ is exactly equal to the ball around α of radius 4^{1-n}. In the case that $n = 2$, as asked, this yields a ball or radius $1/4$.

(ii) Give for each of the following, an example of a topological Markov chain on Ω_3 (endowed with metric d), by means of its transition matrix or Markov graph, that has this property:

1. a topological Markov chain that is not transitive
2. a topological Markov chain that is topologically mixing

Answer: For instance, $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ is a transition matrix for a transitive and topologically mixing Markov chain (full shift) and and $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is a transition matrix for a Markov chain (identity) that is neither transitive nor topologically mixing .

(iii) Show that every <u>topologically transitive</u> topological Markov chain on Ω_3 is <u>topologically mixing</u>.

Answer: See [HK] proposition 7.3.12.