3rd progress test 14/12/2011

Consider the piecewise linear map $F : [0,1] \to [0,1]$ given by

$$F(x) = \begin{cases} 3x & \text{if } x \in \Delta_0 \\ 2-3x & \text{if } x \in \Delta_1 \\ x - \frac{2}{3} & \text{if } x \in \Delta_2 \end{cases}$$

where $\Delta_1 = [0, \frac{1}{3}]$, $\Delta_2 = [\frac{1}{3}, \frac{2}{3}]$, and $\Delta_3 = [\frac{2}{3}, 1]$. [It may be useful to draw the graph of F]

We consider a coding of orbits by means of (half-)inifinite sequences of the form $\omega = \omega_0\omega_1 \ldots \in \Omega_3^R$, where Ω_3^R denotes the metric space of (half-)infinite sequences with symbols $\omega_i \in \{0,1,2\}$ and distance

$$d(\omega, \omega') := \sum_{m\in\mathbb{N}} \frac{\delta(\omega_m, \omega'_m)}{4^m},$$

where $\delta(a,b) = 0$ if $a = b$ and $\delta(a,b) = 1$ if $a \neq b$.

We propose the coding $h : \Omega_3^R \to [0,1]$ to be such that if $x = h(\omega)$, with $\omega = \omega_0\omega_1\omega_2\ldots$ and $y := f^n(x)$ then $y \in \Delta_{\omega_n}$.

(i) Show that the distance d satisfies the triangle inequality.

Answer: $d(x,y) + d(y,z) \geq d(x,z)$ follows from the fact that $\delta(a,b) + \delta(b,c) \geq \delta(a,c)$. This is obviously satisfied if $a = c$. If $a \neq c$ then $b \neq c$ or $a \neq b$ so that the inequality is also satsified.

(ii) Let $\Delta_{\omega_0\ldots\omega_n} := \bigcap_{i=0}^{\infty} f^{-i}(\Delta_{\omega_i})$. Determine Δ_{ab} for all $a, b \in \{0,1,2\}$.

Answer: $\Delta_{00} = [0, \frac{1}{9}]$, $\Delta_{01} = [\frac{1}{9}, \frac{2}{9}]$, $\Delta_{02} = [\frac{2}{9}, \frac{1}{3}]$, $\Delta_{12} = [\frac{1}{3}, \frac{4}{9}]$, $\Delta_{11} = [\frac{4}{9}, \frac{5}{9}]$, $\Delta_{10} = [\frac{5}{9}, \frac{2}{3}]$, $\Delta_{20} = [\frac{2}{3}, 1]$, $\Delta_{21} = \emptyset$, $\Delta_{22} = \emptyset$.

(iii) Determine the codes representing the points 0, $\frac{1}{3}$, $\frac{2}{3}$ and 1, and point out whether or not these codes are unique (and why).

Answer: $h^{-1}(0) = \overline{0}(= 00000\ldots)$, $h^{-1}(\frac{1}{3}) = \{0\overline{2}, 1\overline{20}\}$, $h^{-1}(\frac{2}{3}) = \{1\overline{0}, \overline{20}\}$ and $h^{-1}(1) = \overline{20}$.

(iv) Show that the map F is semi-conjugate to a three-state topological Markov chain determined by the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$.

Answer: A represents the admissible transitions between the labelling regions. Main item to prove is that h is surjective function, where $h(\omega) = \Delta_\omega$. We note that F is expanding on $[0, \frac{2}{3}]$ and non-expanding (translation) on $[\frac{2}{3}, 1]$. We note that F^2 is expanding so that $\lim_{n\to\infty} \Delta_{\omega_0\ldots\omega_n}$ is indeed a single point. Taking into account the fact that F does not expand on Δ_3 (and that $F(\Delta_3) = \Delta_1$) we obtain uniform expansion with rate 3 for F^2, so that $|\Delta_{\omega_0\ldots\omega_n}| \leq \frac{1}{3}\left(\frac{1}{\sqrt{3}}\right)^{n-1}$.

(iv) Discuss if it can be concluded from the result in (iv), perhaps augmented with some further observations, that F is chaotic.

Answer: The Markov chain $\sigma_A : \Omega_A^R \to \Omega_A^R$ is transitive and thus chaotic. The semi-conjugacy is such that h is one-to-one except where codes end on $\overline{0}$ and $\overline{20}$, marking precisely the pre-images of the boundary points analysed in Part (iii). Hence F is transtive and periodic points are dense, so F is chaotic.