Reversible Equivariant Linear Systems

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Abstract

In this paper we classify the structure of linear reversible systems (vector fields) on $\mathbb{R}^n$ that are equivariant with respect to a linear representation of a compact Lie group H. We assume the time-reversal symmetry R also acts linearly and is such that the group G that is generated by H and R is again a compact Lie group. The main tool for the classification is the representation theory of compact Lie groups. The results are applied to some generic eigenvalue movements of linear reversible equivariant systems.

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1 Overview

Linear systems form the basis from which nonlinear dynamical systems are investigated. In many cases, the local dynamics in the neighbourhood of an equilibrium point in phase space is similar to that of a linear system. Moreover bifurcation theory, describing how the dynamics of systems change as parameters are varied, uses linear systems theory as one of its key techniques.

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In this paper we discuss the structure of linear systems that are reversible with respect to a linear time-reversing symmetry R and at the same time equivariant with respect to a linear representation of a compact Lie group H, in such a way that the group G generated by H and R is a compact Lie group. Examples of compact Lie groups are $O(n)$, $U(n)$, T^n, and all their closed subgroups. Our focus on compact Lie groups is in line with many previous works on equivariant bifurcation theory. See Golubitsky *et al.* [8] for examples and a discussion of their properties.

1.1 Introduction

We consider linear differential equations of the form

$$\frac{d\mathbf{x}}{dt} = L(\mathbf{x}) \tag{1.1}$$

where $L : \mathbb{R}^n \mapsto \mathbb{R}^n$ is linear and represented by a matrix in $gl(n;\mathbb{R})$ (the algebra of $n \times n$ matrices with real coefficients).

We focus on linear systems that are both *reversible* and *equivariant*. A linear vector field L is said to be *reversible* when there exists a linear invertible map $R : \mathbb{R}^n \mapsto \mathbb{R}^n$ such that[1]

$$LR = -RL, \tag{1.2}$$

the differential equation (1.1) being invariant under the transformation $(\mathbf{x}, t) \mapsto (R(\mathbf{x}), -t)$. The map R is called a *time-reversal symmetry* or *reversing symmetry* of L. It should be noted that we do not require that R is an involution, i.e. R^2 need not be the identity transformation $\mathbf{I}$.

A linear system L is said to be S*-equivariant* if it commutes with a linear invertible map $S : \mathbb{R}^n \mapsto \mathbb{R}^n$,

$$LS = SL. \tag{1.3}$$

S is called a *symmetry* of L, and correspondingly (1.1) is invariant under the transformation $(\mathbf{x}, t) \mapsto (S(\mathbf{x}), t)$.

The set of symmetries is closed under composition, but the set of reversing symmetries is not. However, since the composition of two reversing symmetries is a symmetry, the set of symmetries and reversing symmetries of L is closed under composition and so forms a group, which we denote by G. If G contains reversing symmetries the subgroup H of symmetries is a normal subgroup of index two, i.e.

$$G/H \simeq \mathbb{Z}_2. \tag{1.4}$$

It should be noted that (1.4) does not imply that G contains a reversing symmetry that is an involution [13]. For example, when $G = \mathbb{Z}_4$ and $H = \mathbb{Z}_2$, the set $G \setminus H$ does not contain an involution. In general G possesses an involutory reversing symmetry if and only if G is isomorphic to a semi-direct product $H \rtimes \mathbb{Z}_2$. [2]

Reversible systems have been studied for many years. It is well known that reversibility has similar implications for the eigenvalues of linear systems as a Hamiltonian structure.

[1] More precisely, a linear map L satisfying (1.2) may be called *infinitesimally reversible* [22], to contrast the definition of reversibility for diffeomorphisms in which a diffeomorphism F is called reversible with respect to an invertible map R when $RF = F^{-1}R$. It is clear that when L satisfies (1.2) then $F = \exp(L)$ is a reversible diffeomorphism. As we will not discuss the situation of diffeomorphisms in this paper, we have chosen to omit the adjective 'infinitesimally'.

[2] Sevryuk [22] noted that for all linear reversible systems one can identify a linear involutory reversing symmetry. Although interesting, this remark is irrelevant in the context of our study. Namely, in the family of

Lemma 1.1 *Let $L \in gl(n;\mathbb{R})$ be reversible and λ be an eigenvalue of L. Then $-\lambda$ and the complex conjugate $\overline{\lambda}$ are also eigenvalues of L.*

Proof Let x be an eigenvector of L with eigenvalue λ. Then $\overline{x}$ is an eigenvector with a eigenvalue $\overline{\lambda}$ and Rx is an eigenvector with eigenvalue $-\lambda$. The latter statement follows from

$$L(R(x)) = -R(L(x)) = -\lambda R(x).$$

□

As a consequence, the eigenvalues of reversible linear systems come in singlets $\lambda = 0$, real or purely imaginary doublets $(\lambda, -\lambda)$ with $\lambda \in \mathbb{R}$ or $\lambda \in i\mathbb{R}$, or complex quadruplets $(\lambda, -\lambda, \overline{\lambda}, -\overline{\lambda})$ with $\lambda \in \mathbb{C}$.

Example 1.2 Consider the purely reversible case with $G = \mathbb{Z}_2$ acting on $\mathbb{R}^2$ by $R : (x, y) \mapsto (x, -y)$ and $H = \{\mathbf{I}\}$. From the reversibility of L it follows that $L = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$ with $b, c \in \mathbb{R}$. The eigenvalues of L are $\lambda_\pm = \pm\sqrt{bc}$, i.e. a pair on the real axis or a pair on the imaginary axis.

From the explicit formula for the eigenvalues we see that $\lambda_\pm = 0$ is a degeneracy of codimension one. Moreover, typically when the eigenvalues meet at the origin, they branch off along the other axis. Staying on the same axis would need adjustment of two parameters and is therefore of codimension two.

The eigenvalue behaviour observed in Example 1.2 is similar to eigenvalue behaviours in linear Hamiltonian systems. However, in the following example, we see that reversibility in combination with equivariance may give rise to eigenvalue behaviours that we are not familiar with in equivariant Hamiltonian contexts.

Example 1.3 Let $G = \mathbb{Z}_4$ act on $\mathbb{R}^2$ by $R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then $H = \mathbb{Z}_2$. From the reversibility of L it follows that $L = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}$. The eigenvalues of L are therefore $\lambda_\pm = \pm\sqrt{a^2 + b^2}$, i.e. always a pair of real eigenvalues of opposite sign. The reversibility here forces the equilibrium point to be of saddle type. From the explicit formula for the eigenvalues we see that $\lambda_\pm = 0$ is a degeneracy of codimension two. Hence, in one-parameter families one typically observes the eigenvalues to move towards each other on the real axis, but bounce off in the opposite direction before they reach the origin. This phenomenon is often referred to as eigenvalue repulsion or avoided crossings.

Example 1.4 Let $G = \mathbb{D}_4$ act as the symmetries of the square in $\mathbb{R}^2$. Then there are two choices for H: either $H = \mathbb{D}_2$ or $H = \mathbb{Z}_4$. In both cases $G \simeq H \rtimes \mathbb{Z}_2$. However the dynamical consequences of the two cases are different.

linear systems with given reversing symmetries and equivariances, the involution identified by Sevryuk will vary over the family when $G \setminus H$ does not contain an involution. Note also that Sevryuk's remark does not apply in the context of nonlinear systems.

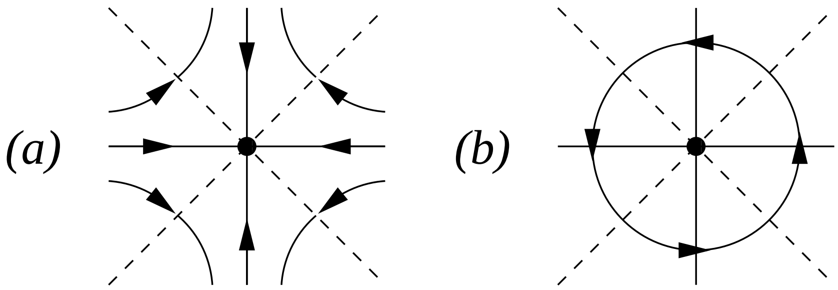

Figure 1: Schematic phase portrait of a typical linear flow (codimension 0) in $\mathbb{R}^2$ with reversing symmetry group $G = \mathbb{D}_4$ and (a) $H = \mathbb{D}_2$, (b) $H = \mathbb{Z}_4$.

When $H = \mathbb{D}_2$, the H-equivariance implies that $L = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ with $a, b \in \mathbb{R}$. The reversibility causes $b = -a$, so that the eigenvalues of L are $\lambda_\pm = \pm a$. Hence, the equilibrium point is typically unstable and of saddle type.

When $H = \mathbb{Z}_4$, the H-equivariance gives $L = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ with $a, b \in \mathbb{R}$. The additional reversibility forces $a = 0$ so that the eigenvalues $\lambda_\pm = \pm ib$ are purely imaginary, and the equilibrium point is typically stable and of centre type. The phase portraits in Fig. 1 illustrate the dynamical consequences of these symmetry properties for the two different choices for H.

In both cases, when the eigenvalues collide at the origin of the complex plane (a generic one-parameter phenomenon), the eigenvalues do not branch off along the other axis (as happens generically when $G = \mathbb{Z}_2$), but pass through the origin and stay on the same axis. In the case of $H = \mathbb{D}_2$ this is the real axis, and in the case of $H = \mathbb{Z}_4$ it is the imaginary axis. Although there is no change of stability in these bifurcations, the passing of the eigenvalues through zero has important consequences for the local dynamics: when $H = \mathbb{D}_2$ the stable and unstable manifolds interchange, and when $H = \mathbb{Z}_4$ the rotating flow around the origin changes its sense of orientation. For a discussion of the changes in the local dynamics in the presence of nonlinear terms, see [14].

The main aim of this paper is to investigate the structure of linear reversible equivariant systems in a systematic way, in order to understand and generalise the behaviours observed in the above examples.

Throughout this paper we will assume that G is a compact Lie group to provide us with the results from representation theory on which our analysis will rely. Recall (eg from [8]) that a *representation* ρ of a group G is a group homomorphism $\rho : G \mapsto GL(n; \mathbb{R})$. In order to use representation theory effectively we will combine conditions (1.2) and (1.3) in

$$L\rho(g) = \sigma(g)\rho(g)L, \tag{1.5}$$

where ρ is a representation of G and $\sigma : G \to \{-1, 1\} \subset GL(1; \mathbb{R})$ is the one-dimensional representation of G, given by $\sigma(g) = 1$ if $g \in H$ and $\sigma(g) = -1$ if $g \in G \backslash H$. The homomorphism σ identifies precisely how G is split into $G \setminus H$ and H. We say that L satisfying (1.5) is (G, σ)-*reversible*, or in short σ-*reversible*. We will denote the representation $\sigma\rho$ by ρ_σ and call it the σ-*dual* of ρ. If ρ_σ is isomorphic to ρ as a representation of G, we say that ρ is σ-*self-dual*.

In the remainder of Section 1 we will summarize our main results (§1.2, §1.3) and briefly discuss some applications (§1.3, §1.4). Detailed statements and proofs are given in Section 3.

Section 2 contains a summary of results we will need on real linear algebras and representation theory.

1.2 Structure results

We will now summarize the main results of this paper. Let G be a compact Lie group acting linearly on $V = \mathbb{R}^N$, H an index two subgroup and $\sigma : G \to \mathbb{Z}_2$ the homomorphism with kernel H. Let $gl_H(V)$ denote the space of H equivariant linear maps from V to itself and $gl_G(V)$ and $gl_\sigma(V)$ the subspaces of $gl_H(V)$ consisting of G-equivariant and σ-reversible maps, respectively.

The action of the compact Lie group G on V can be decomposed as a direct sum of irreducible representations on linear subspaces of V. Taking the sums of all the irreducible representations of the same isomorphism class, ρ say, defines an *isotypic* subspace V_ρ. The *isotypic decomposition* of V into these subspaces

$$V \;=\; V_{\rho_1} \oplus \ldots \oplus V_{\rho_k}$$

is unique up to the order of the summands.

It follows from Schur's Lemma (Lemma 2.5) that every G-equivariant mapping from V to itself leaves each of the isotypic subspaces V_ρ invariant. Moreover, if the action of G on V_ρ is the direct sum of m copies of ρ then $gl_G(V_\rho)$ is isomorphic to $gl(m, \mathbf{k})$ where $\mathbf{k}$ is one of $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. The irreducible representation ρ is correspondingly said to be of type $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.

If the irreducible representation ρ of G is σ-self-dual, then the σ-reversible linear maps from V to itself also preserve the isotypic subspace V_ρ (see §3.2). However, if ρ is not σ-self-dual, then the σ-reversible linear maps map V_ρ to V_{ρ_σ} and vice-versa. Hence the subspace $\widehat{V}_\rho = V_\rho \oplus V_{\rho_\sigma}$ is invariant. We refer to these subspaces as the *σ-isotypic subspaces* of V.

The main result of this paper (Theorem 3.10) is a classification of the possible structures of the spaces $gl_\sigma(V)$ of σ-reversible maps. By the previous paragraph these spaces decompose as the direct sum of the corresponding spaces for the σ-isotypic subspaces. It is therefore sufficient to describe $gl_\sigma(V)$ for the case when V consists of a single σ-isotypic space. To do this we first give a classification of the irreducible representations of G in terms of both their $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$-type as representations of G and how they decompose into irreducible representations when restricted to H (Theorem 3.2 and Corollary 3.4). This classification is summarized in Table 1.

If (U, τ) is an irreducible representation of type $\mathbf{k}$ the action of $gl_G(U) \cong \mathbf{k}$ on U gives it the structure of a (left) $\mathbf{k}$-vector space. [3] As part of the classification of irreducible representations of G we show that if (U, τ) is σ-self-dual then there exists a σ-reversible map $T : U \to U$ which satisfies $T^2 = \pm\mathbf{I}$. The map T is *semilinear* with respect the $\mathbf{k}$-structure induced by $gl_G(U)$. In other words there is an involutory automorphism $\alpha : \mathbf{k} \to \mathbf{k}$ such that $T(kx) = \alpha(k)T(x)$ for all $k \in \mathbf{k}$. These mappings are shown in Table 1 along with some of their properties. In most cases the automorphism α can be taken to be the identity. However, in two cases (SD-CR and SD-CH) with $\mathbf{k} = \mathbb{C}$ the automorphism is complex conjugation.

We can now describe the σ-reversible maps from a representation V to itself when V consists of a single σ-isotypic space. We describe our maps in terms of $\mathbf{k}$-linear maps acting on $\mathbf{k}$-vector spaces, bearing in mind the fact that they represent real-valued matrices.

[3] U is a left $\mathbf{k}$-vector space if for all $w \in U$ and all $\kappa, \lambda \in \mathbf{k}$, $\kappa(\lambda w) = (\kappa\lambda)w$. Since $\mathbb{H}$ is a non-commutative ring, one needs to distinguish between left and right $\mathbb{H}$-vector spaces. In case $k = \mathbb{R}$ or $k = \mathbb{C}$ this is not necessary [19].

	$gl_G(U)$	$gl_H(U)$	T	T^2	α	$\Sigma(L)$
NSD-R	$\mathbb{R}$	$\mathbb{R}$	$-$	$-$	$-$	L
NSD-C	$\mathbb{C}$	$\mathbb{C}$	$-$	$-$	$-$	L
NSD-H	$\mathbb{H}$	$\mathbb{H}$	$-$	$-$	$-$	L
SD-RR	$\mathbb{R}$	$\mathbb{R}\oplus\mathbb{R}$	$(1,-1)$	$\mathbf{I}$	$\mathbb{R}$	$\tau(L)$
SD-CC	$\mathbb{C}$	$\mathbb{C}\oplus\mathbb{C}$	$(1,-1)$	$\mathbf{I}$	$\mathbb{C}$	$\tau(L)$
SD-HH	$\mathbb{H}$	$\mathbb{H}\oplus\mathbb{H}$	$(1,-1)$	$\mathbf{I}$	$\mathbb{H}$	$\tau(L)$
SD-RC	$\mathbb{R}$	$\mathbb{C}$	i	$-\mathbf{I}$	$\mathbb{R}$	$\overline{L}$
SD-CR	$\mathbb{C}$	$gl(2;\mathbb{R})$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\mathbf{I}$	$\overline{\mathbb{C}}$	$-iLi$
SD-CH	$\mathbb{C}$	$\mathbb{H}$	j	$-\mathbf{I}$	$\overline{\mathbb{C}}$	$L^\sharp$
SD-HC	$\mathbb{H}$	$gl(2;\mathbb{C})$	$\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$	$-\mathbf{I}$	$\mathbb{H}$	$-j\overline{L}j$

Table 1: Classification of irreducible representations of a compact Lie group G and their restrictions to an index 2 subgroup H. The first column assigns names to the different cases, indicating at the same time whether (U,τ) is self-dual (SD) or not (NSD). The second column gives the type of (U,τ) as a representation of G. In the third column, the isomorphism class of the algebra $gl_H(U)$, ie the type of (U,τ_H), is displayed. In the σ-self-dual cases, the fourth column gives a σ-reversible map $T : U \to U$ (considered as an element of the algebra in the third column of Table 1) for which $T^2 = \pm\mathbf{I}$, as shown in the fifth column. The map T induces an automorphism α on the $\mathbf{k}$-structure of $gl_G(U)$. This is shown in the sixth column. The last column gives the automorphism Σ of $gl_H(U)$ which fixes $gl_G(U)$ and satisfies $\Sigma(T) = -T$. The notation for the automorphisms is defined in §2.2.

Theorem 1.5 *Let V be a representation of G such that $V = \widehat{V}_\rho$, for some irreducible representation (W,ρ).*

1. *If (U,τ) is not σ-self-dual and is of type $\mathbf{k}$ then there exists a basis for V such that a map L from V to itself is σ-reversible if and only if L acts as a matrix of the form*

$$L = \begin{bmatrix} & 0 & & a_{11}\mathbf{I} & \dots & a_{1n}\mathbf{I} \\ & & & \vdots & \dots & \vdots \\ & & & a_{m1}\mathbf{I} & \dots & a_{mn}\mathbf{I} \\ b_{11}\mathbf{I} & \dots & b_{1m}\mathbf{I} & & & \\ \vdots & \dots & \vdots & & 0 & \\ b_{n1}\mathbf{I} & \dots & b_{nm}\mathbf{I} & & & \end{bmatrix}, \tag{1.6}$$

   where $a_{ij}, b_{ij} \in \mathbf{k}$, m and n are the multiplicities of the irreducible representations (U,τ) and (U,τ_σ) in V, and $\mathbf{I}$ is an identity matrix of the same dimension as W.

2. *If (U,τ) is σ-self-dual of type $\mathbf{k}$ then there exists a basis for V such that a map L from V to itself is σ-reversible if and only if L has a matrix of the form*

$$L = \begin{bmatrix} a_{11}T & \dots & a_{1m}T \\ \dots & \dots & \dots \\ a_{m1}T & \dots & a_{mm}T \end{bmatrix} \tag{1.7}$$

where $a_{ij} \in \mathbf{k}$, m is the multiplicity of (U,τ) in V, and T is the matrix of a fixed σ-reversible map from U to itself as described in Table 1.

It follows from the Theorem that

$$gl_\sigma(V) \;\cong\; \begin{cases} l(n,m;\mathbf{k}) \oplus l(m,n;\mathbf{k}) & \text{if } (W,\rho) \text{ is not } \sigma\text{-self-dual} \\ gl(m;\mathbf{k}) & \text{if } (W,\rho) \text{ is } \sigma\text{-self-dual.} \end{cases} \tag{1.8}$$

Here $l(n,m;\mathbf{k})$ is the space of $n \times m$ matrices with entries in $\mathbf{k}$. In general the isomorphisms are isomorphisms of $\mathbb{R}$-vector spaces. Since $gl_\sigma(V)$ is not closed under composition it can not be isomorphic to $gl(m;\mathbf{k})$ as an algebra.

Examples 1.6 We now will illustrate the content of Theorem 1.5 with reference to the Examples 1.2-1.4 we introduced earlier.

1. In the case of conventional time reversibility in $\mathbb{R}^2$ (Example 1.2), $(G,H) = (\mathbb{Z}_2, 1)$ and the representation of G is the direct sum of two irreducible representations of type $\mathbb{R}$ which are not isomorphic but are σ-self-dual to each other. This example is therefore of type NSD-R. Accordingly, following Theorem 1.5, we find that $L = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}$, with $a, b \in \mathbb{R}$.

2. In case $(G,H) = (\mathbb{Z}_4, \mathbb{Z}_2)$ and ρ is a two-dimensional faithful representation of G (Example 1.3), then ρ is irreducible and of type $\mathbb{C}$ and the representation of H is the direct sum of two isomorphic one-dimensional irreducible representations of type $\mathbb{R}$. It follows that ρ is of type SD-CR and so is σ-self-dual. From Theorem 1.5 we then obtain that
$$L = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}.$$
Here the complex number $A = a + ib \in \mathbb{C}$ $(a, b \in \mathbb{R})$ is represented as a matrix in $gl(2,\mathbb{R})$, and T is the matrix listed in Table 1.

3. When $(G,H) = (D_4, D_2)$, and ρ is a two-dimensional faithful representation of G (Example 1.4), then ρ is irreducible and of type $\mathbb{R}$ and the representation of H is the direct sum of two non-isomorphic one-dimensional irreducible representations of type $\mathbb{R}$. Thus ρ is of type SD-RR. From Theorem 1.5 we now find that
$$L = a\mathbf{I} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix},$$
with $a \in \mathbb{R}$.

4. When $(G,H) = (D_4, \mathbb{Z}_4)$ and the representation of G is the same as in example (3), the representation of H is irreducible and of type $\mathbb{C}$ (cf Example 1.4) and ρ is of type SD-RC. Theorem 1.5 then tells us that
$$L = a\mathbf{I} \cdot \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix},$$
with $a \in \mathbb{R}$, and $T = i$ represented as a matrix in $gl(2,\mathbb{R})$.

One readily verifies that indeed the matrices obtained by direct calculation in Examples 1.2-1.4 are recovered by using the results of Theorem 1.5. For more examples, see Examples 3.7.

1.3 Normal form theory

Our main motivation for investigating the structure of linear equivariant reversible systems is to develop a systematic approach to their normal form theory. A *normal form* $L \in gl(V)$ represents the class of matrices that can be obtained from L by linear coordinate transformations $\phi \in GL(V)$, i.e. the class of matrices of the form

$$\phi.L \;=\; \phi^{-1}L\phi. \tag{1.9}$$

Normal forms (and their versal deformations) are important ingredients of local bifurcation theory.

In the presence of symmetries (and/or other structures such as symplectic forms) we require the coordinate transformations ϕ to preserve the symmetry properties (and/or the other structures) of the system. In the setting of (G,σ)-reversible systems, the natural group of structure preserving transformations is $GL_G(V)$, the group of all G-equivariant invertible linear maps from V to itself. Thus our aim is to describe the action of $GL_G(V)$ on $gl_\sigma(V)$, the space of σ-reversible maps from V to itself.

By Schur's lemma every G-equivariant linear mapping from V to itself preserves the isotypic decomposition of V, and hence also the σ-isotypic decomposition. Moreover, for each isotypic subspace V_ρ we have $GL_G(V_\rho) \cong GL(m;\mathbf{k})$ if V_ρ is the direct sum of m copies of an irreducible representation of G of type $\mathbf{k}$. It follows that if $\widehat{V}_\rho$ is a σ-isotypic subspace then

$$GL_G(\widehat{V}_\rho) \;\cong\; \begin{cases} GL(m;\mathbf{k}) \times GL(n;\mathbf{k}) & \text{if } (W,\rho) \text{ is not } \sigma\text{-self-dual} \\ GL(m;\mathbf{k}) & \text{if } (W,\rho) \text{ is } \sigma\text{-self-dual} \end{cases} \tag{1.10}$$

where m and n are as in the statement of Theorem 1.5 and equation(1.8).

In this respect it is convenient to note that the matrices L described in Theorem 1.5 can be conveniently written as $\mathbf{k}$-tensor products (see Section 2 for a detailed treatment[4]):

$$L = \begin{cases} \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \otimes_{\mathbf{k}} \mathbf{I} & \text{if } (W,\rho) \text{ is not } \sigma\text{-self-dual,} \\ A \otimes_{\mathbf{k}} T & \text{if } (W,\rho) \text{ is } \sigma\text{-self-dual.} \end{cases} \tag{1.11}$$

Accordingly, the isomorphisms are given explicitly by

$$\phi \;\mapsto\; \begin{cases} \begin{pmatrix} \psi_1 & 0 \\ 0 & \psi_2 \end{pmatrix} \otimes_{\mathbf{k}} \mathbf{I} & \text{if } (W,\rho) \text{ is not } \sigma\text{-self-dual} \\ \psi \otimes_{\mathbf{k}} \mathbf{I} & \text{if } (W,\rho) \text{ is } \sigma\text{-self-dual.} \end{cases} \tag{1.12}$$

In the σ-self-dual cases the automorphism α of $gl_G(W) \cong \mathbf{k}$ induces an automorphism of $GL_G(\widehat{V}_\rho) \cong GL(m;\mathbf{k})$ which we also denote by α. If the automorphism of $\mathbf{k}$ is trivial then the automorphism of $GL(m;\mathbf{k})$ is also trivial. In the non trivial cases, when $\mathbf{k} = \mathbb{C}$ and the automorphism is complex conjugation, the corresponding automorphism of $GL(m;\mathbf{k})$ is also complex conjugation.

Theorem 1.7 *Let V be a representation of G such that $V = \widehat{V}_\rho$, for some irreducible representation (W,ρ).*

[4]Some care needs to be taken with the tensor product structure in the the σ-self-dual cases. Namely, it may refer to a tensor product of linear maps or a tensor product of semilinear maps.

1. *If* (W,ρ) *is not* σ*-self-dual and is of type* $\mathbf{k}$ *then the action of* $GL_G(V)$ *on* $gl_\sigma(V)$ *is given by*

$$\phi.L \;=\; \begin{bmatrix} 0 & \psi_1^{-1}A\psi_2 \\ \psi_2^{-1}B\psi_1 & 0 \end{bmatrix} \otimes_{\mathbf{k}} \mathbf{I} \tag{1.13}$$

   where $L \in gl_\sigma(V)$ *is given by* $(A,B) \in l(n,m;\mathbf{k}) \oplus l(m,n;\mathbf{k})$ *as in Theorem 1.5 and* $\phi \in GL_G(V)$ *by* $(\psi_1,\psi_2) \in GL(m;\mathbf{k}) \times GL(n;\mathbf{k})$ *as in (1.12).*

2. *If* (W,ρ) *is* σ*-self-dual of type* $\mathbf{k}$ *then the action of* $GL_G(V)$ *on* $gl_\sigma(V)$ *is given by*

$$\phi.L \;=\; (\psi^{-1}A\alpha(\psi)) \otimes_{\mathbf{k}} T, \tag{1.14}$$

   where $L \in gl_\sigma(V)$ *is given by* $A \in gl(m;\mathbf{k})$ *as in Theorem 1.5 and* $\phi \in GL_G(V)$ *by* $\psi \in GL(m;\mathbf{k})$ *as in (1.12).*

It follows from the Theorem that for the σ-self dual cases in which the automorphism α is trivial, and thus $\alpha(\psi) = \psi$, the normal form theory for L follows directly from the standard Jordan normal form theory for $gl(m,\mathbf{k})$ and the unfolding theory from the standard Arnold unfolding theory [2, 5]. However, it should be noted that the structure of T needs to be taken into account when interpreting the unfoldings of a matrix $L = A \otimes_{\mathbf{k}} T$ given by unfoldings of A. In Section 1.4 we will illustrate how these observations can be used to describe the generic eigenvalue movements of σ-reversible maps in these cases.

If $\alpha(\psi) \neq \psi$ the normal form theory does not follow from the normal form theory for $gl(m;\mathbf{k})$. However it is closely related to the normal form theory for α-semilinear mappings from $\mathbf{k}^m$ to $\mathbf{k}^m$, and in particular to that for $\overline{\mathbb{C}}$ -semilinear mappings from $\mathbb{C}^m$ to $\mathbb{C}^m$. This normal form theory and the corresponding unfolding theory will be presented in a sequel to this paper (with I. Hoveijn) [10]. See also [12] for some results in this direction.

The normal form and unfolding theories for the case when ρ is not σ-self-dual and is of type $\mathbb{R}$ reduce to the standard theories for purely reversible matrices in $gl(m+n;\mathbb{R})$ and have been treated by Hoveijn [9] in the case that $m = n$. The theories for the analogous cases with $\mathbf{k} = \mathbb{C}$ or $\mathbb{H}$ will be discussed in [10].

The normal form and unfolding theories developed in [10] will make use of the *Lie algebra* structure of $gl_H(V)$ and its two subspaces $gl_G(V)$ and $gl_\sigma(V)$. This is described in Lemma 3.1 of this paper. In particular we show that there exists an involutory Lie algebra automorphism Σ of $gl_H(V)$ which has $gl_G(V)$ as its $+1$ eigenspace and $gl_\sigma(V)$ as its -1 eigenspace. The space $gl_\sigma(V)$ is a submodule of $gl_H(V)$ considered as a module over $gl_G(V)$.

1.4 Eigenvalue movements

In this section we show how Theorems 1.5 and 1.7 can be used to describe the generic eigenvalue behaviour of some classes of σ-reversible maps. We will focus on the situation in which we have a σ-self-dual representation of G and the automorphism α is trivial. Under these circumstances we will show that the generic eigenvalue movements are intimately related to the generic eigenvalue movements of matrices in $gl(m,\mathbf{k})$. The study of generic eigenvalue movements when α is nontrivial needs a different approach and will be discussed in [10].

We need a procedure to recover the eigenvalues of matrices in $gl_\sigma(V)$ from those of the corresponding matrices in $gl(m,\mathbf{k})$. The eigenvalues of a matrix in $gl(m,\mathbb{R})$ are by definition those of its complexification. There are m of them and if λ is an eigenvalue then $\overline{\lambda}$ is also an eigenvalue with the same multiplicity as λ. A matrix in $gl(m,\mathbb{C})$ has precisely m eigenvalues if

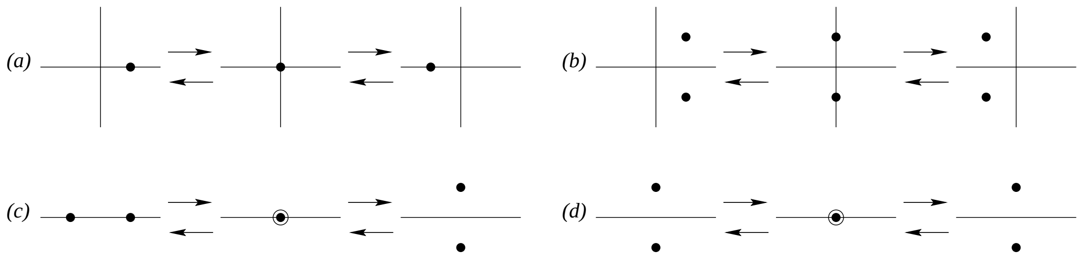

Figure 2: Generic one-parameter eigenvalue movements in the complex plane for linear systems of type (a) $\mathbb{R}$ (steady-state), (b) $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ (Hopf) (c) $\mathbb{R}$, and (d) $\mathbb{C}$.

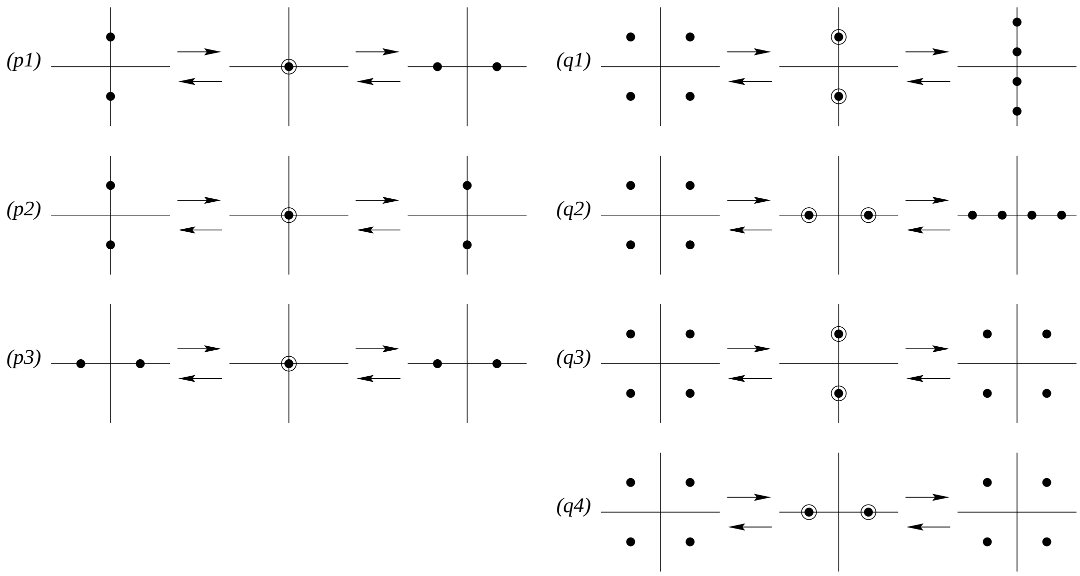

Figure 3: Generic one-parameter eigenvalue movements in the complex plane for reversible equivariant linear systems, as discussed in Theorem 1.9.

each is counted according to its algebraic multiplicity. The eigenvalues of a matrix in $gl(m, \mathbb{H})$ are by definition those obtained by forgetting the $\mathbb{H}$ structure and regarding the matrix as a $\mathbb{C}$-linear map (as a matrix in $gl(2m, \mathbb{C})$). There are therefore $2m$ eigenvalues. The $\mathbb{H}$-structure implies that the eigenvalues occur in complex conjugate pairs. In this case, real eigenvalues must occur with even multiplicities and only occur for maps which have codimension at least three.

In case $\mathbf{k} = \mathbb{C}$ or $\mathbf{k} = \mathbb{H}$ we now note that any matrix in $gl(m, \mathbb{C})$ can be represented as an $\mathbb{R}$-linear matrix in $gl(2m, \mathbb{R})$. If λ is an eigenvalue of the complex matrix with algebraic multiplicity a then λ and $\overline{\lambda}$ are both eigenvalues of the corresponding matrix in $gl(2m, \mathbb{R})$, each with multiplicity equal to a.

Hence, when a linear map L of an $\mathbb{R}$-vector space is obtained from a linear map of a $\mathbb{C}$-vector space with eigenvalues μ_i it will have pairs of complex conjugate eigenvalues $(\mu_i, \bar{\mu}_i)$. If L comes from a linear map of an $\mathbb{H}$-vector space whose complex form has eigenvalue pairs $(\mu_i, \bar{\mu}_i)$, it will have those same pairs of complex conjugate eigenvalues but now occurring with double multiplicity.

The following proposition describes how the eigenvalues of a reversible $\mathbb{R}$-linear map L are related to the eigenvalues of the corresponding matrix A given by Theorem 1.5 in the σ-self-dual cases when the automorphism α is trivial.

Proposition 1.8 (Eigenvalues)

Let V be a representation of G such that $V = \widehat{V}_\rho$, for some σ-self-dual irreducible representation (W, ρ) of type $\mathbf{k}$ for which the automorphism α is trivial. Denote the dimension of W as a vector space over $\mathbf{k}$ by r. Let $L : V \to V$ be an $\mathbb{R}$-linear σ-reversible map given by $L = A \otimes_{\mathbf{k}} T$ as in Theorem 1.5. Then the eigenvalues of L can be determined from those of A as follows.

SD-RR: *If λ is an eigenvalue of A with algebraic multiplicity a then $(\lambda, -\lambda)$ is a pair of eigenvalues of L with algebraic multiplicity $ar/2$. If $\lambda \in \mathbb{R}$ this gives a real pair of eigenvalues. Otherwise λ and $\overline{\lambda}$ together give an eigenvalue quadruplet which degenerates to an imaginary pair when λ is imaginary.*

SD-CC: *If λ is an eigenvalue of A (as a $\mathbb{C}$-linear map) with algebraic multiplicity a then $(\lambda, -\lambda, \bar{\lambda}, -\bar{\lambda})$ is a quadruplet of eigenvalues of L with algebraic multiplicity $ar/2$.*

SD-HH: *If $(\lambda, \bar{\lambda})$ is a conjugate pair of eigenvalues of A (as a $\mathbb{C}$-linear map) with algebraic multiplicity a then $(\lambda, -\lambda, \bar{\lambda}, -\bar{\lambda})$ is a quadruplet of eigenvalues of L with algebraic multiplicity ar.*

SD-RC: *If λ is an eigenvalue of A with algebraic multiplicity a then $(i\lambda, -i\lambda)$ is a pair of eigenvalues of L with algebraic multiplicity ar. If $\lambda \in \mathbb{R}$ this gives an imaginary pair of eigenvalues. Otherwise λ and $\overline{\lambda}$ together give an eigenvalue quadruplet which degenerates to a real pair when λ is imaginary.*

SD-HC: *If $(\lambda, \bar{\lambda})$ is a conjugate pair of eigenvalue of A (as a $\mathbb{C}$-linear map) with algebraic multiplicity a then $(i\lambda, -i\lambda, i\bar{\lambda}, -i\bar{\lambda})$ is a quadruplet of eigenvalues of L with algebraic multiplicity ar. The quadruplet degenerates to an imaginary pair when λ is real and to a real pair when λ is imaginary.*

Proof In the first three cases the tensor product structure implies that L has the form

$$L = \begin{bmatrix} K & 0 \\ 0 & -K \end{bmatrix}$$

where K is an equivariant map from an isotypic component of ρ_H of type $\mathbf{k}$ into itself. Hence the eigenvalues of L follows directly from those of K taking into account its $\mathbf{k}$-structure.

In the remaining two cases it is easily seen that as a complex matrix $L = iK$, where K is either a real matrix or the complex representation of matrix over $\mathbb{H}$. □

The following Theorem shows how we can combine the results of Theorems 1.5 and 1.7 and Proposition 1.8 with the Jordan normal form and Arnold unfolding theories for $\mathbf{k}$-linear maps.

Theorem 1.9 *Let $L_s : V \to V$ be a generic one-parameter family of σ-reversible maps and V_ρ an isotypic subspace of V corresponding to a σ-self-dual irreducible representation (W, ρ) for which the automorphism α is the identity. Then the only possible eigenvalue bifurcations of the restriction of L_s to V_ρ are those shown in Table 2.*

	Codimension one eigenvalue bifurcation
NSD-R $(m = n)$	p_1, q_1, q_2
SD-RR	p_3, q_2, q_3
SD-CC	q_3, q_4
SD-HH	q_3
SD-RC	p_2, q_1, q_4
SD-HC	q_4

Table 2: Generic one parameter eigenvalue bifurcations in the σ-self-dual cases for which the automorphism α of $\mathbf{k}$ induced by T is trivial. For comparison, we also include the previously studied "purely reversible" case NSD-R with $m = n$ (Hoveijn [9]). The labels $p_1, p_2, p_3, q_1, q_2, q_3, q_4$ refer to the bifurcations depicted in Fig. 3.

Proof It follows from Theorems 1.5 and 1.7 that, if (W, ρ) is of type $\mathbf{k}$ and has multiplicity m in V, then the restriction of the generic one-parameter family L_s to V_ρ is isomorphic to $A_s \otimes_{\mathbf{k}} T$ where $T : W \to W$ is the appropriate $\mathbf{k}$-linear map listed in Table 1 and A_s is a generic one-parameter family of $\mathbf{k}$-linear maps from $\mathbf{k}^m$ to itself.

From the Jordan normal form and Arnold unfolding theories the possible eigenvalue bifurcations for the family A_s, considered as a family of $\mathbf{k}$-linear maps, are as follows (see also figure 2).

1. If $\mathbf{k} = \mathbb{R}$ then the possible codimension one eigenvalue bifurcations are:
   (a) A real eigenvalue passing through 0;
   (b) A complex conjugate pair of eigenvalues passing through the imaginary axis;
   (c) Two real eigenvalues colliding and becoming a complex conjugate pair.
2. If $\mathbf{k} = \mathbb{C}$ then the possible codimension one eigenvalue bifurcations are:

   (a) An eigenvalue passing through the real axis;
   (b) An eigenvalue passing through the imaginary axis.

3. If $\mathbf{k} = \mathbb{H}$ then the only possible codimension one eigenvalue bifurcation is a complex conjugate pair of eigenvalues passing through the imaginary axis.

Combining this classification of the eigenvalue movements of A_s with the eigenvalues of T as in Proposition 1.8 gives the possible eigenvalue movements of L_s. $\square$

Remark 1.10 Due to the nature of $\mathbb{C}$ and $\mathbb{H}$ linear maps, certain types of eigenvalues may arise only with relatively high codimension. For instance, because zero eigenvalues have codimension two (resp. four) for $\mathbb{C}$- (resp. $\mathbb{H}$-) linear maps, we note that in the cases SD-CC, SD-RC and SD-HC zero eigenvalues have codimension two (and in the case SD-HH zero eigenvalues have codimension four). Similarly, we note that real eigenvalues have codimension three in $\mathbb{H}$-linear maps and hence also in case SD-HH. It is important to note that for the cases SD-CR and SD-CH we cannot draw these conclusions as they rely on the eigenvalue properties of semilinear rather than linear maps.

Examples 1.11 As an illustration of Theorem 1.9, we review the generic eigenvalue movements of the Examples 1.2-1.4. As all the examples are in $\mathbb{R}^2$, we can only observe the generic eigenvalue movements that involve pairs of eigenvalues. The dimension is too low to observe generic eigenvalue movements involving quadruplets.

1. In the purely reversible case NDS-R in $\mathbb{R}^2$ (Example 1.2) the eigenvalues follow p_1 in Figure 3 (cf [9]).

2. In case $(G, H) = (\mathbb{Z}_4, \mathbb{Z}_2)$ with G acting faithfully on $\mathbb{R}^2$ (Example 1.3), Theorem 1.9 does not apply because in this case SD-CR the map T induces a nontrivial automorphism (complex conjugation) on the $\mathbb{C}$-structure of $gl_G(W)$.

3. In case $(G, H) = (D_4, D_2)$, with G acting faithfully in $\mathbb{R}^2$, we are in the case SD-RR. Theorem 1.9 asserts that we expect generic eigenvalue bifurcations of the type p_3 in Figure 3.

4. In case $(G, H) = (D_4, \mathbb{Z}_4)$, with G acting faithfully in $\mathbb{R}^2$, we are in the case SD-RC. Theorem 1.9 asserts that we expect generic eigenvalue bifurcations of the type p_2 in Figure 3.

One readily verifies that indeed the generic eigenvalue movements of the matrices obtained by direct calculation in Examples 1.2-1.4 are predicted correctly by Theorem 1.9.

1.5 Discussion

In this paper we use representation theory for compact Lie groups to classify the structure of reversible equivariant linear vector fields with a view towards normal form theory.

In some of the cases, the normal form theory for reversible equivariant linear systems can be shown to be equivalent to Jordan normal form theory for non-symmetric systems. Without doing explicit normal form calculations, this classification allows us to predict certain generic eigenvalue movements in one-parameter families.

The non-self-dual cases reduce to *purely reversible* normal form theory over $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$. To our knowledge only part of this normal form theory ($\mathbf{k} = \mathbb{R}$ and $m = n$) has been developed [9]. The self-dual cases reduce to normal form theory for $\mathbf{k}$-(semi)linear maps with $\mathbf{k} = \mathbb{R}, \mathbb{C}$ and $\mathbb{H}$. The normal form theory for $\mathbf{k}$-linear maps is well established (see for example [4]). However, we are not aware of a systematic treatment of $\mathbf{k}$-semilinear maps beyond that given in [12]. This problem will be addressed in a sequel to this paper [10].

The normal form theory that will be developed from this paper will provide a systematic basis for local bifurcation theory in nonlinear reversible equivariant systems. For some results in this direction see for example [7, 6, 15]. In these papers only non self-dual representations of type $\mathbb{R}$ are considered. However, Lamb & Capel [14] do discuss some bifurcations arising in the case of self-dual representations.

Because of the abundance of reversible equivariant systems in classical mechanics, it will be interesting to extend our analysis to the context of symplectic (Hamiltonian) vector fields, extending the symplectic representation theory and normal form theory developed in the equivariant context by Montaldi *et al.* [18] and Melbourne *et al.* [16, 17], and the reversible Hamiltonian context by Hoveijn [9]. We intend to report on this in a forthcoming publication (with I. Hoveijn [11]).

The remaining sections of this paper contain the proofs of our results. In Section 2 we review the necessary background from linear algebra. In Section 3 we develop our theory for σ-reversible maps and prove our main results.

2 Algebras and Representations

In this section we recall some fundamental result on algebras and representations of groups. Useful general references include [1, 3, 19, 20] and [21].

2.1 Real linear algebras

A *real linear algebra* is a real vector space $\mathcal{A}$ together with a product operation, ie a bilinear map from $\mathcal{A} \times \mathcal{A}$ to $\mathcal{A}$. Examples which play an important role in group representation theory include $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$, the algebras of real, complex and quaternionic numbers with their usual products. The real numbers $\mathbb{R}$ form a subalgebra of $\mathbb{C}$

$$\mathbb{R} \;=\; \{a + ib \in \mathbb{C} \;:\; b = 0\}$$

and $\mathbb{C}$ is a subalgebra of $\mathbb{H}$

$$\mathbb{C} \;=\; \{a + ib + jc + kd \in \mathbb{H} \;:\; c = d = 0\}\,.$$

Let $\mathbf{k}$ denote $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ and $\mathbf{k}^m$ the direct sum of m copies of $\mathbf{k}$. The natural action of $\mathbf{k}$ on the right of $\mathbf{k}^m$:

$$(k_1, \ldots, k_m)k \;=\; (k_1 k, \ldots, k_m k)$$

(where here, and in the rest of the paper, the product of two elements in an algebra is denoted by concatenation) gives $\mathbf{k}^m$ the structure of a *right* vector space over $\mathbf{k}$. There is an analogous left vector space structure which coincides with the right structure for $\mathbb{R}$ and $\mathbb{C}$, since they are commutative, but not for $\mathbb{H}$. We will always use the right vector space structure on $\mathbb{H}^m$, except where it is explicitly stated that a left structure is required. Note that the embeddings of $\mathbb{R}$ in $\mathbb{C}$ and $\mathbb{C}$ in $\mathbb{H}$ mean that any vector space over $\mathbb{H}$ is naturally also a vector space over $\mathbb{C}$ and any vector space over $\mathbb{C}$ is also a vector space over $\mathbb{R}$.

In addition to this vector space structure $\mathbf{k}^m$ also has the structure of a real linear algebra, with product

$$(k_1,\ldots,k_m)(\ell_1,\ldots,\ell_m) \;=\; (k_1\ell_1,\ldots,k_m\ell_m).$$

This product will be assumed to be present whenever it is needed.

For any right vector space V over $\mathbf{k}$ we denote the space of $\mathbf{k}$-linear maps from V to itself by $gl(V;\mathbf{k})$, or just $gl(V)$ if this is not likely to lead to confusion. This space is naturally a left vector space over $\mathbf{k}$, but is not a right vector space for $\mathbf{k} = \mathbb{H}$. Similarly, if V is a left vector space over $\mathbf{k}$ the space $gl(V)$ of $\mathbf{k}$-linear maps is a right vector space over $\mathbf{k}$. However we will be primarily concerned with the inherited vector space structures over $\mathbb{R}$. Composition of mappings also defines a product on $gl(V;\mathbf{k})$ and so gives it the structure of a real linear algebra. If $V = \mathbf{k}^m$ we denote this algebra by $gl(m;\mathbf{k})$. We will identify $gl(m;\mathbf{k})$ with the algebra of $m \times m$ matrices with entries in $\mathbf{k}$.

Recall that for $\mathbf{k} = \mathbb{R}$ or $\mathbb{C}$ the tensor product $U \otimes_{\mathbf{k}} W$ of two vector spaces U, W over $\mathbf{k}$ is again a vector space over $\mathbf{k}$. If $A : U \to U$ and $B : W \to W$ are $\mathbf{k}$-linear maps then $A \otimes_{\mathbf{k}} B$ is a $\mathbf{k}$-linear map from $U \otimes_{\mathbf{k}} W$ to itself. Because it is noncommutative more care is needed when $\mathbf{k} = \mathbb{H}$. However, if U is a right vector space over $\mathbb{H}$ and W a left vector space over $\mathbb{H}$ then $U \otimes_{\mathbb{H}} W$ is still well-defined and naturally has the structure of a vector space over the centre of $\mathbb{H}$, namely $\mathbb{R}$. If $A : U \to U$ and $B : W \to W$ are $\mathbb{H}$-linear maps (with respect to the right and left $\mathbb{H}$ structures, respectively) then $A \otimes_{\mathbb{H}} T$ is an $\mathbb{R}$-linear map from $U \otimes_{\mathbf{k}} W$ to itself.

The following natural inclusions of real linear algebras will be used implicitly throughout this paper.

1. $\mathbf{k} \;\subset\; \mathbf{k}^m \;\subset\; gl(m;\mathbf{k})$

The first inclusion is given by the mapping $k \mapsto (k,\ldots,k)$ and the second by $(k_1,\ldots,k_m) \mapsto \mathrm{diag}(k_1,\ldots,k_m)$, the diagonal matrices with entries $k_1,\ldots,k_m$.

2. $gl(m;\mathbf{k}) \;\subset\; gl(rm;\mathbf{k})$ for any $r \geq 1$

This inclusion is given in terms of matrices by the map

$$A \;=\; \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mm} \end{pmatrix} \;\mapsto\; \begin{pmatrix} a_{11}\mathbf{I}_r & \cdots & a_{1m}\mathbf{I}_r \\ \vdots & & \vdots \\ a_{m1}\mathbf{I}_r & \cdots & a_{mm}\mathbf{I}_r \end{pmatrix}.$$

where $\mathbf{I}_r$ is the identity map on $\mathbf{k}^r$.

3. $gl(m;\mathbb{R}) \;\subset\; gl(m;\mathbb{C}) \;\subset\; gl(m;\mathbb{H})$

Because of the natural inclusions of $\mathbb{R}$ in $\mathbb{C}$ and of $\mathbb{C}$ in $\mathbb{H}$, any matrix with entries in $\mathbb{R}$ can be regarded as a matrix with entries in $\mathbb{C}$ and any matrix with entries in $\mathbb{C}$ can be regarded as a matrix with entries in $\mathbb{H}$.

4. $gl(m;\mathbb{C}) \;\subset\; gl(2m;\mathbb{R})$

The real vector space isomorphism between $\mathbb{C}$ and $\mathbb{R}^2$ given by $a+ib \mapsto (a,b)$ extends naturally to an isomorphism between $\mathbb{C}^m$ and $\mathbb{R}^{2m}$. The inclusion of $gl(m;\mathbb{C})$ in $gl(2m;\mathbb{R})$ is obtained by regarding complex linear maps on $\mathbb{C}^m$ as real linear maps on $\mathbb{R}^{2m}$. In terms of matrices it is given by

$$\begin{pmatrix} a_{11}+ib_{11} & \cdots & a_{1m}+ib_{1m} \\ \vdots & & \vdots \\ a_{m1}+ib_{m1} & \cdots & a_{mm}+ib_{mm} \end{pmatrix} \mapsto \begin{pmatrix} a_{11} & -b_{11} & \cdots & a_{1m} & -b_{1m} \\ b_{11} & a_{11} & \cdots & b_{1m} & a_{1m} \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & -b_{m1} & \cdots & a_{mm} & -b_{mm} \\ b_{m1} & a_{m1} & \cdots & b_{mm} & a_{mm} \end{pmatrix}.$$

5. $gl(m;\mathbb{H}) \subset gl(2m;\mathbb{C})$

The identification $\mathbb{H} \cong \mathbb{C}^2$ given by $q = \alpha + j\beta \mapsto (\alpha, \beta)$ induces an isomorphism of $\mathbb{C}$-vector spaces $\mathbb{H}^m \cong \mathbb{C}^{2m}$ where the complex structure on $\mathbb{H}^m$ is that inherited from its right $\mathbb{H}$ structure. Under this isomorphism every $\mathbb{H}$-linear map is $\mathbb{C}$-linear. In terms of matrices the inclusion is given by

$$\begin{pmatrix} \alpha_{11}+j\beta_{11} & \cdots & \alpha_{1m}+j\beta_{1m} \\ \vdots & & \vdots \\ \alpha_{m1}+j\beta_{m1} & \cdots & \alpha_{mm}+j\beta_{mm} \end{pmatrix} \mapsto \begin{pmatrix} \alpha_{11} & -\overline{\beta_{11}} & \cdots & \alpha_{1m} & -\overline{\beta_{1m}} \\ \beta_{11} & \overline{\alpha_{11}} & \cdots & \beta_{1m} & \overline{\alpha_{1m}} \\ \vdots & \vdots & & \vdots & \vdots \\ \alpha_{m1} & -\overline{\beta_{m1}} & \cdots & \alpha_{mm} & -\overline{\beta_{mm}} \\ \beta_{m1} & \overline{\alpha_{m1}} & \cdots & \beta_{mm} & \overline{\alpha_{mm}} \end{pmatrix}.$$

2.2 Automorphisms of algebras

An automorphism of a real linear algebra $\mathcal{A}$ is an invertible $\mathbb{R}$-linear map $\Sigma : \mathcal{A} \to \mathcal{A}$ which preserves the product operation, ie $\Sigma(AB) = \Sigma(A)\Sigma(B)$. The following result, describing the algebra automorphisms of $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$, will play an important role in the next section.

Proposition 2.1

1. *The only algebra automorphism of $\mathbb{R}$ is the identity map.*
2. *The algebra automorphisms of $\mathbb{C}$ are the identity and complex conjugation.*
3. *The algebra automorphisms of $\mathbb{H}$ are precisely the maps $\Sigma_q : \mathbb{H} \mapsto \mathbb{H}$ given by $\Sigma_q(x) = q^{-1}xq$ for some invertible quaternion q.*

For a proof see [19]. It will be convenient to single out the automorphism Σ_i of $\mathbb{H}$ for special treatment and denote $\Sigma_i(x)$ by $x^\sharp$. Note that

$$\Sigma_i(a+ib+jc+dk) = (a+ib+jc+dk)^\sharp = a+ib-jc-dk$$

and so its fixed point set is the subalgebra $\mathbb{C}$ embedded in $\mathbb{H}$ as described in §2.1. This is analogous to the appearance of $\mathbb{R}$ as the fixed point set of the complex conjugation automorphism on $\mathbb{C}$ and so we refer to Σ_i as *quaternionic conjugation.*

An automorphism Σ of $\mathbf{k} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ extends in a natural way to an automorphism of $\mathbf{k}^m$ which we continue to denote by Σ:

$$\Sigma((k_1, \ldots, k_m)) = (\Sigma(k_1), \ldots, \Sigma(k_m))$$

and hence also to an automorphism of $gl(m;\mathbf{k})$ which we denote by κ_Σ:

$$\kappa_\Sigma(A) \;=\; \Sigma^{-1}\circ A\circ\Sigma.$$

If A is regarded as a matrix with entries in $\mathbf{k}$ the matrix $\kappa_\Sigma(A)$ is obtained by applying Σ to each of the entries. If Σ is either complex or quaternionic conjugation we will denote $\kappa_\Sigma(A)$ by $\overline{A}$ or $A^\sharp$, respectively.

For any real linear algebra $\mathcal{A}$ and invertible element B in $\mathcal{A}$ we can define an automorphism Σ_B by $\Sigma_B(A) = B^{-1}AB$. These are called *inner automorphisms*. Examples include the automorphisms of $\mathbb{H}$ and $gl(m;\mathbb{H})$ defined above, but not the complex conjugation automorphisms of $\mathbb{C}$ and $gl(m;\mathbb{C})$.

An *involution* on an algebra $\mathcal{A}$ is an algebra automorphism Σ which is its own inverse, so that Σ^2 is the identity. Many of the automorphisms described above are involutions. Some elementary properties of involutions are summarized in the following proposition.

Proposition 2.2 *Let $\mathcal{A}$ be a real linear algebra with an involution Σ.*

1. *The algebra $\mathcal{A}$ decomposes as a direct sum of linear spaces $\mathcal{A} = \mathcal{A}_+ \oplus \mathcal{A}_-$ where*

$$\mathcal{A}_\pm \;=\; \{A\in\mathcal{A} \;:\; \Sigma(A) = \pm A\}.$$

2. *The subspace $\mathcal{A}_+$ is a subalgebra of $\mathcal{A}$ and $\mathcal{A}$ is a module over $\mathcal{A}_+$.*

3. *The subspace $\mathcal{A}_-$ is a submodule of $\mathcal{A}$ over $\mathcal{A}_+$.*

If U and V are right vector space over $\mathbf{k}$ and α is an automorphism of $\mathbf{k}$, an $\mathbb{R}$-linear mapping A from U to V is said to be $\mathbf{k}^\alpha$*-linear*, or $\mathbf{k}$*-semilinear with respect to* α, if

$$A(xk) \;=\; A(x)\alpha(k) \qquad \text{for all } k\in\mathbf{k},\; x\in U.$$

An analogous definition can be made for maps between left vector spaces. We will denote the space of $\mathbf{k}^\alpha$-linear maps from a $\mathbf{k}$-vector space V to itself by $gl(V;\mathbf{k}^\alpha)$. If V is a right $\mathbf{k}$ space then, like $gl(V;\mathbf{k})$, this is a left $\mathbf{k}$ vector space. However, since the composition of two $\mathbf{k}^\alpha$-linear maps is $\mathbf{k}$-linear, and not $\mathbf{k}^\alpha$-linear if α is not the identity, then in general $gl(V;\mathbf{k}^\alpha)$ is not an algebra under composition. Instead composition gives natural actions of $gl(V;\mathbf{k})$ on both the right and left of $gl(V;\mathbf{k}^\alpha)$ making it into a $gl(V;\mathbf{k})$-module.

When $\mathbf{k} = \mathbb{C}$ and α is complex conjugation we replace the notation $\mathbf{k}^\alpha$ by $\overline{\mathbb{C}}$. If $V = \mathbb{C}^m$ we denote $gl(V;\mathbf{k}^\alpha)$ by $gl(m;\overline{\mathbb{C}})$. Similarly, when $\mathbf{k} = \mathbb{H}$ and α is quaternionic conjugation we replace $\mathbf{k}^\alpha$ by $\mathbb{H}^\sharp$ and denote $gl(\mathbb{H}^m;\mathbb{H}^\sharp)$ by $gl(m;\mathbb{H}^\sharp)$. These examples are discussed in 3 and 5 of Examples 2.4.

Semilinear maps can be combined by tensor products in much the same way as linear maps. In particular, if U and W are right and left $\mathbf{k}$-vector spaces, respectively, and $A : U\to U$ and $B : W\to W$ are $\mathbf{k}^\alpha$-linear maps, then $A\otimes_{\mathbf{k}} B$ is well defined as a semilinear map from $U\otimes_{\mathbf{k}} W$ to itself, considered as a vector space over the centre of $\mathbf{k}$ with the involution obtained by restricting α. If $\mathbf{k}^\alpha = \overline{\mathbb{C}}$ then this means that $A\otimes_{\mathbf{k}} B$ is a well defined $\overline{\mathbb{C}}$-linear map, while if $\mathbf{k}^\alpha = \mathbb{H}^\sharp$ it is an $\mathbb{R}$-linear map.

2.3 Lie algebras and adjoint actions

In addition to the algebra structure induced by the operation of composition, the spaces $gl(m;\mathbf{k})$ also have *Lie algebra* structures, with Lie brackets defined by $[A,B] = AB - BA$. The space $gl(m;\mathbf{k})$ can be identified with the tangent space of the Lie group $GL(m;\mathbf{k})$ at the identity element. As such there is a natural *adjoint action* of $GL(m;\mathbf{k})$ on $gl(m;\mathbf{k})$ given explicitly by

$$\phi.A \; = \; \phi^{-1}A\phi \qquad \forall\phi \in GL(m;\mathbf{k}),\; A \in gl(m;\mathbf{k}).$$

This corresponds to the natural action of linear coordinate changes on linear vector fields or differential equations. More generally, for any Lie algebra $\mathcal{A}$ there is an associated Lie group, $\mathcal{G}$, and a natural adjoint action of $\mathcal{G}$ on $\mathcal{A}$ (see for example [23]).

An involution Σ of a Lie algebra $\mathcal{A}$ is a Lie algebra involution if it preserves the Lie bracket, ie $[\Sigma(A),\Sigma(B)] = \Sigma([A,B])$. If Σ is an algebra involution of $gl(m;\mathbf{k})$ then clearly it is also a Lie algebra involution with the bracket defined as above. An involution Σ of a Lie algebra $\mathcal{A}$ induces a group involution of the corresponding Lie group $\mathcal{G}$ which we will also denote by Σ. Let $\mathcal{A}_+$ and $\mathcal{A}_-$ denote the ± 1 eigenspaces of Σ on $\mathcal{A}$, as in the previous subsection. By Proposition 2.2 $\mathcal{A}_+$ is a Lie subalgebra of $\mathcal{A}$ and $\mathcal{A}_-$ a Lie submodule over $\mathcal{A}_+$. Let $\mathcal{G}_+$ denote the fixed point set of Σ on $\mathcal{G}$. The following results are immediate consequences of the definitions.

Proposition 2.3

1. *The Lie subalgebra $\mathcal{A}_+$ is the Lie algebra of $\mathcal{G}_+$.*
2. *The restriction of the adjoint action of $\mathcal{G}$ on $\mathcal{A}$ to $\mathcal{G}_+$ preserves the subpaces $\mathcal{A}_+$ and $\mathcal{A}_-$. The induced action of $\mathcal{G}_+$ on $\mathcal{A}_+$ is the adjoint action of $\mathcal{G}_+$.*

Examples 2.4

We describe some examples of Lie algebras $\mathcal{A}$ and involutions Σ that will be used in §3.

1. $\mathcal{A} = gl(m+n;\mathbf{k}) \quad \Sigma(A) = R_{m,n}^{-1}AR_{m,n}$

Let $R_{m,n}$ denote the involution on $\mathbf{k}^m \oplus \mathbf{k}^n$ which fixes $\mathbf{k}^m$ and acts by $-\mathbf{I}$ on $\mathbf{k}^n$. Then $\mathcal{A}_+$ is equal to $gl(m;\mathbf{k}) \oplus gl(n;\mathbf{k})$. The space $\mathcal{A}_-$ is given by

$$\mathcal{A}_- \; = \; \left\{ \begin{pmatrix} 0 & B \\ A & 0 \end{pmatrix} \; : \; A \in l(m,n;\mathbf{k}),\; B \in l(n,m;\mathbf{k}) \right\}$$

where $l(m,n;\mathbf{k})$ is the space of $\mathbf{k}$-linear maps from $\mathbf{k}^m$ to $\mathbf{k}^n$. The group $\mathcal{G}_+$ is $GL(m;\mathbf{k}) \times GL(n;\mathbf{k})$, considered as a subgroup of $\mathcal{G} = GL(m+n;\mathbf{k})$. The action of $\mathcal{G}_+$ on $\mathcal{A}_+$ is the product of the adjoint actions of $GL(m;\mathbf{k})$ and $GL(n;\mathbf{k})$. Its action on $\mathcal{A}_- \cong l(m,n;\mathbf{k}) \oplus l(n,m;\mathbf{k})$ is given by

$$(\phi_1,\phi_2).(A,B) \; = \; (\phi_2^{-1}A\phi_1, \phi_1^{-1}B\phi_2).$$

This is the action which underlies the normal form theory of reversible systems studied in [9] and references therein, at least when $m = n$.

2. $\mathcal{A} = gl(m;\mathbb{C}) \quad \Sigma(A) = \overline{A}$

In this case we have

$$\begin{aligned} \mathcal{A}_+ &= \left\{ A \in gl(m;\mathbb{C}) \; : \; \overline{A} = A \right\} \\ \mathcal{A}_- &= \left\{ A \in gl(m;\mathbb{C}) \; : \; \overline{A} = -A \right\}. \end{aligned}$$

Thus $\mathcal{A}_+$ is equal to $gl(m;\mathbb{R})$ while $\mathcal{A}_-$ can be identified with the space of $m \times m$ matrices with imaginary entries. The real vector space isomorphism

$$\begin{array}{rcl} gl(m;\mathbb{C}) & \to & gl(m;\mathbb{R}) \oplus gl(m;\mathbb{R}) \\ A + iB & \mapsto & (A, B) \end{array}$$

restricts to give an isomorphism between $\mathcal{A}_-$ and $gl(m;\mathbb{R})$.

The Lie group $\mathcal{G}$ associated to $\mathcal{A}$ is $GL(m;\mathbb{C})$ and $\mathcal{G}_+$ is $GL(m;\mathbb{R})$. The actions of $\mathcal{G}_+$ on $\mathcal{A}_+$ and $\mathcal{A}_-$ are both isomorphic to the adjoint action of $GL(m;\mathbb{R})$ on $gl(m;\mathbb{R})$.

3. $\mathcal{A} = gl(2m;\mathbb{R}) \quad \Sigma(A) = -iAi$

Here i is being identified with a scalar in $gl(m;\mathbb{C})$ and hence with an element of $gl(2m;\mathbb{R})$. We have

$$\begin{array}{rcl} \mathcal{A}_+ & = & \{A \in gl(2m;\mathbb{R}) \ : \ Ai = iA\} \\ \mathcal{A}_- & = & \{A \in gl(2m;\mathbb{R}) \ : \ Ai = -iA\} \end{array}$$

and so $\mathcal{A}_+$ is equal to $gl(m;\mathbb{C})$. The complementary space $\mathcal{A}_-$ is the space $gl(m;\overline{\mathbb{C}})$ of $\overline{\mathbb{C}}$-linear maps from $\mathbb{C}^m$ to itself, that is $\mathbb{R}$-linear maps which satisfy $A(kx) = \overline{k}A(x)$ for all $k \in \mathbb{C}$, $x \in \mathbb{C}^m$. If $\kappa : \mathbb{C}^m \to \mathbb{C}^m$ is the $\overline{\mathbb{C}}$-linear map given by complex conjugation, the map $B \mapsto B \circ \kappa$ gives a vector space isomorphism from $gl(m;\mathbb{C})$ to $gl(m;\overline{\mathbb{C}})$.

The Lie group $\mathcal{G}$ associated to $\mathcal{A}$ is $GL(2m;\mathbb{R})$ and $\mathcal{G}_+$ is $GL(m;\mathbb{C})$. The action of $\mathcal{G}_+$ on $\mathcal{A}_+$ is the adjoint action of $GL(m;\mathbb{C})$ on $gl(m;\mathbb{C})$. Under the isomorphism $B \circ \kappa \mapsto B$ the action of $\mathcal{G}_+$ on $\mathcal{A}_-$ is given by:

$$\phi.B \ = \ \phi^{-1} B \overline{\phi}.$$

This is the action that underlies the normal form theory for $\overline{\mathbb{C}}$-linear maps.

4. $\mathcal{A} = gl(m;\mathbb{H}) \quad \Sigma(A) = A^\sharp$

This case is similar to 2 above. We have

$$\begin{array}{rcl} \mathcal{A}_+ & = & \left\{A \in gl(m;\mathbb{H}) \ : \ A^\sharp = A\right\} \\ \mathcal{A}_- & = & \left\{A \in gl(m;\mathbb{H}) \ : \ A^\sharp = -A\right\} \end{array}$$

and so $\mathcal{A}_+$ is equal to $gl(m;\mathbb{C})$ while $\mathcal{A}_-$ can be identified with the space of $m \times m$ matrices with entries which are quaternions of the form $cj + dk$ for $c, d \in \mathbb{R}$. The vector space isomorphism

$$\begin{array}{rcl} gl(m;\mathbb{H}) & \to & gl(m;\mathbb{C}) \oplus gl(m;\mathbb{C}) \\ A + j\overline{B} & \mapsto & (A, B) \end{array}$$

restricts to give an isomorphism between $\mathcal{A}_-$ and $gl(m;\mathbb{C})$.

The Lie group $\mathcal{G}$ associated to $\mathcal{A}$ is $GL(m;\mathbb{H})$ and $\mathcal{G}_+$ is $GL(m;\mathbb{C})$. Under the isomorphism $gl(m;\mathbb{H}) \cong gl(m;\mathbb{C}) \oplus gl(m;\mathbb{C})$ described above, the action of $\mathcal{G}_+$ on $\mathcal{A}$ is given by

$$\phi.(A + j\overline{B}) \ = \ \phi^{-1}(A + j\overline{B})\phi \ = \ \phi^{-1}A\phi + j\overline{\phi^{-1}B\overline{\phi}}.$$

Thus the action of $\mathcal{G}_+$ on $\mathcal{A}_+$ is the adjoint action of $GL(m;\mathbb{C})$ on $gl(m;\mathbb{C})$ while that on $\mathcal{A}_-$ is isomorphic to the action on the space of $\overline{\mathbb{C}}$-linear maps described in 3 above.

5. $\mathcal{A} = gl(2m;\mathbb{C}) \quad \Sigma(A) = -j\overline{A}j$

In this case j is being identified with the linear map from $\mathbb{H}^m$ to itself obtained by its action on the right of $\mathbb{H}^m$. We have

$$\begin{aligned}\mathcal{A}_+ &= \{A \in gl(2m;\mathbb{C}) \ : \ Aj = j\overline{A}\} \\ \mathcal{A}_- &= \{A \in gl(2m;\mathbb{C}) \ : \ Aj = -j\overline{A}\}\end{aligned}$$

and so $\mathcal{A}_+$ is equal to $gl(m;\mathbb{H})$ and $\mathcal{A}_-$ is the space $gl(m;\mathbb{H}^\sharp)$ of $\mathbb{H}^\sharp$-linear maps from $\mathbb{H}^m$ to itself, that is $\mathbb{R}$-linear maps which satisfy $A(xk) = A(x)k^\sharp$ for all $k \in \mathbb{H}$, $x \in \mathbb{H}^m$. We can identify $\mathbb{H}^m$ with $\mathbb{C}^{2m}$ so that

$$\begin{aligned}\mathcal{A}_+ &= gl(m;\mathbb{H}) &= \left\{\begin{pmatrix} A & C \\ B & D\end{pmatrix} \in gl(2m;\mathbb{C}) \ : \ C = -\overline{B},\ D = \overline{A}\right\} \\ \mathcal{A}_- &= gl(m;\mathbb{H}^\sharp) &= \left\{\begin{pmatrix} A & C \\ B & D\end{pmatrix} \in gl(2m;\mathbb{C}) \ : \ C = \overline{B},\ D = -\overline{A}\right\}\end{aligned}$$

The map

$$\begin{pmatrix} A & -\overline{B} \\ B & \overline{A}\end{pmatrix} \mapsto i\begin{pmatrix} A & -\overline{B} \\ B & \overline{A}\end{pmatrix}$$

defines a vector space isomorphism between $gl(m;\mathbb{H})$ and $gl(m;\mathbb{H}^\sharp)$.

The Lie group $\mathcal{G}$ associated to $\mathcal{A}$ is $GL(2m;\mathbb{C})$ and $\mathcal{G}_+$ is $GL(m;\mathbb{H})$. The action of $\mathcal{G}_+$ on $\mathcal{A}_+$ is the adjoint action of $GL(m;\mathbb{H})$ on $gl(m;\mathbb{H})$. Under the isomorphism with $gl(m;\mathbb{H})$ described above, the action on $\mathcal{A}_- = gl(m;\mathbb{H}^\sharp)$ is also isomorphic to this adjoint action. Note that this implies that the normal form theory for $\mathbb{H}^\sharp$ linear maps can be deduced from that for $\mathbb{H}$-linear maps.

2.4 Irreducible representations

A *representation* of G on a real vector space V is a group homomorphism $\rho : G \to GL(V)$ from G into $GL(V)$. We refer to the dimension of V as the dimension of the representation. If (V_1, ρ_1) and (V_2, ρ_2) are two representations of G a mapping $\phi : V_1 \to V_2$ is said to be *equivariant* if it commutes with the actions, ie $\phi \circ \rho_1 = \rho_2 \circ \phi$. Two representations (V_1, ρ_1), (V_2, ρ_2) are *isomorphic* if there exists an invertible equivariant linear map from V_1 to V_2. Clearly isomorphic representations have the same dimension. A representation of G is said to be *trivial* if $\rho(g)$ is the identity map for every g in G.

A representation (V, ρ) of G is *irreducible* if V does not contain any proper, non-trivial G-invariant linear subspaces. If G is finite then it has only a finite set of isomorphism classes of irreducible representations. If it is compact but not finite then it has a countable set.

The space of all equivariant linear maps from one representation (V_1, ρ_1) to another (V_2, ρ_2) will be denoted by $l_G((V_1, \rho_1), (V_2, \rho_2))$ or by $l_G(V_1, V_2)$ if this is not likely to cause any confusion. When the two representations are the same we denote $l_G(V, V)$ by $gl_G(V)$. In this case the linear maps can also be composed, giving $gl_G(V)$ the structure of an algebra and not just a real vector space. The following result describes the spaces of linear maps between irreducible representations. It lies at the heart of representation theory and is fundamental to the rest of this paper.

Proposition 2.5 (Schur's Lemma)

1. *If (V_1, ρ_1) and (V_2, ρ_2) are two non-isomorphic irreducible representations of G then $l_G(V_1, V_2) = \{0\}$.*

2. If (V, ρ) is an irreducible representation of G then $gl_G(V)$ is isomorphic (as a real algebra) to $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

We say that an irreducible representation (V, ρ) *is of type* $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ according the isomorphism class of $gl_G(V)$. The natural action of $gl_G(V)$ on an irreducible representation (V, ρ) of type $\mathbf{k}$ gives V the structure of a left vector space over $\mathbf{k}$. With this structure the elements of G act by $\mathbf{k}$-linear maps on V.

2.5 Isotypic decomposition

If G is compact then every representation is isomorphic to a direct sum of irreducible representations. More precisely, if $\rho : G \to GL(V)$ is a representation of G on V then V can be decomposed as a direct sum

$$V \;=\; V_1 \oplus \cdots \oplus V_k \tag{2.1}$$

where V_i is a G-invariant subspace of V and the restriction of the action of G to V_i gives a representation of G which is isomorphic to a direct sum of m_i copies of an irreducible representation (U_i, τ_i). We can suppose U_i is not isomorphic to U_j for $i \neq j$, in which case the subspaces V_i are uniquely defined and the decomposition (2.1) is unique up to a reordering of the summands. We refer to the V_i as the *isotypic blocks* of V and the decomposition 2.1 as an *isotypic decomposition* of V.

It follows from part (1) of Schur's lemma (2.5) that any G-equivariant linear map from V to itself must preserve the isotypic blocks V_i and $gl_G(V)$ decomposes as the direct sum of subspaces which are isomorphic to $gl_G(V_i)$:

$$gl_G(V) \;\cong\; gl_G(V_1) \oplus \cdots \oplus gl_G(V_k). \tag{2.2}$$

Restricting to invertible G-equivariant maps, we obtain a corresponding decomposition

$$GL_G(V) \;\cong\; GL_G(V_1) \times \cdots \times GL_G(V_k). \tag{2.3}$$

The space $gl_G(V)$ is the Lie algebra of $GL_G(V)$ and the adjoint action of $GL(V)$ on $gl(V)$ restricts to the adjoint action of $GL_G(V)$ on $gl_G(V)$. This action decomposes as the direct sum of the adjoint actions of the groups $GL_G(V_i)$ on the spaces $gl_G(V_i)$.

To describe the space $gl_G(V_i)$ we note that since V_i is the direct sum of m_i copies of U_i any equivariant linear map $L : V_i \to V_i$ will have a matrix of the form

$$\begin{pmatrix} L_{11} & \dots & L_{1m_i} \\ \vdots & & \vdots \\ L_{m_i1} & \dots & L_{m_im_i} \end{pmatrix}$$

where each L_{rs} is given by an equivariant map from U_i to itself. If (U_i, ρ_i) is of type $\mathbf{k}_i$ then we can take $L_{rs} = \ell_{rs}\mathbf{I}$ where $\ell_{rs} \in \mathbf{k}_i$ and $\mathbf{I}$ is the identity map on U_i. Thus $L : V_i \to V_i$ is given by an $m_i \times m_i$ matrix with elements in $\mathbf{k}_i$. In other words $gl_G(V_i)$ is isomorphic to $gl(m_i; \mathbf{k}_i)$. Similarly $GL_G(V_i) \cong GL(m_i; \mathbf{k}_i)$ and so the adjoint action of $\mathbf{GL}_G(V)$ on $gl_G(V)$ is isomorphic to the direct sum of the adjoint actions of the groups $GL(m_i; \mathbf{k}_i)$ on their Lie algebras $gl(m_i; \mathbf{k}_i)$.

This result can be very naturally expressed using tensor product. If V_i is the direct sum of m_i copies of U_i and U_i is of type $\mathbf{k}_i$ then

$$V_i \cong \mathbf{k}_i^{m_i} \otimes_{\mathbf{k}_i} U_i. \tag{2.4}$$

The form of the matrix above shows that every G-equivariant linear map L from V_i to itself has the form

$$L \cong \ell \otimes_{\mathbf{k}_i} \mathbf{I} \tag{2.5}$$

where ℓ is the $\mathbf{k}_i$-linear map from $\mathbf{k}_i^{m_i}$ to itself defined by the $m_i \times m_i$ matrix (ℓ_{rs}). The isomorphism from $gl(m_i;\mathbf{k}_i)$ to $gl_G(V_i)$ is given by $\ell \mapsto L = \ell \otimes_{\mathbf{k}_i} \mathbf{I}$. Restricting this to invertible maps gives the isomorphism from $GL(m_i;\mathbf{k}_i)$ to $GL_G(V_i)$.

Finally, we generalise the classification of irreducible representations into types $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$ by defining the *type* of a general representation V (defined over $\mathbb{R}$) to be the isomorphism class of $gl_G(V)$ as a real linear algebra.

3 Proofs of the Main Results

Let H be an index 2 subgroup of G and $\sigma : G \to \mathbb{Z}_2$ the homomorphism with kernel H. Identify $\mathbb{Z}_2$ with the subgroup of $GL(1,\mathbb{R}) \cong \mathbb{R} \setminus \{0\}$ consisting of ± 1, and hence σ with a one dimensional representation $(\mathbb{R}, \sigma)$ which we will call the *sign representation* of the pair (G, H). If (V, ρ) is a representation of G a linear map $L : V \to V$ is said to be σ-reversible if

$$L(\rho(g)v) \;=\; \sigma(g)\rho(g)L(v)$$

for all g in G. Let $gl_\sigma(V,\rho)$, or just $gl_\sigma(V)$, denote the space of all σ-reversible maps from V to itself. If $\phi \in GL_G(V)$ and $L \in gl_\sigma(V)$ then $\phi^{-1} \circ L \circ \phi \in gl_\sigma(V)$. This is the natural change of coordinates action of $GL_G(V)$ on $gl_\sigma(V)$. Our aim is to describe the structure of the space $gl_\sigma(V)$ and this action of $GL_G(V)$ on it.

We start with the natural adjoint action of $GL_H(V)$ on $gl_H(V)$. Let g be an element of $G \setminus H$. Define a group involution $\Sigma : GL_H(V) \to GL_H(V)$ by

$$\Sigma(\phi) \;=\; \rho(g)^{-1}\phi\rho(g).$$

Note that Σ is independent of the choice of $g \in G \setminus H$ and Σ^2 is the identity. We also denote the induced Lie algebra involution of $gl_H(V)$ by Σ:

$$\begin{aligned} \Sigma \;:\; gl_H(V) &\;\to\; gl_H(V) \qquad (3.6)\\ A &\;\mapsto\; \rho(g)^{-1}A\rho(g) \qquad (3.7) \end{aligned}$$

for any $g \in G \setminus H$. A number of elementary properties of this involution are summarised in the following proposition.

Proposition 3.1

1. *The group $GL_G(V)$ is the fixed point set of the action of Σ on $GL_H(V)$.*
2. *The restriction of the adjoint action of $GL_H(V)$ on $gl_H(V)$ to $GL_G(V)$ preserves the subspaces*

$$\begin{aligned} gl_G(V) &= \{A \in gl_H(V) | \Sigma(A) = A\} \\ gl_\sigma(V) &= \{A \in gl_H(V) | \Sigma(A) = -A\}. \end{aligned}$$

3. *The induced action of $GL_G(V)$ on $gl_G(V)$ is the adjoint action.*

4. *The subspace $gl_\sigma(V)$ is a $gl_G(V)$-submodule of $gl_H(V)$ with respect to both the algebra structure induced by composition and that given by the Lie bracket.*

5. *If there exists an invertible σ-reversible map $T : V \to V$ then $gl_\sigma(V)$ is generated by T as a $gl_G(V)$-submodule of $gl_H(V)$ with respect to composition.*

Proof Parts 1 - 4 follow directly from the definitions and Propositions 2.2 and 2.3. For part 5 we note that if $T : V \to V$ is an invertible σ-reversible map and $S : V \to V$ is any σ-reversible map then the composition $S \circ T^{-1}$ is G-equivariant and so $S = (S \circ T^{-1}) \circ T$ is the composition of an element of $gl_G(V)$ with T. □

3.1 Irreducible representations of G

If (U, τ) is an irreducible representation of G of type $\mathbf{k}$ then $gl_G(U) \cong \mathbf{k}$. In this section we classify the possible isomorphism classes of the algebra $gl_H(U)$ and the forms that the automorphisms Σ can take on $gl_H(U)$.

For any representation (U, τ) we define the *σ-dual representation* (U, τ_σ) to be that given by the homomorphism $\tau_\sigma : G \to GL(U)$ with

$$\tau_\sigma(g) \;=\; \sigma(g)\tau(g).$$

Equivalently (U, τ_σ) is the tensor product (over $\mathbb{R}$) of (U, τ) with the sign representation $(\mathbb{R}, \sigma)$. Note that a σ-reversible map $T : U \to U$ can also be regarded as a G-equivariant map, where the action of G on the source U is given by τ and that on the target by the σ - dual τ_σ.

If two representations are isomorphic then so are their σ-duals. Clearly the σ-dual of (U, τ_σ) is again (U, τ) and so the operator $(U, \tau) \mapsto (U, \tau_\sigma)$ defines an involution on the set of all isomorphism classes of representations of G. A representation is said to be *σ-self-dual* if (U, τ_σ) is isomorphic to (U, τ) as a representation of G.

Theorem 3.2 *Let (U, τ) be an irreducible representation of G of type $\mathbf{k}$. Then the following statements are equivalent.*

1. *The representation (U, τ) is σ-self-dual;*

2. *There exists a non-zero σ-reversible linear map $T : U \to U$;*

3. *There exists an invertible σ-reversible linear map $T : U \to U$ satisfying one of the following sets of conditions:*

   (a) $\mathbf{k} = \mathbb{R}$, *T is $\mathbb{R}$-linear and* $T^2 = \mathbf{I}$

   (b) $\mathbf{k} = \mathbb{R}$, *T is $\mathbb{R}$-linear and* $T^2 = -\mathbf{I}$

   (c) $\mathbf{k} = \mathbb{C}$, *T is $\mathbb{C}$-linear and* $T^2 = \mathbf{I}$

   (d) $\mathbf{k} = \mathbb{C}$, *T is $\overline{\mathbb{C}}$-linear and* $T^2 = \mathbf{I}$

   (e) $\mathbf{k} = \mathbb{C}$, *T is $\overline{\mathbb{C}}$-linear and* $T^2 = -\mathbf{I}$

   (f) $\mathbf{k} = \mathbb{H}$, *T is $\mathbb{H}$-linear and* $T^2 = \mathbf{I}$

   (g) $\mathbf{k} = \mathbb{H}$, *T is $\mathbb{H}$-linear and* $T^2 = -\mathbf{I}$

Moreover, if (U,τ) is σ-self-dual and $T : U \to U$ is an invertible σ-reversible linear map, then $gl_\sigma(U)$ is generated as a module over $gl_G(U)$ by T.

Proof

3 ⇒ 2 This follows immediately.

2 ⇒ 1 If (U,τ) is not σ-self-dual then it is not isomorphic to its σ-dual (U,τ_σ) and so by Schur's lemma there are no non-zero G-equivariant linear maps from (U,τ) to (U,τ_σ), and hence no non-zero σ-reversible maps.

1 ⇒ 3 Let $S : U \to U$ be an isomorphism from (U,τ) to (U,τ_σ), ie an invertible σ-reversible linear map. Then so is $T = \phi \circ S$ for any $\phi \in GL_G(U)$. We will show that ϕ can be chosen so that T satisfies one of the sets of conditions in statement *3* of the Theorem. Using the $\mathbf{k}$-vector space structure on U induced from the isomorphism $gl_G(U) \cong \mathbf{k}$ the maps ϕ are given by $\phi = k\mathbf{I}$ for $k \in \mathbf{k} \setminus \{0\}$.

Define a real algebra automorphism $\alpha : gl_H(U) \to gl_H(U)$ by $\alpha(A) = SAS^{-1}$. Note that $\alpha(S) = S$ and α restricts to an algebra automorphism of $gl_G(U) \cong \mathbf{k}$. We also denote the induced automorphism of $\mathbf{k}$ by α.

Lemma 3.3

1. *$S : U \to U$ is semilinear with respect to the automorphism α of $\mathbf{k}$, ie $S \circ k\mathbf{I} = \alpha(k)\mathbf{I} \circ S$.*
2. *$S^2 = k\mathbf{I}$ for some $k \in \mathbf{k} \setminus \{0\}$ with $\alpha(k) = k$.*

Proof

1. This follows immediately from the definition of α:

$$\alpha(k)\mathbf{I} \;=\; S \circ k\mathbf{I} \circ S^{-1}.$$

2. The composition S^2 lies in $GL_G(U)$ and so $S^2 = k\mathbf{I}$ for some non-zero $k \in \mathbf{k}$. Applying α to this gives

$$\alpha(k)\mathbf{I} \;=\; \alpha(S^2) \;=\; (\alpha(S))^2 \;=\; S^2 \;=\; k\mathbf{I}$$

and so $\alpha(k) = k$.

□

If $\mathbf{k} = \mathbb{R}$ then by Proposition 2.1 α must be the identity automorphism. Let $T = \phi \circ S$ where $\phi = |k|^{-\frac{1}{2}}\mathbf{I}$. Then T is $\mathbb{R}$-linear and $T^2 = \pm\mathbf{I}$ as required.

If $\mathbf{k} = \mathbb{C}$ then by Proposition 2.1 α may be either the identity automorphism or conjugation. If α is the identity then put $T = \phi \circ S$ where $\phi = k^{-\frac{1}{2}}\mathbf{I}$. Then T is $\mathbb{C}$-linear and $T^2 = \mathbf{I}$.

If $\mathbf{k} = \mathbb{C}$ and α is conjugation then $S^2 = k\mathbf{I}$ where $\overline{k} = k$ and so k is real. Put $T = \phi \circ S$ where $\phi = |k|^{-\frac{1}{2}}\mathbf{I}$. Then T is $\overline{\mathbb{C}}$-linear and $T^2 = \pm\mathbf{I}$.

If $\mathbf{k} = \mathbb{H}$ then by Proposition 2.1 there exists $q \in \mathbb{H}$ such that for all $r \in \mathbb{H}$ we have $\alpha(r) = qrq^{-1}$. Let $T' = q^{-1}\mathbf{I} \circ S$. Then for all $r \in \mathbb{H}$ we have

$$T' \circ r\mathbf{I} \;=\; q^{-1}\mathbf{I} \circ S \circ r\mathbf{I} \;=\; q^{-1}\mathbf{I} \circ \alpha(r)\mathbf{I} \circ S \;=\; r\mathbf{I} \circ q^{-1}\mathbf{I} \circ S \;=\; r\mathbf{I} \circ T'$$

and so T' is $\mathbb{H}$-linear. The composition $(T')^2$ is also $\mathbb{H}$-linear and lies in $gl_G(U) \cong \mathbb{H}$, and so must be equal to $c\mathbf{I}$ where $c \in \mathbb{R}$. Defining $T = |c|^{-\frac{1}{2}}\mathbf{I} \circ T'$ gives the result and completes the proof that statement *1* implies statement *3*.

The final part of Theorem 3.2 follows immediately from part *5* of Proposition 3.1. □

The results of Theorem 3.2 and the following Corollary are summarised in Table 1.

Corollary 3.4 *Let (U, τ) be an irreducible representation of G of type* $\mathbf{k}$*. Then*

1. *The algebra $gl_H(U)$ is isomorphic to one of the ten algebras listed in column of 3 of Table 1.*
2. *Under this isomorphism, in the σ-self-dual cases, the subspace $gl_\sigma(U)$ is generated as a module over $gl_G(U)$ by the map T shown in column 4 of Table 1.*
3. *Under the isomorphism of 1, the automorphism $\Sigma : gl_H(U) \mapsto gl_H(U)$ is isomorphic to that listed in column 7 of the Table.*

Proof
If U is not σ-self-dual then by Theorem 3.2 $gl_H(U) = gl_G(U)$. If U is σ-self-dual then $gl_H(U)$ is generated by $gl_G(U)$ and any invertible σ-reversible map $T : U \to U$. We choose T to have the properties specified in statement *3* of Theorem 3.2. For each of the seven σ-self-dual cases define an algebra homomorphism from $gl_H(U)$ to the algebra shown in column of 3 of Table 1 by mapping $gl_G(U) \cong \mathbf{k}$ to the subalgebra described in §2.1 and T to the element in column 4. It is straightforward to check that the resulting maps are algebra isomorphisms. This proves statements *1* and *2*.

For *3* note that the automorphism Σ of $gl_H(U)$ is uniquely determined by the conditions that it is the identity on $gl_G(U)$ and satisfies $\Sigma(T) = -T$. Straightforward computations show that the automorphisms listed in column 7 of Table 1 have these properties. □

Remark 3.5 Results closely related to those of Corollary 3.4 have previously been obtained by Simon [21]. In particular he describes the ways in which irreducible representations of G of types $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$ can decompose when restricted to an index two subgroup H. However Simon's approach uses ideas related to the theory of induced representations (see below) rather than the explicit descriptions of the algebras $gl_H(V)$ given here. It is these descriptions that are needed in the next subsection.

Remark 3.6 In some of the cases it is possible to choose the generator T of $gl_\sigma(U)$ to have different properties from those given in Theorem 3.2 and Table 1.

In general let (U, τ) be a σ-self-dual irreducible representation of G of type $\mathbf{k}$ and let T be a generator of $gl_\sigma(U)$. Then so is $T' = k\mathbf{I} \circ T$ for any $k \in \mathbf{k} \cong gl_G(U)$. If T is semilinear with respect to an automorphism α of $\mathbf{k}$ then T' satisfies $T'^2 = k\alpha(k)T^2$ and is semilinear with respect to the composition of α with the automorphism of $\mathbf{k}$ given by conjugation by k.

We consider the possible alternative forms for T obtained by taking k with $k\alpha(k) = -1$. For $\mathbf{k} = \mathbb{R}$ this condition implies $k = 1$ and so there are no alternatives. For $\mathbf{k} = \mathbb{C}$ we must have α equal to the identity and $k = i$. From this case it follows that if (U, τ) is a σ-self-dual representation of G of type SD-CC then the generator T satisfying $T^2 = \mathbf{I}$ can be replaced by a $\mathbb{C}$-linear generator T' satisfying $T'^2 = -\mathbf{I}$.

For $\mathbf{k} = \mathbb{H}$ the condition $k\alpha(k) = -1$ can be satisfied in a number of ways. For example, if α is the identity then k can be any element of $\mathbb{H}$ satisfying $k^2 = -1$. In particular, by taking $k = i$ in the SD-HH and SD-HC cases the $\mathbb{H}$-linear generators T satisfying $T^2 = \pm\mathbf{I}$ can be replaced by $\mathbb{H}^\sharp$-linear generators T' satisfying $T'^2 = -T^2 = \mp\mathbf{I}$.

Examples 3.7 Examples for each of the cases in Theorem 3.2 are given below.

1. Let $G = H \times \mathbb{Z}_2$. Let (U, τ_H) be an irreducible representation of H of type $\mathbf{k}$. Extend this to two non-isomorphic representations of G, denoted (U, τ^+) and (U, τ^-), by requiring $\mathbb{Z}_2$ to act trivially on U in the first case and by $-\mathbf{I}$ in the second case. Then (U, τ^+) and (U, τ^-) are irreducible representations of G of type $\mathbf{k}$. They are σ-dual to each other, and so are not σ-self-dual. They therefore provide examples of cases NSD-R, NSD-C and NSD-H in the Theorem.

2. Let (W, π) be an irreducible representation of type $\mathbf{k}$ of the compact Lie group Γ. Let $H = \Gamma \times \Gamma$ and $U = W \oplus W$. Define a representation τ_H of H on U by $\tau_H(\gamma_1, \gamma_2) = \pi(\gamma_1) \oplus \pi(\gamma_2)$. This is the direct sum of two non-isomorphic representations of H of type $\mathbf{k}$. Let G denote the semi-direct product $\mathbb{Z}_2 \ltimes (\Gamma \times \Gamma)$, where the $\mathbb{Z}_2$ action on $H = \Gamma \times \Gamma$ is generated by the map $r : (\gamma_1, \gamma_2) \mapsto (\gamma_2, \gamma_1)$. Extend the representation τ_H of H on $U = W \oplus W$ to a representation τ of G by requiring $\tau(r)(w_1, w_2) = (w_2, w_1)$. This is an irreducible representation of G of type $\mathbf{k}$, and so gives examples of SD-RR, SD-CC and SD-HH in the Theorem. Note that the case with $G = D_4$ and $H = D_2$ in Example 1.4 is of this type with $\mathbf{k} = \mathbb{R}$.

3. Let $G = D_n$, the dihedral group of order $2n$, with $n \geq 3$. Let G act on $U = \mathbb{R}^2$ as the rotational and reflectional symmetries of a regular n-gon. This is an irreducible representation of type $\mathbb{R}$. Let $H = \mathbb{Z}_n$, the index 2 cyclic subgroup of order n. Then the restriction of the representation of G on U to H is an irreducible representation of type $\mathbb{C}$, an example of SD-RC in the Theorem. The case $G = D_4$ and $H = \mathbb{Z}_4$ in Example 1.4 is the case with $n = 2$.

4. As in Example 1.3, let $G = \mathbb{Z}_4$, the cyclic group of order 4 acting on $\mathbb{R}^2$ by rotations through multiples of $\pi/2$. Let H denote subgroup isomorphic to $\mathbb{Z}_2$. Then the representation of G is irreducible of type $\mathbb{C}$ while the restriction to H is the direct sum of two non-isomorphic irreducible representations of type $\mathbb{R}$, an example for case SD-CR of the Theorem.

5. Let (U, τ_H) denote the natural representation of $H = SU(2)$ on $\mathbb{C}^2$. This is an irreducible representation of H of type $\mathbb{H}$. Let G denote the group generated by $SU(2)$ and the element of $GL(2; \mathbb{C})$ given by the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The natural representation of G on $\mathbb{C}^2$ is an irreducible representation of type $\mathbb{C}$. This therefore gives an example of SD-CH in the Theorem.

6. Finally let $G = Q_8$ denote the 'quaternionic group' of order 8, $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$. This acts naturally on $U = \mathbb{H} \cong \mathbb{R}^4$ by multiplication on the left to give an irreducible representation of type $\mathbb{H}$. The restriction of this representation to the order 4 cyclic subgroup $H = \{\pm 1, \pm i\}$ is the direct sum of two isomorphic representations of type $\mathbb{C}$, yielding an example for case SD-HC.

In the proof of Theorem 3.10 we will need a result that says that if (U_1, τ_1) and (U_2, τ_2) are two irreducible representations of G with restrictions to H which contain isomorphic irreducible representations of H, then (U_2, τ_2) is isomorphic to either (U_1, τ_1) or its σ-dual $(U_1, (\tau_1)_\sigma)$. We will deduce this as a corollary of a result on induced representations that is in turn a corollary of the classification of irreducible representations of G given by Corollary 3.4. First we recall some basic facts about induced representations

Let H be any subgroup of G and (W,π) be a representation of H. The *representation of G induced from* (W,π), denoted $I_H^G(W,\pi)$, is constructed as follows. The underlying vector space is $C_H^0(G,W)$, the vector space of all continuous maps $f : G \to W$ which satisfy $f(gh) = \pi(h)^{-1}f(g)$ for all $g \in G$ and $h \in H$. If H has finite index in G then the dimension of $C_H^0(G,W)$ is given by $\dim C_H^0(G,W) = |G/H| \dim W$. Define a linear action of G on $C_H^0(G,W)$ by $(\gamma.f)(g) = f(\gamma^{-1}g)$ for all γ and g in G. This defines $I_H^G(W,\pi)$.

Denote the restriction of a representation (U,τ) of G to H by $R_H^G(U,\tau)$. The two operators I_H^G and R_H^G are related through the *Frobenius reciprocity* relationship:

$$l_G\left(U, I_H^G W\right) \cong l_H\left(R_H^G U, W\right)$$

for any pair of representations (U,τ) of G and (W,π) of H. For a proof of this and further properties of induced representations see for example [3, 20] or [21].

Corollary 3.8 *Let H be a subgroup of G of index 2 and $\sigma : G \to \mathbb{Z}_2$ the homomorphism with kernel H.*

1. *If (U,τ) is an irreducible representation of G then*

$$I_H^G R_H^G(U,\tau) \;\cong\; (U,\tau) \oplus (U,\tau_\sigma).$$

2. *Let (U_1,τ_1) and (U_2,τ_2) be two irreducible representations of G such that*

$$l_H\left(R_H^G U_1, R_H^G U_2\right) \;\neq\; \{0\}.$$

Then (U_2,τ_2) is isomorphic to either (U_1,τ_1) or its σ-dual $(U_1,(\tau_1)_\sigma)$.

Proof Note first that $\dim I_H^G R_H^G(U,\tau) = 2\dim(U,\tau)$ and that the representations $R_H^G(U,\tau)$ and $R_H^G(U,\tau_\sigma)$ are always the same. We proceed by cases, using the classification of irreducible representations given by Corollary 3.4 and Table 1.

If (U,τ) is of type NSD-$\mathbf{k}$ then (U,τ) is not isomorphic to (U,τ_σ), but, by reciprocity,

$$\begin{array}{rcl} \dim l_G\left(I_H^G R_H^G(U,\tau),(U,\tau)\right) & = & \dim l_H\left(R_H^G(U,\tau), R_H^G(U,\tau)\right) \;>\; 0 \\ \dim l_G\left(I_H^G R_H^G(U,\tau),(U,\tau_\sigma)\right) & = & \dim l_H\left(R_H^G(U,\tau), R_H^G(U,\tau_\sigma)\right) \;>\; 0 \end{array}$$

and so both (U,τ) and (U,τ_σ) must appear in the isotypic decomposition of $I_H^G R_H^G(U,\tau)$. The result now follows from the fact that $\dim I_H^G R_H^G(U,\tau) = 2\dim(U,\tau)$.

In the remaining cases $(U,\tau) \cong (U,\tau_\sigma)$ and it is sufficient to show that if (U,τ) is of type $\mathbf{k}$ as a representation of G then

$$\dim l_G\left(I_H^G R_H^G U, U\right) \;=\; \dim l_H\left(R_H^G U, R_H^G U\right) \;=\; 2\dim \mathbf{k}$$

and so (U,τ) appears with multiplicity 2 in $I_H^G R_H^G(U,\tau)$. This follows easily in each case. For example, if (U,τ) is of type SD-$\mathbf{kk}$ then $R_H^G(U,\tau)$ is the direct sum of two non-isomorphic representations of type $\mathbf{k}$ and so $\dim l_H\left(R_H^G(U,\tau), R_H^G(U,\tau)\right) = 2\dim \mathbf{k}$, as required. The other cases are similar.

For part 2 we note that by reciprocity and part 1

$$\begin{array}{rcl} l_H\left(R_H^G(U_1,\tau_1), R_H^G(U_2,\tau_2)\right) & \cong & l_G\left(I_H^G R_H^G(U_1,\tau_1),(U_2,\tau_2)\right) \\ & \cong & l_G\left((U_1,\tau_1) \oplus (U_1,(\tau_1)_\sigma),(U_2,\tau_2)\right). \end{array}$$

The left hand side is non-trivial if and only if the restriction to H of (U_1, τ_1) has a non-trivial invariant subspace which is isomorphic as a representation of H to a non-trivial invariant subspace of the restriction to H of (U_2, τ_2). The right hand side is non-trivial if and only if (U_2, τ_2) is isomorphic to either (U_1, τ_1) or its σ-dual $(U_1, (\tau_1)_\sigma)$. This proves the result. □

3.2 σ-Reversible maps

Let $\{(U_i, \tau_i) : i = 1, \dots, k\}$ be a set of irreducible representations of G such that each isomorphism class of irreducible representations contains precisely one of the (U_i, τ_i). If G is finite then $k < \infty$, otherwise $k = \infty$. A representation (U, τ) of G is irreducible of type $\mathbf{k}$ if and only if its σ-dual (U, τ_σ) is irreducible of type $\mathbf{k}$. It follows that there is a permutation of $\{1, \dots, k\}$ of order 2, also denoted by σ, such that the σ-dual of (U_i, τ_i) is $(U_{\sigma(i)}, \tau_{\sigma(i)})$.

The isotypic decomposition of a representation (V, ρ) of G can be written as

$$V = \bigoplus_{i=1}^{k} V_i \tag{3.8}$$

where

$$V_i \cong \mathbf{k}_i^{m_i} \otimes_{\mathbf{k}_i} U_i. \tag{3.9}$$

Note that m_i may equal 0 for some i. The action of G on V is the direct sum of the actions on the V_i.

It follows from Schur's lemma (2.5) that any σ-reversible map $L : V \to V$ must map V_i to $V_{\sigma(i)}$ and, since $\sigma^2(i) = i$, vice-versa. It must therefore leave invariant the subspaces defined by

$$\widehat{V_i} \;=\; V_i + V_{\sigma(i)}. \tag{3.10}$$

If (U_i, τ_i) is σ-self-dual then $V_i = V_{\sigma(i)}$ and so $\widehat{V_i} = V_i$. If (U_i, τ_i) is not σ-self-dual then V_i and $V_{\sigma(i)}$ are two different isotypic blocks and $\widehat{V_i}$ is the direct sum $V_i \oplus V_{\sigma(i)}$. The following result, describing the action of G on each of the blocks $\widehat{V_i}$, is a straightforward consequence of the definitions. Let $(\mathbf{k}^m, 1^m)$ denote the trivial representation of G on $\mathbf{k}^m$ and $(\mathbf{k}^m, \sigma^m)$ the direct sum of m copies of the sign representation σ on $\mathbf{k}$.

Lemma 3.9 *Let (V, ρ) be a representation of G and*

$$V = \widehat{V_1} \oplus \cdots \oplus \widehat{V_\ell} \tag{3.11}$$

the decomposition defined above.

1. *If (U_i, τ_i) is not σ-self-dual and is of type $\mathbf{k}$ then $\widehat{V_i} \cong \mathbf{k}^{m+n} \otimes_{\mathbf{k}} U_i$ where $m = m_i$ and $n = m_{\sigma(i)}$. The representation of G on $\widehat{V_i}$ is given by the homomorphism $\widehat{\rho_i} = (1^m \oplus \sigma^n) \otimes_{\mathbf{k}} \tau_i$.*

2. *If (U_i, τ_i) is σ-self-dual and of type $\mathbf{k}$ then $\widehat{V_i} \cong \mathbf{k}^m \otimes_{\mathbf{k}} U_i$ where $m = m_i$. The representation of G on $\widehat{V_i}$ is given by the homomorphism $\widehat{\rho_i} = 1^m \otimes_{\mathbf{k}} \tau_i$.*

We will refer to the decomposition (3.11) of (V, ρ) as its *σ-isotypic decomposition*. Using this we can now state the main result of this paper.

Theorem 3.10 *Let (V, ρ) be a representation of G and $\sigma : G \to \mathbb{Z}_2$ a surjective homomorphism with kernel H. Let $V = \widehat{V_1} \oplus \cdots \oplus \widehat{V_\ell}$ denote the σ-isotypic decomposition of V.*

1. *The space $gl_H(V)$ of H-equivariant maps from V to itself decomposes as*

$$gl_H(V) \;\cong\; gl_H(\widehat{V_1}) \oplus \cdots \oplus gl_H(\widehat{V}_\ell). \tag{3.12}$$

The Lie algebra automorphism Σ of $gl_H(V)$ preserves this decomposition which therefore restricts to decompositions of the subalgebra $gl_G(V)$ and $gl_G(V)$-submodule $gl_\sigma(V)$:

$$gl_G(V) \;\cong\; gl_G(\widehat{V_1}) \oplus \cdots \oplus gl_G(\widehat{V}_\ell) \tag{3.13}$$
$$gl_\sigma(V) \;\cong\; gl_\sigma(\widehat{V_1}) \oplus \cdots \oplus gl_\sigma(\widehat{V}_\ell). \tag{3.14}$$

2. *The algebra $gl_H(\widehat{V}_i)$ is isomorphic to the algebra $\mathcal{A}$ in column 2 of Table 3 determined by the type of the corresponding irreducible representation (U_i, τ_i). The restriction of Σ to $gl_H(\widehat{V}_i)$ is isomorphic to the involution on $\mathcal{A}$ shown in column 3. The ± 1 eigenspaces $\mathcal{A}_\pm$ of this involution are indicated in columns 4 and 5. These are isomorphic to $gl_G(\widehat{V}_i)$ and $gl_\sigma(\widehat{V}_i)$, respectively.*

3. *If (U_i, τ_i) is not σ-self-dual and is of type $\mathbf{k}$, and so $\widehat{V}_i \cong \mathbf{k}^{m+n} \otimes_{\mathbf{k}} U_i$ with $m = m_i$ and $n = m_{\sigma(i)}$, then the isomorphism $\mathcal{A} = gl(m+n;\mathbf{k}) \to gl_H(\widehat{V}_i)$ is given explicitly by $A \mapsto A \otimes_{\mathbf{k}} \mathbf{I}$.*

4. *Let (U_i, τ_i) be σ-self-dual of type $\mathbf{k}$, and so $\widehat{V}_i \cong \mathbf{k}^m \otimes_{\mathbf{k}} U_i$ with $m = m_i$. Let $T : U_i \to U_i$ be the σ-reversible map given in column 4 of Table 1. Then the isomorphism $\mathcal{A} = \mathcal{A}_+ \oplus \mathcal{A}_- \to gl_H(\widehat{V}_i)$ is given by $A = (A_1, A_2) \mapsto (A_1 \otimes_{\mathbf{k}} \mathbf{I}) + (A_2 \otimes_{\mathbf{k}} T)$. Note that A_1 is always $\mathbf{k}$-linear while A_2 is $\mathbf{k}^\alpha$-linear if and only if T is $\mathbf{k}^\alpha$-linear, so the tensor products are well defined.*

5. *The action of $GL_G(\widehat{V}_i)$ on $gl_\sigma(\widehat{V}_i)$ is isomorphic to the natural action of $GL(m;\mathbf{k}) \times GL(n;\mathbf{k})$ on $l(m,n;\mathbf{k}) \oplus l(n,m;\mathbf{k})$ (in the* NSD *cases) or to that of $GL(m;\mathbf{k})$ on $gl(m;\mathbf{k}^\alpha)$ (in the* SD *cases).*

Theorems 1.5 and 1.7 follow immediately from this result.

Proof It follows from Lemma 3.9 and part 2 of Corollary 3.8 that the isotypic decomposition of the restriction of (V, ρ) to H is a refinement of the σ-isotypic decomposition and so (3.12) is a consequence of Schur's lemma. For any g in $G \setminus H$ the linear map $\rho(g) : V \to V$ preserves the σ-isotypic decomposition and so $\Sigma : L \mapsto \rho(g)^{-1} L \rho(g)$ preserves the decomposition (3.12). The decompositions (3.13) and (3.14) follow from the fact that $gl_G(V)$ and $gl_\sigma(V)$ are the ± 1 eigenspaces of Σ (see Lemma 3.1). This proves 1. For the remaining statements of the Theorem we proceed by cases.

NSD-$\mathbf{k}$ If (U_i, τ_i) is of type NSD-$\mathbf{k}$ then its restriction to H, (U_i, τ_i^H), is irreducible of type $\mathbf{k}$ and by part 1 of Lemma 3.9 the representation of H on $\widehat{V}_i$ is isomorphic to $1^{m+n} \otimes_{\mathbf{k}} \tau_i^H$ on $\mathbf{k}^{m+n} \otimes_{\mathbf{k}} U_i$. It follows that $gl_H(\widehat{V}_i)$ is isomorphic to $gl(m+n;\mathbf{k})$, the isomorphism being given explicitly by

$$\begin{array}{ccc} gl(m+n;\mathbf{k}) & \to & gl_H(\widehat{V}_i) \\ A & \mapsto & L = A \otimes_{\mathbf{k}} \mathbf{I}. \end{array}$$

To compute the involution Σ we note that if $g \in G \setminus H$ then its action on $\widehat{V}_i$ is given by

$$\widehat{\rho}_i(g) \;=\; R_{m,n} \otimes_{\mathbf{k}} \tau_i(g)$$

	$\mathcal{A} \cong gl_H(\widehat{V}_i)$	$\Sigma(A)$	$\mathcal{A}_+ \cong gl_G(\widehat{V}_i)$	$\mathcal{A}_- \cong gl_\sigma(\widehat{V}_i)$
NSD-R	$gl(m+n;\mathbb{R})$	$R_{m,n}^{-1}AR_{m,n}$	$gl(m;\mathbb{R}) \oplus gl(n;\mathbb{R})$	$l(m,n;\mathbb{R}) \oplus l(n,m;\mathbb{R})$
NSD-C	$gl(m+n;\mathbb{C})$	$R_{m,n}^{-1}AR_{m,n}$	$gl(m;\mathbb{C}) \oplus gl(n;\mathbb{C})$	$l(m,n;\mathbb{C}) \oplus l(n,m;\mathbb{C})$
NSD-H	$gl(m+n;\mathbb{H})$	$R_{m,n}^{-1}AR_{m,n}$	$gl(m;\mathbb{H}) \oplus gl(n;\mathbb{H})$	$l(m,n;\mathbb{H}) \oplus l(n,m;\mathbb{H})$
SD-RR	$gl(m;\mathbb{R}) \oplus gl(m;\mathbb{R})$	$(A_1,-A_2)$	$gl(m;\mathbb{R})$	$gl(m;\mathbb{R})$
SD-CC	$gl(m;\mathbb{C}) \oplus gl(m;\mathbb{C})$	$(A_1,-A_2)$	$gl(m;\mathbb{C})$	$gl(m;\mathbb{C})$
SD-HH	$gl(m;\mathbb{H}) \oplus gl(m;\mathbb{H})$	$(A_1,-A_2)$	$gl(m;\mathbb{H})$	$gl(m;\mathbb{H})$
SD-RC	$gl(m;\mathbb{C})$	$\overline{A}$	$gl(m;\mathbb{R})$	$gl(m;\mathbb{R})$
SD-CR	$gl(2m;\mathbb{R})$	$-iAi$	$gl(m;\mathbb{C})$	$gl(m;\overline{\mathbb{C}})$
SD-CH	$gl(m;\mathbb{H})$	$A^\sharp$	$gl(m;\mathbb{C})$	$gl(m;\overline{\mathbb{C}})$
SD-HC	$gl(2m;\mathbb{C})$	$-j\overline{A}j$	$gl(m;\mathbb{H})$	$gl(m;\mathbb{H})$

Table 3: The structure of the algebras $gl_H(\widehat{V}_i)$, their subalgebras $gl_G(\widehat{V}_i)$ and $gl_G(\widehat{V}_i)$-submodules $gl_\sigma(\widehat{V}_i)$ for a σ-isotypic block $\widehat{V}_i$ corresponding to an irreducible representation (U_i,τ_i) of G. The first column refers to the classification of the irreducibles (U_i,τ_i) in Table 1. The second column gives the isomorphism class of $gl_H(\widehat{V}_i)$ and the third column that of the automorphism Σ. These automorphisms are defined in §2.3. The fourth column gives the isomorphism class of $gl_G(\widehat{V}_i)$ as a subalgebra of $gl_H(\widehat{V}_i)$ and the fifth column gives the isomorphism class of the submodule $gl_\sigma(V)$ as a space of $\mathbf{k}^\alpha$-linear maps, as defined in §2.2.

where $R_{m,n} = \mathbf{I}_m \oplus -\mathbf{I}_n$, and so

$$\begin{aligned}\Sigma(L) &= \left(R_{m,n}^{-1} \otimes_{\mathbf{k}} \tau_i(g)^{-1}\right)(A \otimes_{\mathbf{k}} \mathbf{I})(R_{m,n} \otimes_{\mathbf{k}} \tau_i(g)) \\ &= R_{m,n}^{-1}AR_{m,n} \otimes_{\mathbf{k}} \mathbf{I}\end{aligned}$$

as indicated in the table. The $\pm I$ eigenspaces of this involution are described in Example 1 of Examples 2.4. It follows that $\mathcal{A}_+$ is isomorphic to $gl(m;\mathbf{k}) \oplus gl(n;\mathbf{k})$ and $\mathcal{A}_-$ is isomorphic to $l(m,n;\mathbf{k}) \oplus l(n,m;\mathbf{k})$. The group $GL_H(\widehat{V}_i)$ is isomorphic to $GL(m+n;\mathbf{k})$ and the subgroup $GL_G(\widehat{V}_i)$ to $GL(m;\mathbf{k}) \times GL(n;\mathbf{k})$. Under these isomorphisms the action of $GL_G(\widehat{V}_i)$ on $\mathcal{A}_-$ is isomorphic to the action of $GL(m;\mathbf{k}) \times GL(n;\mathbf{k})$ on $l(m,n;\mathbf{k}) \oplus l(n,m;\mathbf{k})$ described in Example 1 of Examples 2.4.

SD-**kk** If (U_i,τ_i) is of type SD-**kk** then its restriction (U_i,τ_i^H) decomposes as the direct sum of two non-isomorphic irreducible representations of type $\mathbf{k}$, say $(U_i,\tau_i^H) \cong (U_i^+,\tau_i^+) \oplus (U_i^-,\tau_i^-)$. By part 1 of Lemma 3.9 the representation of H on $\widehat{V}_i$ is therefore isomorphic to $(1^m \otimes_{\mathbf{k}} \tau_i^+) \oplus (1^m \otimes_{\mathbf{k}} \tau_i^-)$ on $(\mathbf{k}^m \otimes_{\mathbf{k}} U_i^+) \oplus (\mathbf{k}^m \otimes_{\mathbf{k}} U_i^-)$ and so we obtain an isomorphism from $gl(m;\mathbf{k}) \oplus gl(m;\mathbf{k})$ to $gl_H(\widehat{V}_i)$ given by $(A_1',A_2') \mapsto (A_1' \otimes_{\mathbf{k}} \mathbf{I}_+, A_2' \otimes_{\mathbf{k}} \mathbf{I}_-)$ where $\mathbf{I}_\pm$ is the identity map on $U_i^\pm$. Equivalently, if $A_1 = \frac{1}{2}(A_1' + A_2')$ and $A_2 = \frac{1}{2}(A_1' - A_2')$, then the isomorphism is given by $(A_1,A_2) \mapsto (A_1 \otimes_{\mathbf{k}} \mathbf{I} + A_2 \otimes_{\mathbf{k}} T)$ where $\mathbf{I}$ is the identity map on U_i and $T : U_i \to U_i$ is $\mathbf{I}_+ \oplus -\mathbf{I}_-$.

If $g \in G \setminus H$ then its action on $\widehat{V}_i$ is given by

$$\widehat{\rho}_i(g) = \mathbf{I} \otimes_{\mathbf{k}} \tau_i(g)$$

and so

$$\Sigma(L) = \left(\mathbf{I} \otimes_{\mathbf{k}} \tau_i(g)^{-1}\right)(A_1 \otimes_{\mathbf{k}} \mathbf{I}_{U_i} + A_2 \otimes_{\mathbf{k}} T)(\mathbf{I} \otimes_{\mathbf{k}} \tau_i(g))$$

$$\begin{aligned} &= A_1 \otimes_{\mathbf{k}} \mathbf{I}_{U_i} + A_2 \otimes_{\mathbf{k}} \tau_i(g)^{-1} T \tau_i(g) \\ &= A_1 \otimes_{\mathbf{k}} \mathbf{I}_{U_i} - A_2 \otimes_{\mathbf{k}} T \end{aligned}$$

since $\tau_i(g)^{-1}T\tau_i(g) = -T$ by part 3 of Corollary 3.4. The $\pm I$ eigenspaces of this involution are both isomorphic to $gl(m;\mathbf{k})$. The group $GL_H(\widehat{V}_i)$ is isomorphic to $GL(m;\mathbf{k}) \times GL(m;\mathbf{k})$ and the subgroup $GL_G(\widehat{V}_i)$ to $GL(m;\mathbf{k})$. Under these isomorphisms the actions of $GL_G(\widehat{V}_i)$ on $\mathcal{A}_\pm$ are both isomorphic to the adjoint action of $GL(m;\mathbf{k})$ on $gl(m;\mathbf{k})$.

SD-RC In this case the restriction (U_i, τ_i^H) is an irreducible representation of H of type $\mathbb{C}$. The representation of H on $\widehat{V}_i \cong \mathbb{R}^m \otimes_{\mathbb{R}} U_i \cong \mathbb{C}^m \otimes_{\mathbb{C}} U_i$ is $1^m \otimes_{\mathbb{C}} \tau_i^H$ and so the map $A \mapsto A \otimes_{\mathbb{C}} \mathbf{I}$ defines a Lie algebra isomorphism from $gl(m;\mathbb{C})$ to $gl_H(\widehat{V}_i)$. Splitting A into its real and imaginary parts, $A = A_R + iA_I$, we have

$$A \otimes_{\mathbb{C}} \mathbf{I} \;=\; (A_R \otimes_{\mathbb{R}} \mathbf{I}) + (A_I \otimes_{\mathbb{R}} T)$$

where $T : U_i \to U_i$ is scalar multiplication by i.

If $g \in G \setminus H$ then $\widehat{\rho}_i = \mathbf{I} \otimes_{\mathbb{R}} \tau_i(g)$ and so

$$\begin{aligned} \Sigma(L) &= \left(\mathbf{I} \otimes_{\mathbb{R}} \tau_i(g)^{-1}\right)(A \otimes_{\mathbb{C}} \mathbf{I})\,(\mathbf{I} \otimes_{\mathbb{R}} \tau_i(g)) \\ &= (A_R \otimes_{\mathbb{R}} \mathbf{I}) + (A_I \otimes_{\mathbb{R}} \tau_i(g)^{-1} T \tau_i(g)) \\ &= (A_R \otimes_{\mathbb{R}} \mathbf{I}) - (A_I \otimes_{\mathbb{R}} T) \\ &= (\overline{A} \otimes_{\mathbb{C}} \mathbf{I}) \end{aligned}$$

again by part 3 of Corollary 3.4. Thus Σ is isomorphic to complex conjugation on $gl(m;\mathbb{C})$. As described in Example 2 of Examples 2.4, both $\mathcal{A}_+$ and $\mathcal{A}_-$ are isomorphic to $gl(m;\mathbb{R})$. The group $GL_H(\widehat{V}_i)$ is isomorphic to $GL(m;\mathbb{C})$ and its subgroup $GL_G(\widehat{V}_i)$ is isomorphic to $GL(m;\mathbb{R})$. The actions of $GL_G(\widehat{V}_i)$ on $\mathcal{A}_+$ and $\mathcal{A}_-$ are both isomorphic to the adjoint action of $GL(m;\mathbb{R})$ on $gl(m;\mathbb{R})$.

SD-CR In this case the restriction (U_i, τ_i^H) decomposes as the direct sum of two copies of an irreducible representation (U_i', τ_i') of H of type $\mathbf{k} = \mathbb{R}$. The representation of H on $\widehat{V}_i \cong \mathbb{C}^m \otimes_{\mathbb{C}} U_i \cong \mathbb{R}^{2m} \otimes_{\mathbb{R}} U_i'$ is $1^{2m} \otimes \tau_i'$ and so $gl_H(\widehat{V}_i)$ is isomorphic to $gl(2m;\mathbb{R})$. Let T denote the map given by complex conjugation on U_i, or equivalently the map $\mathbf{I}' \oplus -\mathbf{I}'$ on $U_i' \oplus U_i'$, where $\mathbf{I}'$ is the identity map on U_i'. Using the isomorphism $gl(2m;\mathbb{R}) \cong gl(m;\mathbb{C}) \oplus gl(m;\overline{\mathbb{C}})$ described in Example 3 of Examples 2.4 we write the elements of $gl_H(\widehat{V}_i)$ as $(A_1 \otimes_{\mathbb{C}} \mathbf{I}) + (A_2 \otimes_{\mathbb{C}} T)$ where $A_1 \in gl(m;\mathbb{C})$ and $A_2 \in gl(m;\overline{\mathbb{C}})$. As in the other self-dual examples the involution Σ is given by

$$\Sigma(L) \;=\; (A_1 \otimes_{\mathbb{C}} \mathbf{I}) - (A_2 \otimes_{\mathbb{C}} T)$$

and so is isomorphic to the involution $A \mapsto -iAi$ described in Example 3 of Examples 2.4. It follows that $\mathcal{A}_+$ is isomorphic to $gl(m;\mathbb{C})$ and $\mathcal{A}_-$ to $gl(m;\overline{\mathbb{C}})$. The group $GL_H(\widehat{V}_i)$ is isomorphic to $GL(2m;\mathbb{R})$ and its subgroup $GL_G(\widehat{V}_i)$ is isomorphic to $GL(m;\mathbb{C})$. The action of $GL_G(\widehat{V}_i)$ on $\mathcal{A}_+$ is the adjoint action of $GL(m;\mathbb{C})$ while that on $\mathcal{A}_-$ is isomorphic to the action of $GL(m;\mathbb{C})$ on $gl(m;\overline{\mathbb{C}})$ described in Example 3 of Examples 2.4.

SD-CH This is similar to case SD-RC. However the restriction (U_i, τ_i^H) is now an irreducible representation of H of type $\mathbb{H}$ and we write $\widehat{V}_i \cong \mathbb{C}^m \otimes_{\mathbb{C}} U_i \cong \mathbb{H}^m \otimes_{\mathbb{H}} U_i$. The map $A \mapsto A \otimes_{\mathbb{H}} \mathbf{I}$ defines a Lie algebra isomorphism from $gl(m;\mathbb{H})$ to $gl_H(\widehat{V}_i)$. As in Example 4 of Examples 2.4 we write $\mathbb{H}^m = \mathbb{C}^m \otimes_{\mathbb{C}} \mathbb{H}$ and split A into $A = A_1 + jA_2$, where $A_1 \in gl(m;\mathbb{C})$ and $A_2 \in gl(m;\overline{\mathbb{C}})$. Then

$$A \otimes_{\mathbb{H}} \mathbf{I} \;=\; (A_1 \otimes_{\mathbb{C}} \mathbf{I}) + (A_2 \otimes_{\mathbb{C}} T)$$

where $T : U_i \to U_i$ is scalar multiplication on the left by j. It follows as for SD-RC that

$$\Sigma(L) \;=\; (A_1 \otimes_{\mathbb{C}} \mathbf{I}) - (A_2 \otimes_{\mathbb{C}} T) \;=\; (A^\sharp \otimes_{\mathbb{H}} \mathbf{I})$$

and so Σ is isomorphic to quaternionic conjugation on $gl(m;\mathbb{H})$. The discussion in Example 4 of Examples 2.4 shows that $\mathcal{A}_+$ is isomorphic to $gl(m;\mathbb{C})$ and $\mathcal{A}_-$ to $gl(m;\overline{\mathbb{C}})$. The group $GL_H(\widehat{V}_i)$ is isomorphic to $GL(m;\mathbb{H})$ and its subgroup $GL_G(\widehat{V}_i)$ is isomorphic to $GL(m;\mathbb{C})$. The action of $GL_G(\widehat{V}_i)$ on $\mathcal{A}_+$ is the adjoint action of $GL(m;\mathbb{C})$ while that on $\mathcal{A}_-$ is isomorphic to the natural action of $GL(m;\mathbb{C})$ on $gl(m;\overline{\mathbb{C}})$.

SD-HC This is similar to case SD-CR, except that here the restriction (U_i, τ_i^H) decomposes as the direct sum of two copies of an irreducible representation (U_i', τ_i') of H of type $\mathbf{k} = \mathbb{C}$ and so $gl_H(\widehat{V}_i)$ is isomorphic to $gl(2m;\mathbb{C})$. As before let T denote the map $\mathbf{I}' \oplus -\mathbf{I}'$ on $U_i' \oplus U_i'$. Using the isomorphism $gl(2m;\mathbb{C}) \cong gl(m;\mathbb{H}) \oplus gl(m;\mathbb{H})$ described in Example 5 of Examples 2.4 we write the elements of $gl_H(\widehat{V}_i)$ as $(A_1 \otimes_{\mathbb{H}} \mathbf{I}) + (A_2 \otimes_{\mathbb{H}} T)$ where $A_1 \in gl(m;\mathbb{H})$ and $A_2 \in gl(m;\mathbb{H})$. As above the involution Σ is given by

$$\Sigma(L) \;=\; (A_1 \otimes_{\mathbb{H}} \mathbf{I}) - (A_2 \otimes_{\mathbb{C}} T)$$

and so is isomorphic to the involution $A \mapsto -j\overline{A}j$. The spaces $\mathcal{A}_\pm$ are both isomorphic to $gl(m;\mathbb{H})$. The group $GL_H(\widehat{V}_i)$ is isomorphic to $GL(2m;\mathbb{C})$ and its subgroup $GL_G(\widehat{V}_i)$ is isomorphic to $GL(m;\mathbb{H})$. The actions of $GL_G(\widehat{V}_i)$ on $\mathcal{A}_+$ and $\mathcal{A}_-$ are both isomorphic to the adjoint action of $GL(m;\mathbb{H})$ on $gl(m;\mathbb{H})$.

□

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