Steady-state bifurcations in reversible equivariant systems

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Abstract. We present results on generic "separable" bifurcations of equilibria of equivariant dynamical systems with finite symmetry groups and time-reversing symmetries. These bifurcation problems are reduced to standard equivariant bifurcation problems using equivariant transversality theory. For each symmetry-breaking subgroup the dimension of the bifurcating family of equilibria with that symmetry is computed in terms of the dimension of the corresponding family of equilibria of the reduced equivariant bifurcation problem and a group theoretically defined "signature". Full details and further results will be published elsewhere.

1. Introduction

Let V be a vector space, G a finite group and ρ a linear representation of G on V. Consider also a representation $\chi : G \to \{\pm 1\}$ and define $\rho_\chi = \chi\rho$. We say that a smooth mapping $f : V \to V$ is *G-reversible-equivariant* if

$$f(\rho(g)x) \;=\; \rho_\chi(g)f(x) \;=\; \chi(g)\rho(g)f(x) \qquad \forall g \in G. \tag{1}$$

Note that this implies that f is H-equivariant in the usual sense for $H = \ker\, \chi$.

We consider smooth families of G-reversible-equivariant mappings parametrised by $\lambda \in \mathbf{R}^r$, our aim being to describe the generic structure of the set of solutions of the bifurcation problem $f(x;\lambda) = 0$ in a neighbourhood of a solution (x_0, λ_0) with $x_0 \in \mathrm{Fix}_V(G)$. Without loss of generality we may assume that $x_0 = 0$ and $\lambda_0 = 0$. In §1.2 we show that if V is a "self-dual" representation (defined in §1.1) then the solution space is identical to that of a standard G-equivariant bifurcation problem on V. In §2 we consider general V and show that if f has a "separable" bifurcation at $x_0 = \lambda_0 = 0$ then the problem can be reduced to a G-reversible equivariant bifurcation problem on a self-dual subspace of V, and hence to a standard G-equivariant bifurcation problem.

1.1. Representation theory

We begin by recalling and developing some representation theory from Lamb and Roberts [8]. For a fixed representation $\chi : G \to \{\pm 1\}$ the representation ρ_χ is called the *dual* of ρ. The represention ρ is said to be *self-dual* if it is isomorphic to its dual. Thus ρ is self-dual if and only if there exists an invertible G-reversible equivariant linear map $T : V \to V$.

We say that two representations of G are *orthogonal* if their respective decompositions as direct sums of irreducible representations do not contain any isomorphic irreducible summands. We define a representation to be *dual-orthogonal* if it is orthogonal to its dual. Every representation can be expressed uniquely, up to isomorphism of representations, as a direct sum $V = W_1 \oplus W_2$ where W_1 is self-dual and W_2 dual-orthogonal. The subspaces W_1 are maximal self-dual subspaces of V in the sense that they are not contained in any strictly larger self-dual subspaces.

If V is not self-dual then every G-reversible-linear map from V to itself has a nontrivial kernel which is G-invariant and contains a dual-orthogonal subspace isomorphic to W_2. If the linear map has maximal possible rank for G-reversible-linear maps from V to itself then the kernel is precisely isomorphic to W_2. We therefore refer (rather loosely) to W_2 as being the *forced kernel* of these maps.

1.2. Bifurcation from self-dual representations

The following result shows that we can apply equivariant bifurcation theory [4, 5] directly to the study of equilibria of reversible-equivariant systems on self-dual spaces.

Theorem 1.1 (Self-Dual Case)**.** *Suppose G acts by a self-dual representation on V and let $T : V \to V$ be an invertible linear G-reversible-equivariant map and f a G-reversible-equivariant bifurcation problem. Then,*

$$(i) \quad \tilde{f} = Tf \text{ is } G\text{-equivariant.} \qquad (ii) \quad \tilde{f}(u,\lambda) = 0 \quad \Leftrightarrow \quad f(u,\lambda) = 0.$$

Proof. (i) $\tilde{f}(gx) = T\chi(g)gf(x) = \chi(g)Tgf(x) = \chi(g)^2 gTf(x) = g\tilde{f}(x)$. (ii) Follows from invertibility of T. □

Note that this is more than just a local bifurcation result since the zero set of f is identical to that of $\tilde{f}$.

2. Bifurcation from non self-dual representations

When V is not self-dual two types of bifurcation can occur from an equilibrium, depending on the representation of G on the kernel of the linearisation.

Definition 2.1. *The bifurcation problem $f(x,\lambda) = 0$ has a* separable bifurcation *at $(0,0)$ if the kernel of its linearisation at $(0,0)$ can be decomposed as $U \oplus W_2$ where W_2 is the forced kernel and U is orthogonal to W_2.*

This definition generalises that of "null-separable bifurcations" in [6, 7]. We only consider separable bifurcations in this paper. Note that this includes the case when the kernel of the linearisation is equal to the forced kernel and so is as small as possible.

The differences between separable and nonseparable bifurcations are illustrated by the following simple example.

Example 2.2. *Let $V = \mathbf{R}^3$ and $G = \mathbf{Z}_2(R)$ where $R.(x, y, z) = (x, y, -z)$. Thus* $\mathrm{Fix}(R) = \{z = 0\}$ *and* $\mathrm{Fix}(-R) = \{x = y = 0\}$. *Let $L = (df)_{(0,0)}$. The forced kernel is one-dimensional and lies in* $\mathrm{Fix}(R)$. *The symmetries imply that L has eigenvalues $\mu, -\mu, 0$. Note that $f : \mathrm{Fix}(R) \to \mathrm{Fix}(-R)$ and so if $f(0) = 0$ there is typically a* $\dim \mathrm{Fix}(R) - \dim \mathrm{Fix}(-R) = 1$ *dimensional curve of solutions passing through 0 If $\mu \neq 0$ then the implicit function theorem guarantees that this curve is smooth at 0 and that the zero set of f near 0 consists of just this curve.*

If $\mu = 0$ then generically L is a Jordan block of rank two and so no bifurcation occurs. However in a one-parameter family of maps, for isolated values of the parameter, the rank of L drops by one and a bifurcation direction is added to the forced kernel. If the additional direction is transverse to $\mathrm{Fix}(R)$ *then the bifurcation is separable. The zero set near the origin in* $\mathrm{Fix}(R)$ *continues to be a smooth curve and a pair of equilibria with trivial symmetry bifurcates from* $\mathrm{Fix}(R)$ *in a pitchfork bifurcation. If the bifurcation direction is contained in* $\mathrm{Fix}(R)$ *then the bifurcation is nonseparable. In this case no symmetry-breaking occurs but the structure of the zero set near the origin in* $\mathrm{Fix}(R)$ *can bifurcate.*

In this example the fact that $\dim \mathrm{Fix}(R) - \dim \mathrm{Fix}(-R) = 1$ implies that the family of solutions with symmetry group $\mathbf{Z}_2(R)$ is typically one-dimensional. We call this the "signature" of $\mathbf{Z}_2(R)$ on V. The following example illustrates what happens if the signature is negative.

Example 2.3. *As in example 2.2 consider $V = \mathbf{R}^3$ and $G = \mathbf{Z}_2(R)$, but this time with $R.(x, y, z) = (x, -y, -z)$. Then an equilibrium at 0 is nongeneric since f maps the one-dimensional space* $\mathrm{Fix}(R)$ *to the two-dimensional space* $\mathrm{Fix}(-R)$ *and so* $\dim \mathrm{Fix}(R) - \dim \mathrm{Fix}(-R) = -1$ *is negative. However generic one-parameter families of maps typically have persistent isolated equilibria.*

In general the signature of an isotropy subgroup of a representation is defined as follows.

Definition 2.4. *Let V be a representation of G and V_χ its dual. Let $\Sigma \subset G$ be an isotropy subgroup of the action of G on V. Then*

$$s(\Sigma, V) = \dim \mathrm{Fix}_V(\Sigma) - \dim \mathrm{Fix}_{V_\chi}(\Sigma)$$

is the signature *of Σ. We write $s(\Sigma)$ when there is no ambiguity about the space.*

Note that the signature of an isotropy subgroup of a self-dual representation is always 0.

2.1. Reduction from reversible-equivariant to equivariant problem

Suppose that V is not self-dual and f has a separable bifurcation at $(0, 0)$. In this case, the strategy is to split the problem into a self-dual part and a complementary non self-dual part. In [2] we show that this strategy leads to a reduction of the reversible-equivariant bifurcation problem to an equivariant bifurcation problem. Here we sketch the main argument.

As described in §1.1 we can decompose a non self-dual representation as $V = W_1 \oplus W_2$ where W_1 is a maximal self-dual subspace and W_2 is dual-orthogonal. The decomposition can be chosen so that $W_2 \subset \ker L$ where $L = (df)_{(0,0)}$. We then split the map f into $f_i : V \to W_i$ with $i = 1, 2$. Jordan normal form theory shows that if f has a separable bifurcation, then the linearization has a block diagonal decomposition $L = (df_1)_{(0,0)} \oplus [0]$. Since W_1 is a maximal self-dual component of V there exists an invertible linear G-reversible-equivariant map $T : W_1 \to W_1$. We will define a G-equivariant map $\tilde{f}_1 : W_1 \to W_1$ by

$$\tilde{f}_1(w_1) = T f_1(w_1, \Phi_\Upsilon(w_1)),$$

where $\Phi_\Upsilon : W_1 \to W_2$ is a G-equivariant map that depends on some parameters Υ and is typically defined only on some subspace of W_1. In order for solutions of $\tilde{f}_1 = 0$ to be solutions of $f = 0$, the map Φ_Υ is defined to satisfy $f_2(x_1, \Phi_\Upsilon(x_1)) = 0$.

Example 2.5. *Let $G = \mathbf{D}_2(\sigma, R)$ act on $V = \mathbf{R}^3$ by $R.(x, y, z) = (x, y, -z)$ and $\sigma.(x, y, z) = (x, -y, -z)$. Let $H = \mathbf{Z}_2(\sigma)$. The general $\mathbf{D}_2(\sigma, R)$-reversible equivariant bifurcation problem f is given by*

$$p(x, y^2, z^2)yz = 0 \quad q(x, y^2, z^2)z = 0 \quad r(x, y^2, z^2)y = 0 \tag{2}$$

where the dependence of p, q and r on parameters has been suppressed from the notation. Note that the signature of the full group is 1 and the line $\{y = z = 0\}$ always lies in the zero set of f. Let $W_1 = \{x = 0\}$ and $W_2 = \{y = z = 0\}$ and set $f_1(x, y, z) = (q(x, y^2, z^2)z, r(x, y^2, z^2)y)$ and $f_2(x, y, z) = p(x, y^2, z^2)yz$. The map $T : W_1 \to W_1$ interchanges the y and z axes. The linearization at a point $(x, 0, 0)$ is

$$(df)_x = 0 \oplus \begin{bmatrix} 0 & q(x,0,0) \\ r(x,0,0) & 0 \end{bmatrix}.$$

The rank of $(df)_x$ will typically drop by one at isolated points along this line, even in the absence of any external parameters. Assume that it does so at $x = 0$. This is then necessarily a separable bifurcation point.

Generically, $p(0, 0, 0) \neq 0$, so the solutions of $f_2 = 0$ near the origin are those with isotropy subgroup $\mathbf{Z}_2(R)$ at $y = 0, z \neq 0$ and those with isotropy $\mathbf{Z}_2(\sigma R)$ at $y \neq 0, z = 0$. By Theorem 1.1, $\tilde{f}_1(y, z; x) = T f_1(x, y, z)$ is $\mathbf{D}_2$-equivariant. The coordinate x can be taken to be a parameter in $\tilde{f}_1$. Substituting the solutions of $f_2 = 0$ in $\tilde{f}_1 = 0$ yields $\tilde{f}_1(0, z; x) = (0, q(x, 0, z^2)z) = 0$ and $\tilde{f}_1(y, 0; x) = (r(x, y^2, 0)y, 0) = 0$. If $(df)_x$ drops rank by one at 0 then either $r(0, 0, 0) = 0$ or $q(0, 0, 0) = 0$. We therefore see a pitchfork bifurcation in phase space parametrized by x with isotropy group equal to $\mathbf{Z}_2(R)$ or $\mathbf{Z}_2(\sigma R)$, respectively. Note that the signatures of both these isotropy subgroups are equal to one and that the bifurcating families typically appear as one-parameter families of equilibria inside the phase space of the dynamical system, without the need to take into account external parameters. We call such bifurcating families "branches" and the associated bifurcation "branching".

In this example the symmetries automatically yielded solutions to the equation $f_2 = 0$. However in general we need to assume that f_2 is G-transversal to 0 at 0, denoted $f_2 \pitchfork_G 0$ at 0. See [1, 3] for the definition of G-transversality. We show in [2] that, for each isotropy subgroup Σ with $s(\Sigma) \geq 0$, the G-transversality of f_2 implies that the zero set with isotropy Σ is generically a manifold with parametrization $\psi_\Sigma(x_1, \mu)$ where $x_1 \in \mathrm{Fix}_{W_1}(\Sigma)$, and $\mu \in \mathrm{Fix}_{W_2}(\Sigma)$ is treated as a $s(\Sigma)$-dimensional vector of parameters. We define the family of maps $\phi_{\Sigma,\mu}$: $\mathrm{Fix}_{W_1}(\Sigma) \to \mathrm{Fix}_{W_2}(\Sigma)$ parametrized by μ by $\phi_{\Sigma,\mu}(x_1) = (\psi_\Sigma(x_1, \mu), \mu)$. Substituting this into $\tilde{f}_1$ gives the equation:

$$\tilde{f}_1(x_1, (\phi_{\Sigma,\mu}(x_1), \mu), \lambda) = 0.$$

It follows that if solutions with isotropy subgroups containing Σ occur in generic l-parameter equivariant bifurcation problems on W_1 then they occur in generic $l - s(\Sigma)$ parameter reversible-equivariant bifurcation problems on V.

This approach can be extended to isotropy subgroups with negative signature by replacing V by $V \times \mathbf{R}^r$ where $\mathbf{R}^r$ is a space of parameters. This effectively adds r to the signatures of the isotropy subgroups and so can be used to make them positive. The following theorem summarises the results and applies to both positive and negative signatures.

Theorem 2.6 (Separable Bifurcation Theorem)**.** *Let $W_1 \subset V$ be a maximal self-dual subspace, Σ an isotropy subgroup of V and $s(\Sigma)$ its signature. If families of equilibria with isotropy subgroups containing Σ generically appear in l-parameter equivariant bifurcation problems on W_1 then $k + s(\Sigma)$-dimensional families of equilibria with isotropy subgroups equal to Σ generically appear in k-parameter reversible-equivariant bifurcation problems on V, provided $k + s(\Sigma) \geq l$.*

The following corollary is obtained by noting that the Equivariant Branching Lemma [5] implies that for a particular class of isotropy subgroups we can take $l = 1$ in Theorem 2.6.

Corollary 2.7. *Let Σ be an isotropy subgroup of V for which there is an absolutely irreducible representation U in W_1 such that* $\dim \mathrm{Fix}_U(\Sigma) = 1$. *Then equilibria with isotropy subgroups Σ occur in $k + s(\Sigma, V)$-dimensional families in generic k-parameter reversible-equivariant bifurcation problems on V, provided $k \geq 1 - s(\Sigma)$.*

In particular this can be applied to equilibria with isotropy subgroups $\mathbf{Z}_2(R)$ in Examples 2.2 and 2.3 and isotropy subgroups $\mathbf{Z}_2(R)$ and $\mathbf{Z}_2(\sigma R)$ in Example 2.5.

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