MATHEMATICAL USES OF GAUGE THEORY

S. K. Donaldson

Imperial College, London

1. Introduction

1.1.

This article surveys some developments in pure mathematics which have, to varying degrees, grown out of the ideas of gauge theory in Mathematical Physics. The realisation that the gauge fields of particle physics and the connections of differential geometry are one and the same has had wide-ranging consequences, at different levels. Most directly, it has lead mathematicians to work on new kinds of questions, often shedding light later on well-established problems. Less directly, various fundamental ideas and techniques, notably the need to work with the infinite-dimensional gauge symmetry group, have found a place in the general world-view of many mathematicians, influencing developments in other fields. Still less direct, the work in this area—between geometry and mathematical physics—has been a prime example of the interaction between these fields which has been so fruitful over the past thirty years.

The body of this paper is divided into three sections: roughly corresponding to *Analysis*, *Geometry* and *Topology*. However the different topics come together in many different ways: indeed the existence of these links between the topics is one the most attractive features of the area.

1.2 Gauge transformations.

We do not have space in this article to review the usual foundational material on connections, curvature and related differential geometric constructions: for these we refer to standard texts. We will however briefly recall the notions of gauge transformations and gauge fixing. The simplest case is that of abelian gauge theory—connections on a $U(1)$-bundle, say over $\mathbf{R}^3$. In that case the connection form, representing the connection in a local trivialisation, is a pure imaginary 1-form A, which can also be identified with a vector field $\mathbf{A}$. The curvature of the connection is the 2-form dA. Changing the local trivialisation by a $U(1)$-valued function $g = e^{i\chi}$ changes the connection form to

$$\tilde{A} = A - dgg^{-1} = A - id\chi.$$

The forms $A, \tilde{A}$ are two representations of the same geometric object: just as the same metric can be represented by different expressions in different co-ordinate

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systems. One may want to fix this choice of representation, usually by choosing A to satisfy the Coulomb gauge condition $d^*A = 0$ (equivalently div $\mathbf{A} = 0$), supplemented by appropriate boundary conditions. Here we are using the standard Euclidean metric on $\mathbf{R}^3$. (Throughout this article we will work with positive definite metrics, regardless of the fact that—at least at the classical level—the Lorentzian signature may have more obvious bearing on physics.) Arranging this choice of gauge involves solving a linear PDE for χ.

The case of a general structure group G is not much different. The connection form A now takes values in the Lie algebra of G and the curvature is given by the expression

$$F = dA + \frac{1}{2}[A, A].$$

The change of bundle trivialisation is given by a G-valued function and the resulting change in the connection form is

$$\tilde{A} = gAg^{-1} - dgg^{-1}.$$

(Our notation here assumes that G is a matrix group, but this is not important.) Again we can seek to impose the Coulomb gauge condition $d^*A = 0$, but now we cannot linearise this equation as before.

We can carry the same ideas over to a global problem, working on a G-bundle P over a general Riemannian manifold M. The space of connections on P is an affine space $\mathcal{A}$: any two connections differ by a bundle-valued 1-form. Now the gauge group $\mathcal{G}$ of automorphisms of P acts on $\mathcal{A}$ and, again, two connections in the same orbit of this action represent essentially the same geometric object. Thus in a sense we would really like to work on the quotient space $\mathcal{A}/\mathcal{G}$. Working locally *in the space of connections*, near to some A_0, this is quite straightforward. We represent nearby connections as A_0+a where a satisfies the analogue of the coulomb condition

$$d_A^* a = 0.$$

Under suitable hypotheses, this condition picks out a unique representative of each nearby orbit. However this gauge fixing condition need not single out a unique representative if we are far away from A_0: indeed the space $\mathcal{A}/\mathcal{G}$ typically has, unlike $\mathcal{A}$, a complicated topology which means that it is impossible to find any such global gauge fixing condition. As we have said, this is one of the distinctive features of gauge theory. The gauge group $\mathcal{G}$ is an infinite-dimensional group, but one of a comparatively straightforward kind—much less complicated than the diffeomorphism groups relevant in Riemannian geometry for example. One could argue that one of the most important influences of gauge theory has been to accustom mathematicians to working with infinite-dimensional symmetry groups in a comparatively simple setting.

2. Analysis and variational methods

2.1 The Yang-Mills functional. A primary object brought to mathematicians attention by physics is the Yang-Mills functional

$$YM(A) = \int_M |F_A|^2 d\mu.$$

Clearly $YM(A)$ is non-negative and vanishes if and only if the connection is flat: it is broadly analogous to functionals such as the area functional in minimal submanifold theory, or the energy functional for maps. As such one can fit into a general framework associated to such functionals. The Euler-Lagrange equations are the *Yang-Mills equations*

$$d_A^* F_A = 0.$$

For any solution (a Yang-Mills connection) there is a "Jacobi operator " H_A such that the second variation is given by

$$YM(A + ta) = YM(A) + t^2 \langle H_A a, a\rangle + O(t^3).$$

The omnipresent phenomenon of gauge invariance means that Yang-Mills connections are never isolated, since we can always generate an infinite-dimensional family by gauge transformations. Thus, as explained in (1.2), one imposes the gauge fixing condition $d_A^* a = 0$. Then the operator H_A can be written as

$$H_A a = \Delta_A a + [F_A, a],$$

where Δ_A is the bundle-valued "Hodge Laplacian" $d_A d_A^* + d_A^* d_A$ and the expression $[F_A, a]$ combines the bracket in the Lie algebra with the action of Λ^2 on Λ^1. This is a self-adjoint elliptic operator and, if M is compact, the span of the negative eigenspaces is finite dimensional, the dimension being defined to be the *index* of the Yang-Mills connection A.

In this general setting, a natural aspiration is to construct a "Morse Theory" for the functional. Such a theory should relate the topology of the ambient space to the critical points and their indices. In the simplest case, one could hope to show that for any bundle P there is a Yang-Mills connection with index 0, giving a minimum of the functional. More generally, the relevant ambient space here is the quotient $\mathcal{A}/\mathcal{G}$ and one might hope that the rich topology of this is reflected in the solutions to the Yang-Mills equations.

2.2 Uhlenbeck's Theorem.

The essential foundation needed to underpin such a "Direct Method" in the calculus of the Calculus of Variations is an appropriate compactness theorem. Here the dimension of the base manifold M enters in a crucial way. Very roughly, when a connection is represented locally in a Coulomb gauge the Yang-Mills action combines the L^2 norm of the derivative of the connection form A with the L^2-norm of the quadratic term $[A, A]$. The latter can be estimated by the L^4-norm of A. If dim $M \leq 4$ then the Sobolev inequalities allow the L^4 norm of A to be controlled by the L^2 norm of its derivative, but this is definitely not true in higher dimensions. Thus dim $M = 4$ is the "critical dimension" for this variational problem. This is related to the fact that the Yang-Mills equations (and Yang-Mills functional) are conformally invariant in 4-dimensions. For any non-trivial Yang-Mills connection over the 4-sphere one generates a 1-parameter family of Yang-Mills connections, on which the functional takes the same value, by applying conformal transformations corresponding to dilations of $\mathbf{R}^4$. In such a family of connections the integrand $|F_A|^2$—the "curvature density"—converges to a delta-function at the origin. More generally, one can encounter sequences of connections over 4-manifolds for which YM is bounded but which do not converge, the Yang-Mills density converging to delta functions. There is a detailed analogy with the theory of the harmonic maps energy functional, where the relevant critical dimension (for the domain of the map) is 2.

The result of K. Uhlenbeck [65], which makes these ideas precise, considers connections over a ball $B^n \subset \mathbf{R}^n$. If the exponent $p \geq 2n$ then there are positive constants $\epsilon(p,n), C(p,n) > 0$ such that any connection with $\|F\|_{L^p(B^n)} \leq \epsilon$ can be represented in Coulomb gauge over the ball, by a connection form which satisfies $d^*A = 0$, together with certain boundary conditions, and

$$\|A\|_{L^p_1} \leq C\|F\|_{L^p}.$$

In this Coulomb gauge the Yang-Mills equations are elliptic and it follows readily that, in this setting, if the connection A is Yang-Mills one can obtain estimates on all derivatives of A.

2.3 Instantons in four dimensions.

This result of Uhlenbeck gives the analytical basis for the direct method of the calculus of variations for the Yang-Mills functional over base manifolds M of dimension ≤ 3. For example, any bundle over such a manifold must admit a Yang-Mills conection, minimising the functional. Such a statement is definitely false in dimensions ≥ 5. For example, an early result of Bourguignon, Lawson and Simons [12] asserts that there is no minimising connection on any bundle over S^n for $n \geq 5$. The proof exploits the action of the conformal transformations of the sphere. In the critical dimension 4, the situation is much more complicated. In four dimensions there are the renowned "instanton" solutions of the Yang-Mills equation. Recall that if M is an oriented 4-manifold the Hodge $*$-operation is an involution of $\Lambda^2 T^*M$ which decomposes the two forms into self-dual and anti-self dual parts, $\Lambda^2 T^*M = \Lambda^+ \oplus \Lambda^-$. The curvature of a connection can then be written as

$$F_A = F_A^+ + F_A^-,$$

and a connection is a self-dual (respectively anti-self-dual) *instanton* if F_A^- (respectively F_A^+) is 0. The Yang-Mills functional is

$$YM(A) = \|F_A^+\|^2 + \|F_A^-\|^2,$$

while the difference $\|F_A^+\|^2 - \|F_A^-\|^2$ is a *topological invariant* $\kappa(P)$ of the bundle P, obtained by evaluating a 4-dimensional characteristic class on $[M]$. Depending on the sign of $\kappa(P)$, the self-dual or anti-self-dual connections (if any exist) minimise the Yang-Mills functional among all connections on P. These instanton solutions of the Yang-Mills equations are analogous to the holomorphic maps from a Riemann surface to a Kahler manifold, which minimise the harmonic maps energy functional in their homotopy class.

2.4 Moduli spaces.

The instanton solutions typically occur in *moduli spaces*. To fix ideas let us consider bundles with structure group $SU(2)$ in which case $\kappa(P) = -8\pi^2 c_2(P)$. For each $k > 0$ we have a moduli space $\mathcal{M}_k$ of anti-self-dual instantons on a bundle $P_k \to M^4$, with $c_2(P_k) = k$. It is a manifold of dimension $8k - 3$. The general goal of the calculus of variations in this setting is to relate three things:

(1) The topology of the space $\mathcal{A}/\mathcal{G}$ of equivalence classes of connections on P_k;

(2) The topology of the moduli space $\mathcal{M}_k$ of instantons;
(3) The existence and indices of other, non-minimal, solutions to the Yang-Mills equations on P_k.

In this direction, a very influential conjecture was made by Atiyah and Jones [9]. They considered the case when $M = S^4$ and, to avoid certain technicalities, work with spaces of "framed" connections, dividing by the restricted group $\mathcal{G}_0$ of gauge transformations equal to the identity at infinity. Then for any k the quotient $\mathcal{A}/CalG_0$ is homotopy equivalent to the third loop space $\Omega^3 S^3$ of based maps from the 3-sphere to itself. The corresponding "framed" moduli space $\tilde{\mathcal{M}}_k$ is a manifold of dimension $8k$ (a bundle over $\mathcal{M}_k$ with fibre $SO(3)$). Atiyah and Jones conjectured that the inclusion $\tilde{\mathcal{M}}_k \to \mathcal{A}/\mathcal{G}_0$ induces an isomorphism of homotopy groups π_l in a range of dimensions $l \leq l(k)$ where $l(k)$ increases with k. This would be consistent with what one might hope to prove by the calculus of variations if there were no other Yang-Mills solutions, or if the indices of such solutions increased with k.

The first result along these lines was due to Bourguignon and Lawson [12], who showed that the instanton solutions are the only local minima of the Yang-Mills functional over the 4-sphere. Subsequently, Taubes [58] showed that the index of an non-instanton Yang-Mills connection P_k is at least $k+1$. Taubes' proof used ideas related to the action of the quaternions and the hyperkahler structure on the $\tilde{\mathcal{M}}_k$, see (3.6). Contrary to some expectations, it was shown by Sibner, Sibner and Uhlenbeck [56] that non-minimal solutions do exist; some later constructions were very explicit [55]. Taubes' index bound gave ground for hope that an analytical proof of the Atiyah-Jones conjecture might be possible, but this is not at all straightforward. The problem is that in the critical dimension 4 a mini-max sequence for the Yang-Mills functional in a given homotopy class may diverge, with curvature densities converging to sums of delta functions as outlined above. This is related to the fact that the $\tilde{\mathcal{M}}_k$ are not compact. In a series of papers culminating in [60], Taubes succeeded in proving a partial version of the Atiyah-Jones conjecture, together with similar results for general base manifolds M^4. Taubes showed that if the homotopy groups of the moduli spaces stabilise as $k \to \infty$ then the limit must be that predicted by Atiyah and Jones. Related analytical techniques were developed for other variational problems at the critical dimension involving "critical points at infinity". The full Atiyah-Jones conjecture was established by Boyer,Hurtubise, Mann and Milgram [13] but using geometrical techniques: the "explicit" description of the moduli spaces obtained from the ADHM construction (see (3.2) below). A different geometrical proof was given by Kirwan [41], together with generalisations to other gauge groups.

There was a parallel story for the solutions of the *Bogomolony equation* over $\mathbf{R}^3$, which we will not recount in detail. Here the base dimension is below the critical case but the analytical difficulty arises from the non-compactness of $\mathbf{R}^3$. Taubes succeeded in overcoming this difficulty and obtained relations between the topology of the moduli space, the appropriate configuration space and the higher critical points. Again, these higher critical points exist but their index grows with the numerical parameter corresponding to k. At about the same time, Donaldson showed that the moduli spaces could be identified with spaces of rational maps [19] (subsequently extended to other gauge groups). The analogue of the Atiyah-Jones conjecture is a result on the topology of spaces of rational maps proved earlier by Segal, which had been one of the motivations for Atiyah and Jones.

2.5 Higher dimensions.

While the scope for variational methods in Yang-Mills theory in higher dimensions is very limited, there are useful analytical results about solutions of the Yang-Mills equations. An important monotonicity result was obtained by Price [53]. For simplicity, consider a Yang-Mills connection over the unit ball $B^n \subset \mathbf{R}^n$. Then Price showed that the normalised energy

$$E(A, B(r)) = \frac{1}{r^{n-4}} \int_{|x| \leq r} |F|^2 d\mu$$

decreases with r. Nakajima and Uhlenbeck [49] used this monotonicity to show that for each n there is an ϵ such that if A is a Yang-Mills connection over a ball with $E(A, B(r)) \leq \epsilon$ then all derivatives of A, in a suitable gauge, can be controlled by $E(A, B(r))$. Tian [64] showed that if A_i is a sequence of Yang-Mills connections over a compact manifold M with bounded Yang-Mills functional, then there is a subsequence which converges away from a set Z of Haussdorf codimension at least 4 (extending the case of points in a 4-manifold). Moreover, the singular set Z is a minimal subvariety, in a suitably generalised sense.

In higher dimensions, important examples of Yang-Mills connections arise within the framework of "calibrated geometry". Here we consider a Riemannian n-manifold M with a covariant constant calibrating form $\Omega \in \Omega^{n-4}_M$. There is then an analogue of the instanton equation

$$F_A = \pm * (\Omega \wedge F_A),$$

whose solutions minimise the Yang-Mills functional. This includes the Hermitian Yang-Mills equation over a Kahler manifold (see (3.4) below) and also certain equations over manifolds with special holonomy groups [27]. For these "higher dimensional instantons", Tian shows that the singular sets Z that arise are calibrated varieties.

2.6 Gluing techniques.

Another set of ideas from PDE and analysis which has had great impact in gauge theory involves the construction of solutions to appropriate equations by the following general scheme.

(1) Constructing an "approximate solution", formed from some standard models using cut-off functions;
(2) showing that the approximate solution can be deformed to a true solution by means of an implicit function theorem.

The heart of the second step is usually made up of estimates for the relevant linear differential operator. Of course the success of this strategy depends on the particular features of the problem. This approach, due largely to Taubes, has been particularly effective in finding solutions to the first-order instanton equations and their relatives. (The applicability of the approach is connected to the fact that such solutions typically occur in moduli spaces and one can often "see" local co-ordinates in the moduli space by varying the parameters in the approximate solution.) Taubes applied this approach to the Bogomolny monpole equation over $\mathbf{R}^3$ [37] and to construct instantons over general 4-manifolds [57]. In the latter case the approximate solutions are obtained by transplanting standard solutions over $\mathbf{R}^4$—with curvature density concentated in a small ball—to small balls on the

4-manifold, glued to the trivial flat connection over the remainder of the manifold. These kind of techniques have now become a fairly standard part of the armoury of many differential geometers, working both within gauge theory and other fields. An example of a problem where similar ideas have been used is Joyce's construction of constant of manifolds with exceptional holonomy groups [39]. (Of course, it is likely that similar techniques have been developed over the years in many other areas, but Taubes' work in gauge theory has done a great deal to bring them into prominence.)

3. Geometry: integrability and moduli spaces

3.1 The Ward correspondence.

Suppose that S is a complex surface and ω is the 2-form corresponding to a Hermitian metric on S. Then S is an oriented Riemannian 4-manifold and ω is a self-dual form. The orthogonal complement of ω in Λ^+ can be identified with the real parts of forms of type $(0,2)$. Hence if A is an anti-self-dual instanton connection on a principle $U(r)$-bundle over S the $(0,2)$ part of the curvature of A vanishes. This is the *integrability condition* for the $\overline{\partial}$-operator defined by the connection, acting on sections of the associated vector bundle $E \to S$. Thus, in the presence of the connection, the bundle E is naturally a holomorphic bundle over S.

The *Ward correspondence* [67] builds on this idea to give a complete translation of the instanton equations over certain Riemannian 4-manifolds into holomorphic geometry. In the simplest case, let A be an instanton on a bundle over $\mathbf{R}^4$. Then for any choice of a linear complex structure on $\mathbf{R}^4$, compatible with the metric, A defines a holomorphic structure. The choices of such a complex structure are parametrised by a 2-sphere; in fact the unit sphere in $\Lambda^+(\mathbf{R}^4)$. So for any $\lambda \in S^2$ we have a complex surface S_λ and a holomorphic bundle over S_λ. This data can be viewed in the following way. We consider the projection $\pi : \mathbf{R}^4 \times S^2 \to \mathbf{R}^4$ and the pull-back $\pi^*(E)$ to $\mathbf{R}^4 \times S^2$. This pull-back bundle has a connection which defines a holomorphic structure along each fibre $S_\lambda \subset \mathbf{R}^4 \times S^2$ of the other projection. The product $\mathbf{R}^4 \times S^2$ is the *twistor space* of $\mathbf{R}^4$ and it is in a natural way a 3-dimensional complex manifold. It can be identified with the complement of a line L_∞ in $\mathbf{CP}^3$ where the projection $\mathbf{R}^4 \times S^2 \to S^2$ becomes the fibration of $\mathbf{CP}^3 \setminus L_\infty$ by the complex planes through L_∞. One can see then that $\pi^*(E)$ is naturally a holomorphic bundle over $\mathbf{CP}^3 \setminus L_\infty$. The construction extends to the conformal compactification S^4 of $\mathbf{R}^4$. If S^4 is viewed as the quaternionic projective line $\mathbf{HP}^1$ and we identify $\mathbf{H}^2$ with $\mathbf{C}^4$ in the standard way we get a natural map $\pi : \mathbf{CP}^3 \to \mathbf{HP}^1$. Then $\mathbf{CP}^3$ is the twistor space of S^4 and an anti-self-dual instanton on a bundle E over S^4 induces a holomorphic structure on the bundle $\pi^*(E)$ over $\mathbf{CP}^3$.

In general, the twistor space Z of an oriented Riemannian 4-manifold M is defined to be the unit sphere bundle in Λ^+_M. This has a natural almost-complex structure which is integrable if and only if the self-dual part of the Weyl curvature of M vanishes [7]. The antipodal map on the two sphere induces an antiholomorphic involution of Z. In such a case, an ASD instanton over M lifts to a holomorphic bundle over Z. Conversely, a holomorphic bundle over Z which is holomorphically trivial over the fibres of the fibration $Z \to M$ (projective lines in Z), and which satisfies a certain reality condition with respect to the antipodal map, arises from a unitary instanton over M. This is the Ward correspondence, part of Penrose's twistor theory.

3.2 The ADHM construction.

The problem of describing all solutions to the Yang-Mills instanton equation over S^4 is thus reduced to a problem in algebraic geometry, of classifying certain holomorphic vector bundles. This was solved by Atiyah, Drinfeld, Hitchin and Manin [5]. The resulting *ADHM construction* reduces the problem to certain matrix equations. The equations can be reduced to the following form. For a bundle Chern class k and rank r we require a pair of $k \times k$ matrices α_1, α_2 a $k \times r$ matrix a and an $r \times k$ matrix b. Then the equations are

$$[\alpha_1, \alpha_2] = ab$$
$$[\alpha_1^*, \alpha_1] + [\alpha_2^*, \alpha_2] = aa^* - b^*b.$$

We also require certain open, nondegeneracy conditions. Given such matrix data, a holomorphic bundle over $\mathbf{CP}^3$ is constructed via a "monad": a pair of bundle maps over $\mathbf{CP}^3$

$$\mathbf{C}^k \otimes \mathcal{O}(-1) \xrightarrow{D_1} \mathbf{C}^k \oplus \mathbf{C}^k \oplus \mathbf{C}^r \xrightarrow{D_2} \mathbf{C}^k \otimes \mathcal{O}(1),$$

with $D_2 D_1 = 0$. That is, the rank r holomorphic bundle we construct is $KerD_2/ImD_1$. The bundle maps D_1, D_2 are obtained from the matrix data in a straightforward way, in suitable co-ordinates. It is this matrix description which was used by Boyer *et al* to prove the Atiyah-Jones conjecture on the topology of the moduli spaces of instantons. The only other case when the twistor space of a compact 4-manifold is an algebraic variety is the complex projective plane, with the non-standard orientation. An analogue of the ADHM description in this case was given by Buchdahl [16].

3.3 Integrable systems.

The Ward correspondence can be viewed in the general framework of integrable systems. Working with the standard complex structure on $\mathbf{R}^4$, the integrability consdition for the $\overline{\partial}$-operator takes the shape

$$[\nabla_1 + i\nabla_2, \nabla_3 + i\nabla_4] = 0$$

where ∇_i are the components of the covariant derivative in the co-ordinate directions. So the instanton equation can be viewed as a family of such commutator equations parametrised by $\lambda \in S^2$. One obtains many reductions of the instanton equation by imposing suitable symmetries. Solutions invariant under translation in one variable correspond to the Bogomolny *monopole equation* [37]. Solutions invariant under three translations correspond to solutions of *Nahm's equations*

$$\frac{dT_i}{dt} = \epsilon_{ijk}[T_j, T_k],$$

for matrix valued functions T_1, T_2, T_3 of one variable t. Nahm [48] and Hitchin [32] developed an analogue of the ADHM construction relating these two equations. This is now seen as a part of a general "Fourier-Mukai-Nahm transform" [26]. The instanton equations for connections invariant under two translations, Hitchin's equations [33], are locally equivalent to the harmonic map equation for a surface into the symmetric space dual to the structure group. Changing the signature of the metric on $\mathbf{R}^4$ to $(2,2)$ one gets the harmonic mapping equations into Lie groups [35]. More complicated reductions yield almost all the known examples of integrable PDE as special forms of the instanton equations [45].

3.4 Moment maps: the Kobayashi-Hitchin conjecture.

Let Σ be a compact Riemann surface. The Jacobian of Σ is the complex torus $H^1(\Sigma, \mathcal{O})/H^1(\Sigma, \mathbf{Z})$: it parametrises holomorphic line bundles of degree 0 over Σ. The Hodge theory (which was of course developed long before Hodge in this case) shows that the Jacobian can also be identified with the torus $H^1(\Sigma, \mathbf{R})/H^1(\Sigma, \mathbf{Z})$ which parametrises *flat* $U(1)$-connections. That is, any holomorphic line bundle of degree 0 admits a unique compatible flat unitary connection.

The generalisation of these ideas to bundles of higher rank began with Weil. He observed that any holomorphic vector bundle of degree 0 admits a flat conection, not necessarily unitary. In 1964, Narasimhan and Seshadri [50] showed that (in the case of degree 0) the existence of a flat, irreducible, unitary connection was equivalent to an algebro-geometric condition of *stability* which had been introduced shortly before by Mumford, for quite different purposes. Mumford introduced the stability condition in order to construct separated moduli spaces of holomorphic bundles—generalising the Jacobian—as part of his general Geometric Invariant Theory. For bundles of non-zero degree the discussion is slightly modified by the use of projectively flat unitary connections. The result of Narasimhan and Seshadri asserts that there are two different descriptions of the same moduli space $\mathcal{M}^{d,r}(Sigma)$: either as parametrising certain irreducible projectively flat unitary connections (representations of $\pi_1(\Sigma)$), or parametrising stable holomorphic bundles of degree d and rank r. While Narasimhan and Seshadri probably did not view the ideas in these terms, another formulation of their result is that a certain nonlinear PDE for a Hermitian metric on a holomorphic bundle—analogous to the Laplace equation in the abelian case—has a solution when the bundle is stable.

In the early 1980's, Atiyah and Bott cast these results in the framework of gauge theory [4]. (The Yang-Mills equations in 2-dimensions essentially reduce to the condition that the connection be flat, so they are rather trivial locally but have interesting global structure.) They made the important observation that the curvature of a connection furnishes a map

$$F : \mathcal{A} \to \mathrm{Lie}(\mathcal{G})^*,$$

which is an equivariant *moment map* for the action of the gauge group on $\mathcal{A}$. Here the symplectic form on the affine space $\mathcal{A}$ and the map from the adjoint bundle-valued 2-forms to the dual of the Lie algebra of $\mathcal{G}$ are both given by integration of products of forms. From this point of view, the Narasimhan-Seshadri result is an infinite-dimensional example of a general principle relating symplectic and complex quotients. At about the same time, Hitchin and Kobayashi independently proposed an extension of these ideas to higher dimensions. Let E be a holomorphic bundle over a complex manifold V. Any compatible unitary connection on E has curvature F of type $(1,1)$. Let ω be the $(1,1)$-form corresponding to a fixed Hermitian metric on V. The *Hermitian Yang-Mills equation* is the equation

$$F.\omega = \mu 1_E,$$

where μ is a constant (determined by the topological invariant $c_1(E)$). The Kobayashi-Hitchin conjecture is that, when ω is Kahler, this equation has a irreducible solution if and only if E is a stable bundle in the sense of Mumford. Just as in the Riemann surface case, this equation can be viewed as a nonlinear second order PDE of Laplace type for a metric on E. The moment map picture of Atiyah and Bott also extends to this higher dimensional version. In the case when V has complex dimension 2 (and μ is zero) the Hermitian Yang-Mills connections are exactly the anti-self-dual instantons, so the conjecture asserts that the moduli spaces of instantons can be identified with certain moduli spaces of stable holomorphic bundles.

The Kobayashi-Hitchin conjecture was proved in the most general form by Uhlenbeck and Yau [66], and in the case of algebraic manifolds in [21]. The proofs in [20],[21] developed some extra structure surrounding these equations, conected with the moment map point of view. The equations can be obtained as the Euler-Lagrange equations for a non-local functional, related to the renormalised determinants of Quillen and Bismut. The results have been extended to nonkahler manifolds and certain noncompact manifolds. There are also many extensions to equations for systems of data comprising a bundle with additional structure such as a holomorphic section or Higgs' field [15], or a parabolic structure along a divisor. Hitchin's equations [33] are a particularly rich example.

3.5 Topology of moduli spaces.

The moduli spaces $\mathcal{M}_{r,d}(\Sigma)$ of stable holomorphic bundles/projectively flat unitary connections over Riemann surfaces Σ have been studied intensively from many points of view. They have natural kahler structures: the complex structure being visible in the holomorphic bundles guise and the symplectic form as the "Marsden-Weinstein quotient" in the unitary connections guise. In the case when r and d are coprime they are compact manifolds with complicated topologies. There is an important basic construction for producing cohomology classes over these (and other) moduli spaces. One takes a universal bundle U over the product $\mathcal{M} \times \Sigma$ with Chern classes

$$c_i(U) \in H^{2i}(\mathcal{M} \times \Sigma).$$

Then for any class $\alpha \in H_p(\Sigma)$ we get a cohomology class $c_i(U)/\alpha \in H^{2i-p}(\mathcal{M})$. Thus if R_Σ is the graded ring freely generated by such classes we have a homomorphism $\nu : R_\Sigma \to H^*(\mathcal{M})$. The questions about the topology of the moduli spaces which have been studied include:

(1) Find the Betti numbers of the moduli space $\mathcal{M}$;
(2) Identify the kernel of ν;
(3) Give an explicit system of generators and relations for the ring $H^*(\mathcal{M})$;
(4) Identify the Pontrayagin and Chern classes of $\mathcal{M}$ within $H^*(\mathcal{M})$;
(5) Evaluate the pairings

$$\int_{\mathcal{M}} \nu(W)$$

for elements W of the appropriate degree in R.

All of these questions have now been solved quite satisfactorily. In early work, Newstead [51] found the Betti numbers in the rank 2 case. The main aim of Atiyah and Bott was to apply the ideas of Morse Theory to the Yang-Mills functional over a Riemann surface and they were able to reproduce Newstead's results in this way and extend them to higher rank. They also showed that the map ν is a surjection, so the universal bundle construction gives a system of generators for the cohomology. Newstead made conjectures on the vanishing of the Pontrayagin and Chern classes above a certain range which were established by Kirwan and extended to higher rank by Earl and Kirwan [28]. Knowing that R_Σ maps on to $H^*(M)$, a full set of relations can (by Poincare duality) be deduced in principle from a knowledge of the integral pairings in (5), but this is not very explicit. A solution to (5) in the case of rank 2 was found by Thaddeus [62]. He used results from the Verlinde theory (see (4.5) below) and the Riemann-Roch formula. Another point of view was developed by Witten [71], who showed that the volume of the moduli space was related to the theory of torsion in algebraic topology and satisfied simple gluing axioms. These different points of view are compared in [23]. Using a non-rigorous localisation principle in infinite dimensions, Witten wrote down a general formula [72] for the pairings (5) in any rank, and this was established rigorously by Jeffrey and Kirwan, using a finite-dimensional version of the same localisation method. A very simple and explicit set of generators and relations for the cohomology (in the rank 2 case) was given by King and Newstead [40]. Finally, the *quantum cohomology* of the moduli space, in the rank 2 case, was identified explicitly by Munoz [47].

3.6 Hyperkahler quotients.

Much of this story about the structure of moduli spaces extends to higher dimensions and to the moduli spaces of connections and Higgs fields . A particularly notable extension of the ideas involves *hyperkahler* structures. Let M be a hyperkahler 4-manifold, so there are three covariant-constant self-dual forms $\omega_1, \omega_2, \omega_3$ on M. These correspond to three complex structures I_1, I_2, I_3 obeying the algebra of the quaternions. If we single out one structure, say I_1, the instantons on M can be viewed as holomorphic bundles with respect to I_1 satisfying the moment map condition (Hermitian-Yang-Mills equation) defined by the form ω_1. Taking a different complex structure interchanges the role of the moment map and integrability conditions. This can be put in a general framework of hyperkahler quotients due to Hitchin et al. [36]. Suppose initially that M is compact (so either a K3 surface or a torus). Then the ω_i components of the curvature define three maps

$$F_i : \mathcal{A} \to \mathrm{Lie}(\mathcal{G})^*,$$

The structures on M make $\mathcal{A}$ into a flat hyperkahler manifold and the three maps F_i are the moment maps for the gauge group action with respect to the three symplectic forms on $\mathcal{A}$. In this situation it is a general fact that the hyperkahler quotient—the quotient by $\mathcal{G}$ of the common zero set of the three moment maps—has a natural hyperkahler structure. This hyperkahler quotient is just the moduli space of instantons over M. In the case when M is the noncompact manifold $\mathbf{R}^4$ the same ideas apply except that one has to work with the based gauge group $\mathcal{G}_0$. The conclusion is that the framed moduli spaces $\tilde{\mathcal{M}}$ of instantons over $\mathbf{R}^4$ are naturally hyperkahler manifolds. One can also see this hyperkahler structure through the ADHM matrix description. A variant of these matrix equations was used by Kronheimer to construct "gravitational instantons". The same ideas also apply to the moduli spaces of monopoles, where the hyperkahler metric, in the simplest case, was studied by Atiyah and Hitchin [6].

4. Low-dimensional topology

4.1 Instantons and four-manifolds.

Gauge theory has had unexpected applications in low-dimensional topology, particularly the topology of smooth 4-manifolds. The first work in this direction, in the early 1980's, involved the Yang-Mills instantons. The main issue in 4-manifold theory at that time was the correspondence between the diffeomeorphism classification of simply connected 4-manifolds and the classification up to homotopy. The latter is determined by the *intersection form*, a unimodular quadratic form on the second integral homology group (i.e. a symmetric matrix with integral entries and determinant ± 1, determined up to integral change of basis). The only known restriction was that Rohlin's Theorem, which asserts that if the form is even the signature must be divisible by 16. The achievement of the first phase of the theory was to show that

(1) There are unimodular forms which satisfy the hypotheses of Rohlin's Theorem but which do not appear as the intersection forms of smooth 4-manifolds. In fact no non-standard definite form, such as a sum of copies of the E_8 matrix, can arise in this way.
(2) There are simply connected smooth 4-manifolds which have isomorphic intersection forms, and hence are homotopy equivalent, but which are not diffeomorphic.

These results stand in contrast to the *homeomorphism* classification which was obtained by Freedman shortly before and which is almost the same as the homotopy classification.

The original proof of item (1) above argued with the moduli space $\mathcal{M}$ of anti-self-dual instantons $SU(2)$ instantons on a bundle with $c_2 = 1$ over a simply connected Riemannian 4-manifold M with a negative definite intersection form [18]. In the model case when M is the 4-sphere the moduli space $\mathcal{M}$ can be identified explicitly with the open 5-ball. Thus the 4-sphere arises as the natural boundary of the moduli space. A sequence of points in the moduli space converging to a boundary point corresponds to a sequence of connections with curvature densities converging to a delta-function, as described in (2.2) above. One shows that in the general case (under our hypotheses on the 4-manifold M) the moduli space $\mathcal{M}$ has a similar behaviour, it contains a collar $M \times (0, \delta)$ formed by instantons made using Taubes' gluing construction, described in (2.6). The complement of this collar is compact. In the interior of the moduli space there are a finite number of special points corresponding to $U(1)$-reductions of the bundle P. This is the way in which the moduli space "sees" the integral structure of the intersection form since such reductions correspond to integral homology classes with self-intersection -1. Neighbourhoods of these special points are modelled on quotients $\mathbf{C}^3/U(1)$; *i.e* cones on copies of $\mathbf{CP}^2$. The upshot is that (for generic Riemannian metrics on M) the moduli space gives a cobordism from the manifold M to a set of copies of $\mathbf{CP}^2$ which can be counted in terms of the intersection form, and the result follows easily from standard topology. More sophisticated versions of the argument extended the results to rule out some indefinite intersection forms.

On the other side of the coin, the original proofs of item (2) used *invariants* defined by instanton moduli spaces [22]. The general scheme exploits the same construction outlined in (3.5) above. We suppose that M is a simply connected 4-manifold with $b^+(M) = 1 + 2p$ where $p > 0$ is an integer. (Here $b^+(M)$ is, as usual, the number of positive eigenvalues of the intersection matrix). Ignoring some technical restrictions, there is a map

$$\nu : R_M \to H^*(\mathcal{M}_k),$$

where R_M is a graded ring freely generated by the homology (below the top dimension) of the 4-manifold M and $\mathcal{M}_k$ is the moduli space of ASD $SU(2)$-instantons on a bundle with $c_2 = k > 0$. For an element W in R_M of the appropriate degree one obtains a number by evaluating, or integrating, $\nu(W)$ on $\mathcal{M}_k$. The main technical difficulty here is that the moduli space $\mathcal{M}_k$ is rarely compact, so one needs to make sense of this "evaluation". With all the appropriate technicalities in place, these invariants could be shown to distinguish various homotopy equivalent, homeomorphic 4-manifolds. All these early developments are described in detail in the book [26].

4.2 Basic classes.

Until the early 1990's these instanton invariants could only be calculated in isolated favourable cases. (Although the calculations which were made, through the work of many mathematicians, lead to a large number of further results about 4-manifold topology). Deeper understanding of their structure came with the work of Kronheimer and Mrowka. This work was in large part motivated by a natural question in geometric topology. Any homology class $\alpha \in H_2(M; \mathbf{Z})$ can be represented by a embedded, connected, smooth surface. One can define an integer $\underline{g}(\alpha)$ to be the minimal genus of such a representative. The problem is to find $\underline{g}(\alpha)$, or at least bounds on it. A well-known conjecture, ascribed to Thom, was that when M is the complex projective plane the minimal genus is realised by a complex curve; i.e.

$$\underline{g}(\pm dH) = \frac{1}{2}(d-1)(d-2),$$

where H is the standard generator of $H_2(\mathbf{CP}^2)$ and $d \geq 1$.

The new geometrical idea introduced by Kronheimer and Mrowka was to study instantons over a 4-manifold M with singularities along a surface $\Sigma \subset M$. For such connections there is a real parameter: the limit of the trace of the holonomy around small circles linking the surface. By varying this parameter they were able to interpolate between moduli spaces of non-singular instantons on different bundles over M and obtain relations between the different invariants. They also found that if the genus of Σ is suitably small then some of the invariants are forced to vanish so, conversely, getting information about $\underline{g}$ for 4-manifolds with non-trivial invariants. For example, they showed that if M is a K3 surface then $\underline{g}(\alpha) = \frac{1}{2}(\alpha.\alpha + 2)$.

The structural results of Kronheimer and Mrowka [42] introduced the notion of a 4-manifold of "simple type". Write the invariant defined above by the moduli space $\mathcal{M}_k$ as $I_k : R_M \to \mathbf{Q}$. Then I_k vanishes except on terms of degree $2d(k)$ where $d(k) = 4k - 3(1+p)$. We can put all these together to define $\underline{I} = \sum I_k : R_M \to \mathbf{Q}$. The ring R_M is a polynomial ring generated by classes $\alpha \in H_2(M)$, which have degree 2 in R_M, and a class X of degree 4 in R_M, corresponding to the generator of $H_0(M)$. The 4-manifold is of simple type if

$$\underline{I}(X^2 W) = 4\underline{I}(W),$$

for all $W \in R_M$. Under this condition, Kronheimer and Mrowka showed that all the invariants are determined by a finite set of "basic " classes $K_1, \dots, K_s \in H_2(M)$ and rational numbers $\beta_1, \dots \beta_s$. To express the relation they form a generating function

$$\mathcal{D}_M(\alpha) = \underline{I}(e^{\alpha}) + \underline{I}(\frac{X}{2}e^{\alpha}).$$

This is *a priori* a formal power series in $H^2(M)$ but *a posteriori* the series converges and can be regarded as a function on $H_2(M)$. Kronheimer and Mrowka's result is that

$$\mathcal{D}_M(\alpha) = \exp(\frac{\alpha.\alpha}{2}) \sum_{r=1}^{s} \beta_r e^{K_r.\alpha}.$$

It is not known whether all simply connected 4-manifolds are of simple type, but Kronheimer and Mrowka were able to show that this is the case for a multitude of examples. They also introduced a weaker notion of "finite type", and this condition was shown to hold in general by Munoz and Froyshov. The overall result of this work of Kronheimer and Mrowka was to make the calculation of the instanton invariants for many familiar 4-manifolds a comparatively straightforward matter.

4.3 Three-manifolds: Casson's invariant.

Gauge theory has also entered into 3-manifold topology. In 1985, Casson introduced a new integer-valued invariant of oriented homology 3-spheres which "counts" the set Z of equivalence classes of irreducible flat $SU(2)$-connections, or equivalently irreducible representations $\pi_1(Y) \to SU(2)$ Casson's approach [1] was to use a Heegard splitting of a 3-manifold Y into two handle-bodies Y^+, Y^- with common boundary a surface Σ. Then $\pi_1(\Sigma)$ maps onto $\pi_1(Y)$ and a flat $SU(2)$ connection on Y is determined by its restriction to Σ. Let M_Σ be the moduli space of irreducible flat connections over Σ (as discussed in (3.4) above) and let $L^\pm \subset M_\Sigma$ be the subsets which extend over $Y^\pm$. Then $L^\pm$ are submanifolds of half the dimension of $\mathcal{M}_\Sigma$ and the set Z can be identified with the intersection $L^+ \cap L^-$. The Casson invariant is one half the algebraic intersection number of L^+ and L^-. Casson showed that this is independent of the Heegard splitting (and is also in fact an integer, although this is not obvious). He showed that when Y is changed by Dehn surgery along a knot the invariant changes by a term computed from the Alexander polynomial of the knot. This makes the Casson invariant computable in examples. (For a discussion of Casson's formula see [24].) Taubes showed that the Casson invariant could also be obtained in a more differential-geometric fashion, analogous to the instanton invariants of 4-manifolds [59].

4.4 Three-manifolds: Floer Theory.

Independently, at about the same time, Floer introduced more sophisticated invariants—the Floer homology groups—of homology 3-spheres, using gauge theory [30]. This development ran parallel to his introduction of similar ideas in symplectic geometry. Suppose for simplicity that the set Z of equivalence classes of irreducible flat connections is finite. For pairs ρ_-, ρ_+ in Z, Floer considered the instantons on the tube $Y \times \mathbf{R}$ asymptotic to $\rho^\pm$ at $\pm\infty$. There is an infinite set of moduli spaces of such instantons, labelled by a relative Chern class, but the dimensions of these moduli spaces agree modulo 8. This gives a relative index $\delta(\rho_-, \rho_+) \in \mathbf{Z}/8$. If $\delta(\rho_-, \rho_+) = 1$ there is a moduli space of dimension 1 (possibly empty), but the translations of the tube act on this moduli space and dividing by translations we get a finite set. The number of points in this set, counted with suitable signs, gives an integer $n(\rho_-, \rho_+)$. Then Floer considers the free abelian groups

$$C_* = \bigoplus_{\rho \in Z} \mathbf{Z}\langle \rho \rangle,$$

generated by the set Z and a map $\partial : C_* \to C_*$ defined by

$$\partial(\langle\rho_-\rangle) = \sum n(\rho_-, \rho_+)\langle\rho_+\rangle.$$

Here the sum runs over the ρ_+ with $\delta(\rho_-, \rho_+) = 1$. Floer showed that $\partial^2 = 0$ and the homology $HF_*(Y) = \ker\partial/\mathrm{Im}\,\partial$ is independent of the metric on Y (and various other choices made in implementing the construction in detail). The chain complex C_* and hence the Floer homology can be graded by $\mathbf{Z}/8$, using the relative index, so the upshot is to define 8 abelian groups $HF_i(Y)$: invariants of the 3-manifold Y. The casson invariant appears now as the Euler characteristic of the Floer homology. There has been quite a lot of work on extending these ideas to other 3-manifolds (not homology spheres) and gauge groups, but this line of research does not yet seem to have reached a clear-cut conclusion.

Part of the motivation for Floer's work came from Morse Theory, and particularly the approach to this theory expounded by Witten [68]. The *Chern-Simons functional* is a map

$$CS : \mathcal{A}/\mathcal{G} \to \mathbf{R}/\mathbf{Z},$$

from the space of $SU(2)$-connections over Y. Explicitly, in a trivialisation of the bundle

$$CS(A) = \int_Y A \wedge dA + \frac{3}{2} A \wedge A \wedge A.$$

It appears as a boundary term in the Chern-Weil theory for the second Chern class, in a similar way as holonomy appears as a boundary term in the Gauss-Bonnet Theorem. The set Z can be identified with the critical points of CS and the instantons on the tube as integral curves of the gradient vector field of CS. Floer's definition mimics the definition of homology in ordinary Morse theory, taking Witten's point of view. It can be regarded formally as the "middle dimensional" homology of the infinite dimensional space $\mathcal{A}/\mathcal{G}$. See [2], [17] for discussions of these ideas.

The Floer Theory interacts with 4-manifold invariants, making up a structure approximating to a $3+1$-dimensional *Topological Field Theory*[3]. Roughly, the numerical invariants of closed 4-manifolds generalise to invariants for a 4-manifold M with boundary Y taking values in the Floer homology of Y. If two such manifolds are glued along a common boundary the invariants of the result are obtained by a pairing in the Floer groups. There are however, at the moment, some substantial technical restrictions on this picture. This theory, and Floer's original construction, is developed in detail in the book [25]. At the time of writing, the Floer homology groups are still hard to compute in examples. One important tool is a surgery exact sequence found by Floer [14], related to Casson's surgery formula.

4.5 Three-manifolds: Jones-Witten Theory.

There is another, quite different, way in which ideas from gauge theory have entered 3-manifold topology. This is the Jones-Witten theory of knot and 3-manifold invariants. This theory falls outside the main line of this article, but we will say a little about it since it draws on many of the ideas we have discussed. The goal of the theory is to construct a family of $2+1$-dimensional Topological Field Theories indexed by an integer k, asssigning complex vector space $H_k(\Sigma)$ to a surface Σ and an invariant in $H_k(\partial Y)$ to a 3-manifold-with-boundary Y. If ∂Y is empty, the vector space $H_k(\partial Y)$ is taken to be $\mathbf{C}$, so one seeks numerical invariants of closed

3-manifolds. Witten's idea [70] is that these invariants of closed 3-manifolds are Feynmann integrals

$$\int_{\mathcal{A}/\mathcal{G}} e^{i2\pi k CS(A)} \mathcal{D}A.$$

This functional integral is probably a schematic rather than a rigorous notion. The data associated to surfaces can however be defined rigorously. If we fix a complex structure I on Σ we cane define a vector space $H_k(\Sigma, I)$ to be

$$H_k(\Sigma, I) = H^0(\mathcal{M}(\Sigma); L^k),$$

where $\mathcal{M}(\Sigma)$ is the moduli space of stable holomorphic bundles/flat unitary connections over Σ and L is a certain holomorphic line bundle over $\mathcal{M}(\Sigma)$. These are the spaces of "conformal blocks" whose dimension is given by the Verlinde formulae. Recall that $\mathcal{M}(\Sigma)$, as a *symplectic* manifold, is canonically associated to the surface Σ, without any choice of complex structure. The Hilbert spaces $H_k(\Sigma, I)$ can be regarded as the quantisation of this symplectic manifold, in the general framework of Geometric Quantisation: the inverse of k plays the role of Planck's constant. What is not obvious is that this quantisation is independent of the complex structure chosen on the Riemann surface: i.e. that there is a natural identification of the vector spaces (or at least the associated projective spaces) formed using different complex structures. This was established rigorously by Hitchin [34] and Axelrod *et al* [10], who constructed a projectively flat connection on the bundle of spaces $H_k(\Sigma, I)$ over the space of complex structures I on Σ. At a formal level, these constructions are derived from the construction of the metaplectic representation of a linear symplectic group, since the $\mathcal{M}_\Sigma$ are symplectic quotients of an affine symplectic space.

The Jones-Witten invariants have been rigorously established by indirect means, but it seems that there is still work to be done in developing Witten's point of view. If Y^+ is a 3-manifold with boundary one would like to have a geometric definition of a vector in $H_k(\partial Y^+)$. This should be the quantised version of the submanifold L^+ (which is Lagrangian in $\mathcal{M}_\Sigma$) entering into the Casson theory.

4.6 Seiberg-Witten invariants.

The instanton invariants of a 4-manifold can be regarded as the integrals of certain natural differential forms over the moduli spaces of instantons. In [69], Witten showed that these invariants could be obtained as functional integrals, involving a variant of the Feynmann integral, over the space of connections and certain auxiliary fields (insofar as this latter integral is defined at all). A geometric explanation of Witten's construction was given by Atiyah and Jeffrey [8]. Developing this point of view, Witten made a series of predictions about the instanton invariants, many of which were subsequently verified by other means. This line of work culminated in 1994 where, applying developments in supersymmetric Yang-Mills QFT, Seiberg and Witten introduced a new system of invariants and a precise prediction as to how these should be related to the earlier ones.

The Seiberg-Witten invariants [73] are associated to a $Spin^c$ structure on a 4-manifold M. If M is simply connected this is specified by a class $K \in H^2(M; \mathbf{Z})$ lifting $w_2(M)$. One has spin bundles $S^+, S^- \to M$ with $c_1(S^\pm) = K$. The Seiberg-Witten equation is for a spinor field ϕ—a section of S^+ and a connection A on

the complex line bundle $\Lambda^2 S^+$. This gives a connection on S^+ and hence a Dirac operator
$$D_A : \Gamma(S^+) \to \Gamma(S^-).$$
The Seiberg-Witten equations are
$$D_A\phi = 0 \quad , \quad F_A^+ = \sigma(\phi),$$
where $\sigma : S^+ \to \Lambda^+$ is a certain natural quadratic map. The crucial differential-geometric feature of these equations arises from the Weitzenbock formula
$$D_A^* D_A \phi = \nabla_A^* \nabla_A \phi + \frac{R}{4}\phi + \rho(F^+)\phi,$$
where R is the scalar curvature and ρ is a natural map from Λ^+ to the endomophisms of S^+. Then ρ is adjoint to σ and
$$\langle \rho(\sigma(\phi))\phi, \phi\rangle = |\phi|^4.$$
It follows easily from this that the moduli space of solutions to the Seiberg-Witten equation is *compact.* The most important invariants arise when K is chosen so that
$$K.K = 2\chi(M) + 3\text{sign } (M),$$
where $\chi(M)$ is the Euler characteristic and sign (M) is the signature. (This is just the condition for K to correspond to an *almost complex* ctructure on M.) In this case the moduli space of solutions is 0 dimensional (after generic perturbation) and the Seiberg-Witten invariant $SW(K)$ is the number of points in the moduli space, counted with suitable signs.

Witten's conjecture relating the invariants, in its simplest form, is that when M has simple type the classes K for which $SW(K)$ is non-zero are exactly the basic classes K_r of Kronheimer and Mrowka and that
$$\beta_r = 2^{C(M)} SW(K_r),$$
where $C(M) = 2 + \frac{1}{4}(7\chi(M) + 11\text{sign } (M))$. This asserts that the two sets of invariants contain exactly the same information about the 4-manifold.

The evidence for this conjecture, via calculations of examples, is very strong. A somewhat weaker statement has been proved rigorously by Feehan and Leness [29]. They use an approach suggested by Pidstragatch and Tyurin, studying moduli spaces of solutions to a non-abelian version of the Seiberg-Witten equations. These contain both the instanton and abelian Seiberg-Witten moduli spaces and the strategy is to relate the topology of these two sets by standard localisation arguments. (This approach is related to ideas introduced by Thaddeus [63] in the case of bundles over Riemann surfaces.) The serious technical difficulty in this approach stems from the lack of compactness of the nonabelian moduli spaces. The more general versions of Witten's conjecture [46] (for example when $b^+(M) = 1$) contain very complicated formulae, involving modular forms, which presumeably arise as contributions from the compactification of the moduli spaces.

4.7 Applications.

Regardless of the connection with the instanton theory, one can go ahead directly to apply the Seiberg-Witten invariants to 4-manifold topology, and this has been the main direction of research over the past decade. The features of the Seiberg-Witten theory which have lead to the most prominent developments are

(1) The reduction of the equations to 2-dimensions is very easy to understand. This has lead to proofs of the Thom conjecture and wide-ranging generalisations [52].
(2) The Weitzenboch formula implies that if M has positive scalar curvature solutions to the Seiberg-witten equations must have $\phi = 0$. This has lead to important interactions with 4-dimensional Riemannian geometry [44].
(3) In the case when M is a symplectic manifold there is a natural deformation of the Seiberg-Witten equations, discovered by Taubes [61], who used it to show that the Seiberg-Witten invariants of M are non-trivial. More generally, Taubes showed that for large values of the deformation parameter the solutions of the deformed equation localise around surfaces in the 4-manifold and used this to relate the Seiberg-Witten invariants to the Gromov theory of pseudo-holomorphic curves. These results of Taubes have completely transformed the subject of 4-dimensional symplectic geometry.

Bauer and Furuta [11] have combined the Seiberg-Witten theory with more sophisticated algebraic topology to obtain further results about 4-manifolds. They consider the map from the space of conections and spinor fields defined by the formulae on the left hand side of the equations. The general idea is to obtain invariants from the homotopy class of this map, under a suitable notion of homotopy. A technical complication arises from the gauge group action, but this can be reduced to the action of a single $U(1)$. Ignoring this issue, Bauer and Furuta obtain invariants in the stable homotopy groups $\lim_{N\to\infty} \pi_{N+r}(S^N)$, which reduce to the ordinary numerical invraints when $r = 1$. Using these invariants they obtain results about connected sums of 4-manifolds, for which the ordinary invariants are trivial. Using refined cobordism invariants ideas, Furuta made great progress towards resolving the question of which intersection forms arise from smooth, simply-connected 4-manifolds. A well known conjecture is that if such a manifold is spin then the second Betti number satisfies

$$b_2(M) \geq \frac{11}{8}\ |\text{sign }(M)|.$$

Furuta [31] proved that $b_2(M) \geq \frac{10}{8}\ |\text{sign }(M)| + 2$.

An important and very recent achievement, bringing together many different lines of work, is the proof of "Property P" in 3-manifold topology by Kronheimer and Mrowka [43]. This asserts that one cannot obtain a homotopy sphere (counterexample to the Poincaré conjecture) by $+1$-surgery along a non-trivial knot in S^3. The proof uses work of Gabai and Eliashberg to show that the manifold obtained by 0-framed surgery is embedded in a symplectic 4-manifold; Taubes' results to show that the Seiberg-Witten invariants of this 4-manifold are non-trivial; Feehan and Leness' partial proof of Witten's conjecture to show that the same is true for the instanton invariants; the gluing rule and Floer's exact sequence to show that the Floer homology of the $+1$-surgered manifold is non-trivial. It follows then from the definition of Floer homology that the fundamental group of this manifold is not trivial; in fact it must have an irreducible representation in $SU(2)$.

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