Walk through for Number Ladder

It helps to be familiar with the integer powers in the range 100-1000:

Cubes: [0, 1, 8, 27, 64,] 125, 216, 343, 512 729, [1000] **Fourth powers**: 256, 625.
Fifth powers: 243. **Sixth powers**: 729, **Seventh, eighth, ninth powers**: 128, 256, 512.
Squares: 100 121 144 169 196 225 256 289 324 361 400 441 484 529 576 625 676 729 784 841 900 961.

So many of the rungs can immediately be filled in. It is also easier if one knows the 4 numbers which equal the sum of their digits cubed – these are of course listed on the web, but we work them out below. There are many ways of solving the puzzle, but here is one.

The ladder starts with **128.** To reach a prime it is necessary to change the 8, and the only prime is **127.** The reachable cube is **125,** and 4th power is **625.** Skipping one rung, we must reach **729** in two steps, so we need only consider 725 and 629 as the intermediate rung, and **725** is divisible by 29. From 729, the only square in reach is **529,** and we then must reach **512** in two steps. 522 or 519 are the only possibilities, and 29 divides **522.**

From 512, a multiple of 15 must be 510 or 515 and only **510** works. Half a cube is then **500.** To reach a multiple of 39, we must change one of the zeroes into a 1, 4 or 7 for divisibility by 3, and only 507 is also divisible by 13.

We now have the first of four numbers equal to the sum of the cubes of its digits. It's easiest to calculate the rung below first, but suppose we don't. If the 0 of 507 were changed to x. Then $5^3 + x^3 + 7^3$ must end in a 7, so x^3 would end in a 9, so x = 9. But that doesn't work. It follows that either the 5 or 7 must change. If 5 changes to y, $y^3 + 343 = 100y + 7$. So y^3 ends in a 4, so that y = 4. This does in fact work, 343 + 64 = 407. If 7 changes to z, $125 + z^3 = 500 + z$. But one side is even and the other is odd so that is not possible. So the next rung is **407.**

If C = 1, then $C^3 - 2C^2 + 2C = 1$, which is too small. If C = 27, it is more than 3 digits, and so we must have C = 8.Then 512 - 2(64) + 16 = **400**.

From **400,** 3 times a square must be **300.** We then reach another "sum of cubes" number. Clearly a zero must change to something whose cube is about 300. $7^3 + 3^3 = 370$, so the next rung is **370.** The next rung is also a "sum of cubes" number and must be **371.**

Three rungs further on we have the 5th power **243**. Above that, we have a cube which must be **343**. Between 371 and 343 must be either 373 or 341, and 31 divides **341.** Below 243, we have **143** which is one less than a square. We then have the fourth "four cubes number."

2 rungs below that, we have the 8th power **256.** So the 3 possibilities for the rung under 143 are 243, 153 and 146. The sums of cubes of the first is too small, of the last is too big, but the middle is just right, as **153** = 1 + 125 + 27. Between 153 and 256 we have **253**, which is divisible by 23. Finally, the only unused cube next to 256 has to be **216**.