Pieces of Eight walkthrough

We can find the integer either by constructing it digit by digit, or by seeking the whole answer, which requires some mathematical skills. We give both methods:

Digit by digit construction:

If an integer has the same number of digits when it is multiplied by 8, it must start with a 1:

First Digit is 1

and the product must begin with an 8 or 9. We are told to choose the smaller of two possible solutions, so we assume the product starts with an 8. This means the

Last Digit is 8.

When we multiply by 8 we find 8*8 = 64 and the last digit of the product must be 4, so

Penultimate Digit is 4

Continuing in this manner we have 8 x 48 = 384 and so

8 x 1…..48 = 8……..84

8 x 1…..848 = 8…….784

8 x 1….7848 = 8……2784

8 x 1…27848 = 8…22784

8 x 1…227848 = 8…822784

8 x 1….8227848 = 8…5822784

8 x 1…58227848 = 8…65822784

8 x 1…658227848 = 8..265822784

8 x 1..2658227848 = 8..1265822784

8 x 1..12658227848 = 8..01265822784

8 x 1012658227848 = 8101265822784

This is the solution, giving the digit triples for insertion in the solution grid

012, 658, 227, 848

Construction of entire integer

To construct an integer with the required multiplication property, call it x and suppose it has k digits, with a last digit N. Then removing the last digit is equivalent to subtracting N and dividing by 10. We then put N at the front which is adding N x 10^{k-1}. We deduce that

$8x= (x-N)/10 + N*10^{k-1}$

Or $80x = x-N +N*10^{k}$

And solving for x

$x = N* (10^{k}-1) / 79$

This requires $10^{k}-1$ to be divisible by 79. Using "Fermat's Little Theorem", since 79 is prime, this happens for k = 78. We were in fact told that x has 13 digits. (Note 13 divides 78.)

Indeed

$(10^{13}-1)/79 = 0126582278481$

where we've added a leading zero to give a 13 digit number. We now must multiply by the smallest value of N which doesn't give a leading zero. This is N=8, giving

8 * 126582278481 = 1012658227848

which is the smallest solution : **8 * 1012658227848 = 8101265822784**

N = 9 would give the slightly larger

9 * 126582278481 = 1139240506329 and 8 * 1139240506329 = 9113924050632

(Incidentally, if we try k = 78 we find by straight division $(10^{78}-1)/79$ =9999999……99999999999/79 =

12658227848101265822784810126582278481012658227848101265822784810126582278481

which clearly repeats itself with the string of 13 digits we have been using.)