ull7a.tex **Week 7: am, 9.11.2016**

In the insurance context (below), the Poisson points represent the claim arrivals, so the Poisson rate λ is the rate at which claims arrive; μ is the mean claim size. So $\lambda\mu$ has the interpretion of a *claim rate* – rate at which money goes *out* of the company in claims.

Just as the mathematics of the Black-Scholes model (Ch. VI) is dominated by *Brownian motion*, that of insurance is dominated by the *Poisson* and *compound Poisson* processes. These are the basic prototypes, and all we have time to cover in detail in this course. However, these are models, of reality, and reality is always more complicated than any model! Box's dictum (George Box, British statistician, 1919-2013): *All models are wrong. Some models are useful.* In more advanced work, more complicated and detailed models are needed. So there is plenty of scope for useful applications in the real world of any probability or statistics you know, or will learn! At the end of the course (VII.5), we discuss briefly some generalisations. But to note for now: the principal weakness of our assumptions here is the *independence* of claims. This is reasonable under normal conditions, but not during a crisis. Think of natural disasters such as major hurricanes, etc.

§3. Renewal theory

Renewal Processes

Suppose we use components – light-bulbs, say – whose lifetimes $X_1, X_2, \ldots$ are independent, all with law F on $(0, \infty)$. The first component is installed new, used until failure, then replaced, and we continue in this way. Write

$$S_n := \sum_1^n X_i, \qquad N_t := \max\{k : S_k \leq t\}.$$

Then $N = (N_t : t \geq 0)$ is called the *renewal process* generated by F; it is a *counting process*, counting the number of failures seen by time t. Note that

$$S_{N(t)} \leq t.$$

Note. For stochastic processes, notations such as N_t and $N(t)$ are used interchangeably.

Renewal processes are often used, but the only ones we need here are the

Poisson processes – those for which the lifetime law is exponential.

The renewal function

We saw above that

$$N_t/t \to 1/\mu \qquad (t \to \infty), \ a.s.$$

If we apply the expectation operator $E[.]$ formally, this suggests that

$$E[N_t]/t \to 1/\mu \qquad (t \to \infty).$$

This is indeed true, but although its conclusion seems weaker than that of the a.s. result, its proof if harder (though not as hard as that of the SLLN!).

Theorem. If the mean lifetime length μ is finite, the renewal function $E[N_t]$ satisfies

$$E[N_t]/t \to 1/\mu \quad (t \to \infty).$$

Proof. The conclusion with $\geq$ in place of $=$ does indeed follow from the a.s. result by taking expectations. This is by *Fatou's lemma*, which we quote from Measure Theory. [For proof, see e.g. a book on Measure Theory, or my homepage, Stochastic Processes, I.5 Lecture 8.] For the $\leq$ part, choose $a > 0$, and truncate the X_n at level a:

$$\tilde{X}_n := min(X_n, a).$$

Write $\tilde{N}_t$, $\tilde{\mu}$ for the 'tilde' analogues of N_t, μ. By Wald's identity,

$$E[\tilde{X}_1 + \cdots + \tilde{X}_{\tilde{N}_t}] = E[\tilde{X}].E[\tilde{N}_t] = \tilde{\mu}.E[\tilde{N}_t].$$

Now $\tilde{N}_t \geq N_t$ (because of the truncation, there will be more renewals if anything), and $\tilde{S}_{\tilde{N}_t - 1} + \tilde{X}_{\tilde{N}_t} \leq t + a$ (the 't' from the first term, the 'a' from the second). So

$$\begin{aligned} E[N_t]/t &\leq E[\tilde{N}_t]/t && (N_t \leq \tilde{N}_t) \\ &= \tilde{\mu}^{-1} E[\tilde{X}_1 + \cdots + \tilde{X}_{\tilde{N}_t}]/t && (\text{above – Wald's identity}) \\ &= \tilde{\mu}^{-1} E[\tilde{S}_{\tilde{N}_t}]/t && (\text{ definition of } \tilde{S}_n) \\ &\leq \tilde{\mu}^{-1} && (S_{N(t)} \leq t, \text{ and similarly for } \tilde{S}_n,\ \tilde{N}_t). \end{aligned}$$

So

$$\limsup E[N_t]/t \leq \tilde{\mu}^{-1}.$$

Now let $a \uparrow \infty$: $\tilde{\mu} \to \mu$, giving the $\leq$ part and the result. //

With F the lifetime distribution function – that of each X_i – the distribution function of $S_n := X_1 + \cdots + X_n$ is $F * \cdots * F$ (n Fs), the n-fold convolution of F with itself, written F^{*n}. Define

$$U(t) := \sum_{n=0}^{\infty} F^{*n}(t).$$

This is called the *renewal function* of F. For, it gives the mean number $E[N_t]$ of renewals up to time t:

Theorem. The renewal function gives the mean number of renewals:

$$U(t) = E[N_t].$$

So if the mean lifetime is μ,

$$U(t)/t \to 1/\mu \qquad (t \to \infty).$$

Proof.

$$\begin{aligned} E[N_t] &= \sum_{0}^{\infty} nP(N_t = n) \\ &= \sum n[P(N_t \geq n) - P(N_t \geq n+1)] \\ &= \sum P(N_t \geq n), \end{aligned}$$

by *partial summation* (or *Abel's lemma*). [This is the discrete analogue of integration by parts. See e.g. a book on Analysis, or my homepage, M3P16 Analytic Number Theory, I.3.] But $\{N_t \geq n\} = \{S_n \leq t\}$, so

$$E[N_t] = \sum P(S_n \leq t) = \sum F_n^*(t) = U(t),$$

giving the first part; the second part follows from the result above. //

The renewal theorem

Renewal theory needs a distinction between two cases. If the X_i are integer-valued (when so are the S_n), or are supported by an arithmetic progression (AP), we are in the *lattice case*, otherwise in the *non-lattice case*.

The next result looks like a differenced form of the last one. It is due to David Blackwell (1919-2010) in 1953. We state it for the non-lattice case and $\mu < \infty$, but it extends to the lattice case and $\mu = \infty$ also.

Theorem (Blackwell's renewal theorem). In the non-lattice case,

$$U(t+h) - U(t) \to h/\mu \quad (t \to \infty) \quad \forall h > 0.$$

This famous result has a number of different proofs, but we do not include one here (my favourite is only a few lines, but needs a prerequisite beyond our scope here).

Blackwell's theorem has a number of variants. The one we need (which we also quote) is due to W. L. Smith and W. Feller. Recall the *Riemann integral* (defined for functions on a finite interval), and the *Lebesgue integral* which generalises it (defined for functions on e.g. the line, plane etc.). We need a new concept.

Definition. Divide the line into intervals $I_{n,h} := (nh, (n+1)h]$. For a function z on $\mathbb{R}$ and $x \in I_{n,h}$, write

$$\overline{z}_h := sup\{z(y) : y \in I_{n,h}\}, \qquad \underline{z}_h := inf\{z(y) : y \in I_{n,h}\}.$$

Call z *directly Riemann integrable (dRi)* if $\int \overline{z_h} := \int_{-\infty}^{\infty} \overline{z}_h(x)dx$ is finite for some (equivalently, for all) $h > 0$, and similarly for $\int \underline{z_h}$, and

$$\int \overline{z}_h - \int \underline{z}_h \to 0 \qquad (h \to 0).$$

This is the same as Riemann integrability if z is supported on some finite interval, but for z of unbounded support is stronger than Lebesgue integrability: z is dRi iff it is Lebesgue integrable, and both $\int \overline{z}_h$ and $\int \underline{z}_h$ have a common limit $\int z$ as $h \to 0$. Condition dRI will hold whenever we need it. We quote that dRi needs z bounded and a.e. continuous (w.r.t. Lebesgue measure), and that this plus z of bounded support implies dRi. Also, z non-increasing and Lebesgue integrable imples dRi.

The *renewal equation* for F and z (both known) is the integral equation

$$Z(t) = z(t) + \int_0^t Z(t-u)dF(u) \quad (t \geq 0): \quad Z = z + F * Z. \qquad (RE)$$

Here F (for us, the lifetime distribution above) and z are given, and (RE) is to be solved for Z.

Theorem (Key Renewal Theorem). If z in (RE) is dRi, then for U the renewal function of F as above,

$$\lim_{t\to\infty} Z(t) = \lim_{t\to\infty} U * z(t) = \frac{1}{\mu}\int_0^\infty z(x)dx.$$

The proof of the Key Renewal Theorem from Blackwell's Renewal Theorem is not long or hard, but as it is Analysis rather than probability or insurance mathematics, we omit it. For a proof, see e.g. [RSST, 6.1.4 p216-219.

§4. The Ruin Problem.

Consider the cash flow of an insurance company. The premium income comes in from the policy holders at constant rate, c say (to a first approximation: the company hopes to attract more policy holders, and premium rates will typically vary on renewal – but are constant during the lifetime of the policy). So income over time t is ct. If the company has initial capital u, its capital at time t is thus $u + ct$. Meanwhile, claims occur. We model these as occurring at the instants of a Poisson process of rate λ, the claims being independent and identically distributed (iid) with claim distribution F, with CF ϕ, mean μ and variance σ^2. So the number of claims over the interval $[0, t]$ is $N(t)$, which is Poisson distributed with parameter λt: $N(t) \sim P(\lambda t)$. So by the Theorem of VII.2 above, the total claim has mean $\lambda\mu t$. Thus cash comes in at rate c, but goes out at rate $\lambda\mu$. This simple argument suggests – what is indeed true – that a *necessary* condition for the company to avoid bankruptcy is

$$c > \lambda\mu :$$

money should come in *faster* than it goes out. The proof is by the Strong Law of Large Numbers (LLN, as above). In the critical case $c = \lambda\mu$ the company is 'balanced on a knife-edge', and will soon go bankrupt.

The company thus *must* have $c > \lambda\mu$, so we assume this from now on. But, any insurance company has only finite funds; it may face arbitrarily severe runs of bad luck; combining these, bankruptcy is always a possibility. (Indeed, this is true for all companies, not just insurance companies! This is why bankruptcy needs to be recognised as a possibility, and governed by bankruptcy law. This varies from time to time and from country to country – a very interesting and important subject, but not one we can pursue here.)

Clearly the company's best defence against bankruptcy is to have a large cash reserve u, to act as a buffer, or 'insurance policy', against such runs of bad luck. Clearly the probability of ruin – ruin probability – decreases with u. How fast? The classical *ruin problem* is to investigate this question, to which we return below.

Note. We may if we wish take $c = 1$ for convenience. This (slightly) simplifies the formulae. It amounts to changing from real time to *operational* or *business* time – looking at the situation in the time-scale most natural to it. Recall that there are *no natural units of time or space* (except the Planck scale, at subatomic level, for those with a background in Physics!): time is measured in seconds, minutes, hours, days (60 s to the m, 60 m to the h, 24 h to the day – pre-decimal), and length in metres (metric system – mm, cm, m, km) or inches/feet/yards/miles (Imperial measure) – neither is natural, both are conventional.

The Net Profit Condition (NPC)

With c the premium rate, X_i the claim sizes and W_i the inter-claim waiting times, write

$$Z_i := X_i - cW_i.$$

Then

$$E[Z_i] := E[X_i] - cE[W_i] = \mu - c/\lambda.$$

The first term on the right measures money *out* (of the company), the second measures money *in*. As we have seen, to avoid bankruptcy we need ('more in than out')

$$E[Z_i] := E[X_i] - cE[W_i] = \mu - c/\lambda < 0 : \qquad c > \mu\lambda. \qquad (NPC)$$

This is called the *net profit condition (NPC)*. For as we have seen, $\lambda\mu$ is the claim rate (rate at which cash goes *out* to claims); c is the *premium rate* (rate at which cash comes *in*, through premiums); we need (NPC) – 'more in than out' for survival.

Safety loading and premium calculation

The first duty of any company is to stay solvent – to avoid bankruptcy.

To do this, an insurance company has to have its premium rate $c > \mu\lambda$ so as to satisfy (NPC).

But, like any other business, the insurance business is competitive. If premiums are too *low*, the firm goes bankrupt (above) because its premium income fails to meet its outgoings on claims. But if premiums are too *high*, the firm will not be competitive with other firms; over time, it will lose market share to them, and will eventually go bankrupt (or otherwise go out of business – e.g., be taken over) as premium income declines to be too small to meet overheads. So the firm needs to take a policy decision as to how much to charge in premiums. This is measured by the *safety loading (SL)*, ρ, defined by

$$c = (1+\rho)\frac{E[X_i]}{E[W_i]} = (1+\rho)\lambda\mu : \qquad \rho := \frac{c - \lambda\mu}{\lambda\mu}. \qquad (SL)$$

Thus $\rho > 0$ in (SL) is equivalent to (NPC).

Lundberg's inequality

Before, we used the characteristic function (CF), defined for a random variable X by $\phi(t) := E[e^{itX}]$, for t real. The reason for using complex numbers here – for the $i := \sqrt{-1}$ – is to ensure that the CF *always exists*. It does, because

$$|\phi(t)| = |E[e^{itX}]| \leq E[|e^{itX}|] = E[1] = 1.$$

(Recall Euler's formula: for θ real, $e^{i\theta} = cos\,\theta + isin\,\theta$, so $|e^{i\theta}| = \sqrt{cos^2\theta + sin^2\theta} = 1$. Recall also that expectation is integration (w.r.t. a probability distribution), so 'mod of integral $\leq$ integral of mod'.) But we now find it convenient to use real numbers, and switch to the *moment-generating function (MGF)*,

$$M(s) := E[e^{sX}].$$

This is certainly defined for $s = 0$: $M(0) = E[e^0] = E[1] = 1$. But it may not be defined (finite) for all (or even any) $s \neq 0$. (Example: the *exponential* distribution $E(\lambda)$ with parameter λ has MGF $\lambda/(\lambda - s)$, but this is only finite for $s < \lambda$.) We now assume the *small claim condition (SCC)*,

$$M(s) := E[e^{sX_1}] < \infty \quad \forall s \in (-s_0, s_0), \quad \text{for some } s_0 > 0. \qquad (SCC)$$

This implies that the tail of X_1 decays exponentially. For (*Markov's Inequality*): for $s \in (0, s_0)$ and $x > 0$,

$$M(s) = E[e^{sX_1}] \geq E[e^{sX_1}; X_1 > x] \geq e^{sx}E[1; X_1 > x] = e^{sx}P(X_1 > x) :$$

$$P(X_1 > x) \leq e^{-sx} M(s) \qquad \forall x > 0.$$

Differentiating the MGF twice (and writing X for X_1 for convenience):

$$M(s) = E[e^{sX}], \qquad M'(s) = E[Xe^{sX}], \qquad M''(s) = E[X^2 e^{sX}] \geq 0.$$

Also, the MGF M(s) is smooth (we can differentiate it as often as we like, where it is defined). So its graph has a tangent, and as $M'' \geq 0$, the *tangent is increasing* – the *graph bends upwards*. Such functions are called *convex*. Also, as $M(0) = 1$, the graph goes through 1 at the origin. Now smooth convex functions can intersect any line at most twice (e.g., a parabola may not cut a line, or can cut it once (double point of contact), or twice, but not more).

The crucial assumption is that $M(s)$ cuts the line $y = 1$ *twice*, once (necessarily) at the origin and once at a positive point r.

Definition.

The *Lundberg coefficient* (or *adjustment coefficient*) r, which we assume to exist in what follows, is the point $r > 0$ (we assume r exists; it is then unique) such that $r = s$ satisfies

$$M_{Z_1}(s) := E[\exp\{s(X_1 - cW_1)\}] = 1. \qquad (LC)$$

The right is (writing X, W for X_1, W_1) $M_X(s).M_W(-cs)$. Now as $W \sim E(\lambda)$, W has Laplace-Stieltjes transform (LST) $E[e^{-tW}] = M_W(-t) = \int_0^\infty e^{-tx}.\lambda e^{-\lambda x} dx = \lambda/(\lambda + t)$. So the defining property of the Lundberg (adjustment) coefficient is (writing M for M_X for short)

$$M(r).\frac{\lambda}{\lambda + cr} = 1 : \qquad M(r) = \frac{\lambda + cr}{\lambda} = 1 + \frac{cr}{\lambda}. \qquad (LC')$$

Theorem (Lundberg's Inequality). Assuming that the Small-Claims Condition (SCC) holds and that the Lundberg coefficient r in (LC) exists, the ruin probability $\psi(u)$ with initial capital u and over all time satisfies

$$\psi(u) \leq e^{-ru}.$$

Proof. Write

$$S_n := Z_1 + \cdots + Z_n, \qquad Z_i := X_i - cW_i.$$

Then $S = (S_n)$ is a *random walk*, with *step-lengths* $Z_i := X_i - cW_i$. As the ruin probability increases with time, the ruin probability $\psi(u)$ is the increasing limit of the ruin probability $\psi_n(u)$ with just the first n claims X_i and waiting times W_i involved:

$$\psi_n(u) = P(\max_{1 \le k \le n} S_k > u) = P(S_k > u \text{ for some } k \in \{1, \cdots, n\}\ ).$$

We prove that

$$\psi_n(u) \le e^{-ru} \qquad \forall n \in \mathbb{N}, u > 0. \tag{$*$}$$

The result follows from this by letting $n \to \infty$; we prove $(*)$ by induction (on n).

The induction starts, by Markov's Inequality:

$$\psi_1(u) \le e^{-ru} M_{Z_1}(r) = e^{-ru},$$

by definition of the Lundberg coefficient: $M_{Z_1}(r) = 1$.

Assume that $(*)$ holds for n, and write F for F_{Z_1}, the distribution function of Z_1. Then

$$\begin{aligned}
\psi_{n+1}(u) &= P(\max\{S_k : 1 \le k \le n+1\} > u) \\
&= P(Z_1 > u) + P(Z_1 \le u, \max\{Z_1 - (S_k - Z_1) : 2 \le k \le n+1\} > u) \\
&= p_1 + p_2,
\end{aligned}$$

say.

We now make our first use of the *renewal argument*, which will allow us to reduce the proof of our main results to an application of the Key Renewal Theorem. The idea is to *condition* on the value of the first claim Z_1, and let the process 'renew itself' with the first claim, starting afresh thereafter. So, starting the random walk after $Z_1 = x$ in the p_2-term above and conditioning on the value x of Z_1,

$$p_2 = \int_{(-\infty,u]} P(\max_{1 \le k \le n}(x + S_k) > u) dF(x).$$

In full, this is a use of the Conditional Mean Formula. For an event A, the random variable I_A (its indicator function: 1 if $\omega \in A$, 0 if not) has mean

$$E[I_A] = P(A).$$

Then conditioning on information $\mathcal{B}$ (size of first claim here),

$$P(A) = E[I_A] = E[E[I_A|\mathcal{B}]].$$

Now

$$p_1 = \int_{(u,\infty)} dF(x) \leq \int_{(u,\infty)} e^{r(x-u)} dF(x),$$

as $r > 0$, while

$$\begin{aligned} p_2 &= \int_{(-\infty,u]} P(\max_{1 \leq k \leq n}(x + S_n) > u) dF(x) \\ &= \int_{(-\infty,u]} \psi_n(u-x) dF(x) \\ &\leq \int_{(-\infty,u]} e^{r(x-u)} dF(x) \quad \text{(by the induction hypothesis).} \end{aligned}$$

Combining the domains $(-\infty, u]$ and (u, ∞) of integration here,

$$p_1 + p_2 \leq \int_{-\infty}^{\infty} e^{r(x-u)} dF(x) = e^{-ru} \int e^{rx} dF(x) = e^{-ru} M(r) = e^{-ru},$$

as $M(r) = 1$ by definition of the Lundberg coefficient r, completing the induction. //

Example: Exponential claims.

Recall the exponential distribution $E(\lambda)$ with parameter λ, which has mean $1/\lambda$ and MGF $\lambda/(\lambda - s)$. With the arrival process Poisson with rate λ as above (so the inter-claim waiting times are $E(\lambda)$), consider now the simplest case, when the claim sizes are also exponential, $E(\gamma)$ say. So W_i has MGF $\gamma/(\gamma - s)$, cW_i has MGF $\gamma/(\gamma - cs)$, and $Z_i = X_i - cW_i$ has MGF

$$M_Z(s) = M_X(s) M_{cW}(-s) = \frac{\gamma}{\gamma - s} . \frac{\lambda}{\lambda + cs}.$$

As usual, we assume the Net-Profit Condition (NPC):

$$E[X]/E[W] = \lambda/\gamma < c.$$

Then the Lundberg coefficient r is the (unique, positive) root of

$$M_Z(r) = \frac{\gamma}{\gamma - r} . \frac{\lambda}{\lambda + cr} = 1.$$

This is a quadratic,

$$Q(r) := -[(cr + \lambda)(-r + \gamma) - \lambda\gamma] = cr^2 + (\lambda - c\gamma)r = r(cr + \lambda - c\gamma) = 0,$$

with positive root

$$r = \gamma - \frac{\lambda}{c} > 0,$$

by (NPC). In terms of the safety loading ρ,

$$c = \frac{E[X]}{E[W]}(1+\rho) = \frac{\lambda}{\gamma}(1+\rho).$$

So in terms of the safety loading ρ rather than the premium rate c,

$$r = \gamma \frac{\rho}{(1+\rho)},$$

and the Lundberg inequality is

$$\psi(u) \le \exp\{-u\gamma\rho/(1+\rho)\}.$$

This is nearly exact: in this case, there is a constant C with

$$\psi(u) = C \exp\{-u\gamma\rho/(1+\rho)\}.$$

Note. This example is unusually simple: in general, there is no closed form for r, and we have to find it by numerical methods. This is typically the case for solutions of transcendental (rather than algebraic) equations.

Cumulant-generating function (CGF)
Definition. The cumulant-generating function (CGF) $\kappa(s)$ of a distribution is the logarithm of the MGF M:

$$\kappa(s) := \log M(s).$$

Thus the Lundberg (adjustment) coefficient may also be defined by

$$\kappa_{Z_1}(s) = \log M_{Z_1}(s) := \log E[\exp\{s(X_1 - cW_1)\}] = 0. \qquad (LC')$$

Like the MGF, the CGF is also *convex*. For,

$$\kappa = \log M, \qquad \kappa' = M'/M, \qquad \kappa'' = [MM'' - (M')^2]/M^2.$$

By the Cauchy-Schwarz inequality,

$$(M')^2 = (E[Xe^{sX}])^2 \le MM'' = E[e^{sX}].E[X^2 e^{sX}]$$

($E[.]$ is an integral, over the probability space Ω w.r.t. the probability measure P, or $dP(\omega)$; here we apply C-S for the measure $e^{sX(\omega)}dP(\omega)$). So $\kappa'' \ge 0$. So κ is convex. The graph of $\kappa(s)$ has two roots, $s = 0$ and $s = r$, the Lundberg (adjustment) coefficient.