Problems for the Msc in Mathematical Finance

1. Write down the definition of a probability space, $(\Omega, \mathcal{F}, P)$. Prove that countable additivity of the probability measure, P, is equivalent to the following; if (E_n) is a sequence of sets in $\mathcal{F}$ with

$$E_1 \supseteq E_2 \supseteq E_3 \ldots$$

and $\cap_n E_n = \emptyset$, then $lim_n P(E_n) = 0$.

2. Define what it means for two events, E and F in $\mathcal{F}$, to be *independent* with respect to P. Prove that if X and Y are *simple* random variables and X and Y are independent then

$$E(XY) = E(X)E(Y).$$

Is it true that $\Omega \setminus E$ and $\Omega \setminus F$ are independent?

3. Define the *Conditional Expectation*, M, of $L^2(\Omega, \mathcal{F}, P)$ onto $L^2(\Omega, \mathcal{G}, P)$, where $\mathcal{G}$ is a sub-σ field of the σ-field $\mathcal{F}$. Prove, from your definition, that for X in $L^2(\Omega, \mathcal{F}, P)$ and Y in $L^2(\Omega, \mathcal{G}, P)$ we have

$$\begin{aligned} E(M(X)) &= E(X) \\ M(Y) &= Y \\ If\ X &\geq 0\ then\ M(X) \geq 0. \end{aligned} \tag{1}$$

4. Let $(\mathcal{F}_t)$, $t \in [0, T]$ be a *filtration*[1] of σ-fields on the probability space, $(\Omega, \mathcal{F}_T, P)$. We have a family of conditional expectations, (M_t), associated with this filtration and here M_t is the conditional expectation of $\mathcal{F}_T$ onto $\mathcal{F}_t$. Prove that for $X \in L^2(\Omega, \mathcal{F}, P)$

$$M_s(M_t(X)) = M_s(X)$$

when $s \leq t$.

For $X \in L^2(\Omega, \mathcal{F}, P)$, prove that the process (X_t) given by

$$X_t = M_t(X)$$

is a martingale adapted to $(\mathcal{F}_t)$.

[1] So this is an increasing right continuous family of σ-fields

5. Recall that an event E is independent of a σ-field, $\mathcal{G}$, if E is independent of G for every $G \in \mathcal{G}$. So, a random variable, X, is independent of $\mathcal{G}$ if $X^{-1}(H)$ is independent of G, for every G in $\mathcal{G}$ and every Borel set $H \subseteq \mathcal{R}$. Now let M denote the conditional expectation of $\mathcal{F}$ onto a sub-σ-field, $\mathcal{G}$. Suppose also that X is independent of $\mathcal{G}$. Prove that

$$M(X) = E(X)I_\Omega = (\int_\Omega X dP)I_\Omega.$$

You will need properties of the conditional expectation described in Theorem M1 and you must use the independence of X and $\mathcal{G}$.

6. Recall the definition of Brownian Motion, we will denote the Brownian Motion by W. If we choose the filtration of σ-fields, $(\mathcal{F}_t)$, to be that generated by the Brownian Motion;

$$\mathcal{F}_t = \sigma\{W_s^{-1}(H) : 0 \leq s \leq t, H \text{ a Borel set in } \mathcal{R}\}$$

then, because "the increments of Brownian Motion are independent" it is easy to see that for $u \geq t$, $W_u - W_t$ is independent of the events

$$\{W_s^{-1}(H) : 0 \leq s \leq t, H \text{ a Borel set in } \mathcal{R}\}.$$

Why is this 'easy' to see? Further, can you prove that $W_u - W_t$ is independent of $\mathcal{F}_t$? You might like to consider the set of all events in $\mathcal{F}_t$ which are independent of $W_u - W_t$. Perhaps this is a σ-field.

7. By writing Brownian motion, W, as $W_t = W_t - W_s + W_s$ and using Questions 5 and 6, prove that W is a martingale adapted to the filtration , $(\mathcal{F}_t)$, generated by W.

8. Prove the *Isometry Property* for the Stochastic Integral of a simple process with respect to Brownian Motion. Let W be Brownian Motion and f a simple process. Prove that the process, X, defined by

$$X_t = \int_0^t f(s) dW_s$$

is a martingale adapted to the filtration, $\mathcal{F}_t)$, generated by Brownian Motion.

9. Let (X_t) be an L^2-martingale on $[0,T]$ adapted to $(\mathcal{F}_t)$. Suppose also that we can write
$$X_t^2 = U_t + A_t$$
for $t \in [0,T]$, where (U_t) is a martingale and (A_t) is an *increasing process*, that is the paths of A are increasing functions of $t \in [0,T]$. Let f be a simple process. Prove a generalisation of the Isometry Property of Question 8 for the stochastic integral of f with respect to X. Do this by following carefully the steps you took in Question 8 with the integral of f with respect to X.

10. Let (E_n) be a sequence of events in the probability space, $(\Omega, \mathcal{F}, P)$. We define the two limiting sets
$$\limsup_n E_n = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} E_k$$
and
$$\liminf_n E_n = \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} E_k.$$
Prove that each of these sets lies in $\mathcal{F}$. Prove that
$$\limsup_n E_n = \{\omega : \omega \in E_n, for\ infinitely\ \ many\ n \in \mathcal{N}\}$$
and
$$\liminf_n E_n = \{\omega : \omega \in E_n,\ \ for\ all\ but\ finitely\ many\ n \in \mathcal{N}\}.$$
Now suppose that the series, $\sum_n P(E_n)$ converges. Prove that
$$P(lim\, sup_n E_n) = 0.$$
Suppose now that the sequence of events, (E_n) are independent of one another and that the series $\sum_n P(E_n)$ is divergent. Prove that
$$P(lim\, sup_n E_n) = 1.$$
Hint: It is enough to show that $P(lim\, inf_n\, (\Omega \setminus E_n)) = 0$ and hence enough to show that $P(\cap_{k=n}^{\infty} (\Omega \setminus E_k)) = 0$ for each n.