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Analytical solutions for uniform potential flow past multiple cylinders

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Abstract

The problem of uniform potential flow past a circular cylinder is a basic one in fluid dynamics and the solution is well-known. In this paper, an analytical construction is presented to generalize this fundamental result to find solutions for steady irrotational uniform flow past a multi-cylinder configuration in a planar flow in the case when the circulations around the obstacles is taken to vanish. More generally, if a conformal mapping from a canonical multiply connected circular region to the unbounded fluid region exterior to a finite collection of non-cylindrical obstacles of more general shape is known, the formulation also provides solutions for the uniform flow past those obstacles.
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Keywords: Uniform potential flow; Multiply connected; Obstacles

1. Introduction

One of the most basic problems in elementary fluid dynamics is to find the streamlines associated with uniform irrotational flow past a cylindrical obstacle. Indeed, a diagram of such a streamline distribution is the very first image in Van Dyke's classic album of photographs depicting paradigmatic fluid motions [1]. Being an incompressible and irrotational flow, it has an associated complex potential [2,3]. The complex potential for uniform flow (aligned with the x-axis in an (x, y)-plane) past a unit-radius cylinder is known [2] to be given by

$$W(z) = U\left(z + \frac{1}{z}\right) \tag{1}$$

where $z = x + \mathrm{i}y$. This represents a linear superposition of a uniform flow of speed U with a dipole of strength $-2\pi U$ positioned at the centre of the cylinder. Such a solution has obvious application to basic problems in aerodynamics, to name just one of many applications.

It is natural to ask about the mathematical solution for uniform flow around multiple cylinders or, more generally, around a collection of obstacles with more general shapes. Such considerations are relevant, for example, to the calculation of flows, and associated flow diagnostics such as lift, around a stack of aerofoils. A first step is to investigate the case of just two obstacles and this problem has been a topic of recurring interest to fluid dynamicists. Analytical

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results in this case appear to have been independently rediscovered several times. The earliest investigations of the flow around two cylinders are apparently due to Hicks [4] and Greenhill [5]. Their work is discussed in the treatise by Basset [6]. Burnside [7] was aware of the work of Hicks and Greenhill and it was precisely such problems in hydrodynamics that led him to undertake his more general investigations on the theory of automorphic functions [8] (which, in fact, is related to the mathematical approach we adopt in the present paper). Later, Lagally [9] and Ferrari [10] also wrote down the solution to the problem of uniform streaming flow past two circular obstacles with specifiable round-obstacle circulations. More recently, Johnson and McDonald [11] and Burton, Gratus and Tucker [12] have independently considered the same basic two-obstacle problem with the additional ingredient of a finite collection of point vortices evolving in the fluid region exterior to two cylinders. With various civil engineering applications in mind, Burton, Gratus and Tucker [12] also study what happens when the two cylindrical obstacles are not fixed in space but move around in the flow. Concerning the problem of flow past more than two discs, hardly any analytical results exist although Burnside [8] comments on this more general case. There is some numerical work, however. Yamamoto [13] has devised an iterative numerical scheme to compute hydrodynamic forces on multiple cylinders based on the Milne-Thomson circle theorem [14].

This paper presents a new analytical framework in which to compute steady irrotational flow past a multi-cylinder array. While the focus here is on presenting solutions for flow past cylindrical/circular obstacles, the conformal invariance of the flow problem means that, up to conformal mapping, the solution procedure is also applicable to an arbitrary finite array of obstacles of general shape.

The analysis is performed in a *circular domain* D_ζ of a parametric ζ-plane consisting of the unit ζ-disc with some smaller interior circular discs excised. Such circular domains are a canonical class of multiply connected domains [15]; that is, it is known that some choice of bounded multiply connected circular domain D_ζ can be conformally mapped (by some function $z(\zeta)$) to the unbounded fluid region, D_z say, exterior to any given collection of obstacles. If $w_1(z)$ is the complex potential in D_z then we define $W_1(\zeta) = w_1(z(\zeta))$ and find explicit expressions for $W_1(\zeta)$ in the domain D_ζ. This implies that, up to knowledge of the appropriate conformal mapping $z(\zeta)$ (the existence of which is guaranteed by the Riemann mapping theorem), the solution for the complex potential associated with the flow is known.

2. Uniform flow past a cylinder

To motivate our approach, first consider uniform irrotational flow in an (x, y)-plane, with speed U parallel to the x-axis, past a single unit-radius cylinder (of infinite extent in the direction normal to the (x, y)-plane). Let $z = x + \mathrm{i}y$ be the usual complex coordinate. The complex potential $W(z)$ for the flow must tend to Uz as $z \to \infty$ and satisfy the condition that $\mathrm{Im}[W(z)]$ is constant on the surface of the cylinder. The first condition is the uniform flow condition in the far-field; the second condition ensures that the surface of the cylinder is a streamline of the flow. The solution for $W(z)$ is given in (1) and the most common derivation of it makes use of the Milne-Thomson circle theorem [2,14]. This theorem says that if the complex potential of the flow exterior to a cylinder is given by $w(z)$ in the absence of the cylinder, then the actual complex potential, $W(z)$ say, satisfying the same conditions exterior to the (unit-radius) cylinder but also the streamline condition on its surface is given by

$$W(z) = w(z) + \overline{w}\left(z^{-1}\right) \tag{2}$$

where the conjugate analytic function $\overline{w}(z)$ is defined as

$$\overline{w}(z) \equiv \overline{w(\overline{z})}. \tag{3}$$

This result is easy to verify. Solution (1) follows on use of $w(z) = Uz$ in (2).

There are (at least) two other ways to derive (1) which we will now discuss. First, let $z(\zeta)$ be a conformal mapping from the interior of the unit ζ-disc to the exterior of the unit z-disc mapping $\zeta = \beta$ to $z = \infty$ with

$$z \sim \frac{a}{\zeta - \beta}, \quad \text{as } \zeta \to \beta \tag{4}$$

where a is some constant which can be assumed to be real by a rotational degree of freedom of the Riemann mapping theorem. The point β can be chosen arbitrarily using a degree of freedom in the Riemann mapping theorem. Now, the complex potential – in a complex ζ-plane – due to a single unit-circulation point vortex at position β *inside* a unit-disc $|\zeta| < 1$ is given by $-(\mathrm{i}/2\pi)W_0(\zeta, \beta)$ where

$$W_0(\zeta, \beta) \equiv \log\left(\frac{(\zeta - \beta)}{|\beta|(\zeta - \bar{\beta}^{-1})}\right). \tag{5}$$

Provided $|\beta| < 1$, (5) has a single logarithmic singularity inside the unit disc at position β corresponding to the point vortex singularity. It can easily be verified that (5) satisfies the streamline condition on the unit circle, i.e.,

$$\mathrm{Re}\big[W_0(\zeta, \beta)\big] = c, \quad \text{on } |\zeta| = 1 \tag{6}$$

where, in fact, the constant $c = 0$. Now let $\beta = \beta_x + i\beta_y$ where β_x, β_y are the real and imaginary parts of β respectively. Define

$$W(\zeta, \beta) \equiv -\frac{Ua}{\mathrm{i}}\frac{\partial}{\partial \beta_y} W_0(\zeta, \beta). \tag{7}$$

Owing to the differentiation, $W(\zeta, \beta)$ has a simple pole at $\zeta = \beta$ instead of the logarithmic singularity exhibited by $W_0(\zeta, \beta)$. It is also easy to see from (6) that it satisfies

$$\mathrm{Im}\big[W(\zeta, \beta)\big] = d, \quad \text{on } |\zeta| = 1 \tag{8}$$

where d is some real constant. Since

$$-\frac{1}{\mathrm{i}}\frac{\partial}{\partial \beta_y} = \frac{\partial}{\partial \bar{\beta}} - \frac{\partial}{\partial \beta}, \tag{9}$$

and given that

$$\frac{\partial W_0(\zeta, \beta)}{\partial \beta} = \left(-\frac{1}{\zeta - \beta} - \frac{1}{2\beta}\right), \qquad \frac{\partial W_0(\zeta, \beta)}{\partial \bar{\beta}} = \left(-\frac{1}{\bar{\beta}^2(\zeta - \beta^{-1})} - \frac{1}{2\bar{\beta}}\right), \tag{10}$$

it follows that

$$W(\zeta, \beta) = Ua\left(\frac{\partial}{\partial \bar{\beta}} - \frac{\partial}{\partial \beta}\right)W_0(\zeta, \beta) = Ua\left(\frac{1}{\zeta - \beta} + \frac{1}{2\beta} - \frac{1}{\bar{\beta}^2(\zeta - \bar{\beta}^{-1})} - \frac{1}{2\bar{\beta}}\right). \tag{11}$$

Then (11) is the complex potential, written as a function of ζ, for uniform flow past the cylinder in the z-plane. Finally, a simple exercise reveals that taking the limit $\beta \to 0$ in (4) and (11) and setting $a = 1$ retrieves (1) to within an unimportant constant having no effect on the velocity field.

Finally, a third approach proceeds as follows. Consider the unit-radius disc in a ζ-plane and let $z(\zeta)$ be the conformal map

$$z(\zeta) = \zeta^{-1} \tag{12}$$

taking the interior of the unit ζ-disc to the exterior of the unit z-disc. Now consider the conformal mapping to a complex η-plane given by

$$\eta(\zeta) = \frac{1}{\zeta} + \zeta. \tag{13}$$

Such a mapping is the familiar *Joukowski map* [16,2] taking the interior of the unit ζ-disc to the unbounded region exterior to the finite-length slit, between ± 2, on the real axis of the η-plane. Now consider the complex potential given, as a function of η, by

$$W(\eta) = U\eta. \tag{14}$$

$W(\eta)$ has a simple pole at $\eta = \infty$ corresponding to $\zeta = 0$ which, in turn, corresponds to $z = \infty$. (14) therefore has the required simple pole at $z = \infty$ giving rise to the sought-after uniform flow there. Moreover it is clear that, everywhere on the real η-axis – corresponding to the unit circle $|\zeta| = 1$ which, in turn, corresponds to the boundary of the circular cylinder in the z-plane – $W(\eta)$ is purely real. It therefore satisfies the streamline condition on the boundary of the cylinder. It follows that $W(\eta)$ is the required complex potential. That is,

$$W\big(\eta(\zeta(z))\big) = U\eta\big(\zeta(z)\big) = U\left(\frac{1}{\zeta(z)} + \zeta(z)\right) = U\left(z + \frac{1}{z}\right) \tag{15}$$

where we have back-substituted (12) and (13).

These three different methods have been documented here for completeness. However, it is the second of these derivations that will now be generalized to yield formulae for the complex potential corresponding to uniform flow past multiple cylinders.

3. Generalized formulation

Let D_ζ be a bounded circular domain with the outer boundary, called C_0, given by $|\zeta| = 1$. Let M be a non-negative integer and let the boundaries of M smaller circular discs enclosed by C_0 be denoted $\{C_j \mid j = 1, \ldots, M\}$. $M = 0$ will give the simply connected case – that is, the single obstacle case. Let the radius of circle C_j be $q_j \in \mathbb{R}$ and let its centre be at $\zeta = \delta_j \in \mathbb{C}$. Such a domain D_ζ is $(M + 1)$-connected. Such a quadruply connected circular domain is shown in Fig. 1.

Solving the problem of uniform flow in the unbounded fluid region D_z exterior to a finite collection of obstacles is equivalent to finding a function $w_1(z)$, a complex potential that is analytic everywhere in the fluid region D_z except at infinity where it satisfies the condition that

$$w_1(z) \sim U\,\mathrm{e}^{-\mathrm{i}\chi} z + \mathcal{O}(1), \quad \text{as } z \to \infty \tag{16}$$

where U and χ are real constants. Condition (16) ensures that the flow speed at infinity is U while its direction makes an angle χ with the positive real axis. $w_1(z)$ must also satisfy the boundary conditions that $\mathrm{Im}[w_1(z)]$ is constant on the boundaries of all the obstacles. This ensures that all the obstacle boundaries are streamlines. These constant values can, in general, be different on the different obstacle boundaries and these degrees of freedom in the choice of constants are associated with the freedom to specify the flow circulations around each of the obstacles. Here we elect to specify these constants by imposing the additional conditions that, in the solutions we seek, all circulations around the obstacles will be zero.

It will be supposed that $z(\zeta)$ is a conformal mapping from the circular domain D_ζ to the fluid region D_z exterior to the collection of obstacles (e.g., a collection of circular cylinders). Suppose $\zeta = \beta$ is the point in D_ζ mapping to $z = \infty$ and that, as $\zeta \to \beta$,

$$z(\zeta) = \frac{a}{\zeta - \beta} + \mathcal{O}(1) \tag{17}$$

for some constant a. A rotational degree of freedom of the Riemann mapping theorem allows us to assume a is real. In what follows, we shall find the complex potential as a function of ζ, i.e., in the form $W_1(\zeta, \beta) \equiv w_1(z(\zeta))$.

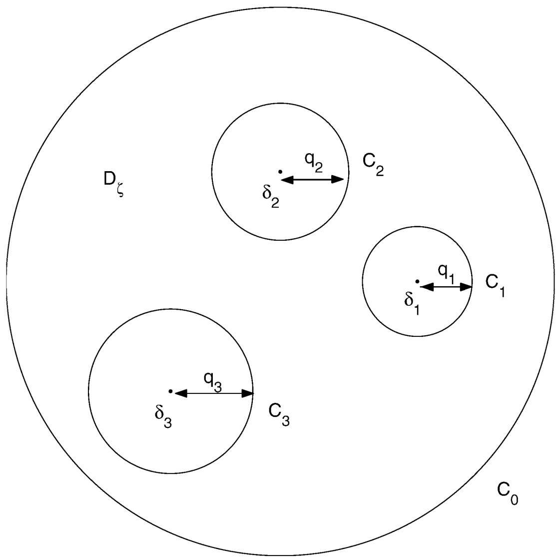

Fig. 1. Schematic of a typical multiply connected circular region D_ζ. The case shown is quadruply connected.

To make progress, we now emulate the second derivation of the simply connected result (1) described in the previous section. Let $-(\mathrm{i}/2\pi)W_0(\zeta,\beta)$ be the complex potential associated with an incompressible flow in D_ζ which is irrotational except for a single point vortex singularity at $\zeta=\beta$. Assume also that all the circulations around the M excised circular discs are zero. The function $W_0(\zeta,\beta)$ must be analytic (but not necessarily single-valued) everywhere in D_ζ except for a logarithmic singularity at $\zeta=\beta$ corresponding to the point vortex. It must also be such that

$$\mathrm{Re}\big[W_0(\zeta,\beta)\big]=0, \quad \text{on } |\zeta|=1, \tag{18}$$

and

$$\mathrm{Re}\big[W_0(\zeta,\beta)\big]=c_j, \quad \text{on } C_j,\ j=1,\ldots,M, \tag{19}$$

where c_j are some real constants. Conditions (18) and (19) ensure that all boundaries are streamlines. The choice (18) provides a normalization which uniquely determines $W_0(\zeta,\beta)$.

It is demonstrated in Crowdy and Marshall [17] that an explicit formula for the complex potential $W_0(\zeta,\beta)$ satisfying all the conditions above is

$$W_0(\zeta,\beta)=\log\left(\frac{\omega(\zeta,\beta)}{|\beta|\omega(\zeta,\bar{\beta}^{-1})}\right). \tag{20}$$

The function $\omega(\zeta,\cdot)$ is defined as follows. For each interior circle $\{C_j \mid j=1,\ldots,M\}$ of the domain D_ζ (see Fig. 1), define the conformal map

$$\theta_j(\zeta)=\frac{a_j\zeta+b_j}{c_j\zeta+d_j}, \quad j=1,\ldots,M, \tag{21}$$

where

$$a_j=q_j-\frac{|\delta_j|^2}{q_j}, \quad b_j=\frac{\delta_j}{q_j}, \quad c_j=-\frac{\overline{\delta_j}}{q_j}, \quad d_j=\frac{1}{q_j}. \tag{22}$$

Conformal maps of the linear-fractional form (21) are known as Möbius maps [16]. With the M basic Möbius maps (21) together with their M inverses $\{\theta_j^{-1} \mid j=1,\ldots,M\}$ (which are also easily shown to be Möbius maps), an infinite number of additional maps can be generated by composition of these $2M$ basic maps (it is easy to verify that the composition of two Möbius maps is another Möbius map). The infinite set of maps constructed in this way can be categorized according to their *level*. The *level one* maps will be the $2M$ maps

$$\theta_1,\theta_2,\ldots,\theta_M,\theta_1^{-1},\theta_2^{-1},\ldots,\theta_M^{-1}, \tag{23}$$

the *level two* maps will be all compositions of any *two* of the above level one maps that do not reduce to a lower level map (i.e., the identity). Some level two maps include

$$\theta_1^2,\theta_1\theta_2,\theta_1\theta_2^{-1},\ldots,\theta_{M-1}\theta_M,\theta_M^2. \tag{24}$$

The *level three* maps are all those compositions of any *three* of the basic maps which do not reduce to a lower-level map. All higher level maps are defined similarly. It is a straightforward matter to algorithmically generate the maps at each level given the basic (level one) generating maps.

Given this group of Möbius maps, the function $\omega(\zeta,\gamma)$ is defined to be

$$\omega(\zeta,\gamma)=(\zeta-\gamma)\omega'(\zeta,\gamma) \tag{25}$$

where

$$\omega'(\zeta,\gamma)=\prod_{\theta_k}\frac{(\theta_k(\zeta)-\gamma)(\theta_k(\gamma)-\zeta)}{(\theta_k(\zeta)-\zeta)(\theta_k(\gamma)-\gamma)} \tag{26}$$

and where the product is over all compositions of the basic maps $\{\theta_j,\theta_j^{-1} \mid j=1,\ldots,M\}$ *excluding* the identity map and all inverse maps. This means, for example, that if it is decided to include the level two map $\theta_1(\theta_2(\zeta))$, then the map $\theta_2^{-1}(\theta_1^{-1}(\zeta))$ must *not* be included. Note that the prime notation is not used in this paper to denote derivatives. In writing a numerical subroutine to compute the above functions, it is necessary to truncate the infinite products defining them. This can be done in a natural way by including all maps up to some level (as defined earlier). All the examples in this paper are computed by truncating the infinite products at level four, keeping all maps up to level three in the product.

Equipped with $W_0(\zeta,\beta)$ satisfying (18) and (19) and, following the result given in the simply connected case in Section 2, first consider the function

$$\phi(\zeta,\beta) = -\frac{1}{\mathrm{i}}\frac{\partial W_0(\zeta,\beta)}{\partial \beta_y} = \left(\frac{\partial}{\partial\bar{\beta}} - \frac{\partial}{\partial\beta}\right)W_0(\zeta,\beta) \tag{27}$$

where $\beta = \beta_x + \mathrm{i}\beta_y$. Owing to the differentiation, $\phi(\zeta,\beta)$ has a simple pole, with unit residue, at $\zeta = \beta$. It also satisfies

$$\mathrm{Im}\big[\phi(\zeta,\beta)\big] = d_j, \quad \text{on } C_j,\ j = 0,1,\ldots,M, \tag{28}$$

where $\{d_j \mid j = 0,1,\ldots,M\}$ are some real constants. It should also be clear that all logarithmic dependence of the function has now disappeared after the differentiation so that the function is single-valued. Next, consider the function

$$\psi(\zeta,\beta) = -\frac{\partial W_0(\zeta,\beta)}{\partial\beta_x} = -\left(\frac{\partial}{\partial\beta} + \frac{\partial}{\partial\bar{\beta}}\right)W_0(\zeta,\beta). \tag{29}$$

Owing to the differentiation, $\psi(\zeta,\beta)$ also has a simple pole, with unit residue, at $\zeta = \beta$. Furthermore, it satisfies

$$\mathrm{Im}\big[\mathrm{i}\psi(\zeta,\beta)\big] = e_j, \quad \text{on } C_j,\ j = 0,1,\ldots,M, \tag{30}$$

where $\{e_j \mid j = 0,1,\ldots,M\}$ are some real constants.

Now consider the linear combination of $\phi(\zeta,\beta)$ and $\psi(\zeta,\beta)$ given by

$$\Phi_\chi(\zeta,\beta) = \cos\chi\,\phi(\zeta,\beta) - \mathrm{i}\sin\chi\,\psi(\zeta,\beta). \tag{31}$$

This function has a simple pole, with residue $\mathrm{e}^{-\mathrm{i}\chi}$, at $\zeta = \beta$ and satisfies the conditions that

$$\mathrm{Im}\big[\Phi_\chi(\zeta,\beta)\big] = \cos\chi\,d_j - \sin\chi\,e_j, \quad \text{on } C_j,\ j = 0,1,\ldots,M. \tag{32}$$

It follows, on use of (17), that the required complex potential for uniform flow with speed U in a direction making angle χ with the positive real axis is given by

$$W_1(\zeta,\beta) = Ua\Phi_\chi(\zeta,\beta) = Ua\big[\cos\chi\,\phi(\zeta,\beta) - \mathrm{i}\sin\chi\,\psi(\zeta,\beta)\big] \tag{33}$$

since this function tends to $U\mathrm{e}^{-\mathrm{i}\chi}z$ as $z\to\infty$. It is also single-valued as ζ makes a traversal of any of the circles $\{C_j \mid j = 0,1,\ldots,M\}$ implying that the round-obstacle circulations are all zero. (33) can be written in the more convenient form

$$W_1(\zeta,\beta) = Ua\left[\mathrm{e}^{\mathrm{i}\chi}\frac{\partial}{\partial\bar{\beta}} - \mathrm{e}^{-\mathrm{i}\chi}\frac{\partial}{\partial\beta}\right]W_0(\zeta,\beta). \tag{34}$$

This is the required solution for the complex potential.

3.1. The simply connected case

First, consider the flow around a single cylinder so that $M = 0$. The fluid region is then simply connected and there are no enclosed circles to generate any Möbius maps. This means that $\omega(\zeta,\gamma) = (\zeta - \gamma)$. Substituting this into (20) and (34) and taking $\beta = 0$ with $\chi = 0$ yields (1).

3.2. The doubly connected case

The simplest non-trivial case of the above general formulation is given by the example of uniform flow past two cylinders. In such a case the flow region is an unbounded doubly connected domain. Any doubly connected domain can be obtained by a conformal mapping from some annulus $q < |\zeta| < 1$ in a parametric ζ-plane (the value of the parameter q is determined by the domain itself [15]). In this case, $\delta_1 = 0$ and $q_1 = q$, so that the single Möbius map given by (21) is $\theta_1(\zeta) = q^2\zeta$. Then it can be shown that

$$\omega(\zeta,\gamma) = -\frac{\gamma}{C^2}P\left(\frac{\zeta}{\gamma}, q\right) \tag{35}$$

where

$$P(\zeta,q)\equiv(1-\zeta)P'(\zeta,q) \tag{36}$$

and

$$P'(\zeta,q)\equiv\prod_{k=1}^{\infty}\left(1-q^{2k}\zeta\right)\left(1-q^{2k}\zeta^{-1}\right),\qquad C\equiv\prod_{k=1}^{\infty}\left(1-q^{2k}\right). \tag{37}$$

Therefore

$$W_0(\zeta,\beta)=\log\left(\frac{|\beta|P(\zeta\beta^{-1},q)}{P(\zeta\bar{\beta},q)}\right). \tag{38}$$

On use of (34), it is found that

$$W_1(\zeta,\beta)=\frac{Ua\mathrm{i}}{\beta}\sin\chi+\frac{Ua}{\beta}\left(\mathrm{e}^{-\mathrm{i}\chi}K\left(\zeta\beta^{-1},q\right)-\mathrm{e}^{\mathrm{i}\chi}K(\zeta\beta,q)\right) \tag{39}$$

where we have taken β to be real and where we define

$$K(\zeta,q)\equiv\frac{\zeta P_\zeta(\zeta,q)}{P(\zeta,q)}. \tag{40}$$

$P_\zeta(\zeta,q)$ denotes the derivative of $P(\zeta,q)$ with respect to its first argument. Of course, the first term in (39) is just a constant and can be discarded.

The solution for uniform flow past two cylinders has been known for a long time. For example, both Lagally [9] and Johnson and McDonald [11] use a formulation in terms of elliptic function theory to write down the analytical solution. Appendix A gives details of the correspondence between the solution (39) and that presented in [11]. This provides a useful check on the preceding general analysis.

Fig. 2 gives an example of the streamlines for steady uniform flow around two equal circular cylinders. In this case, the conformal mapping from the annulus $q<|\zeta|<1$ is given by the Möbius mapping

$$z(\zeta)=R\left(\frac{\zeta+\sqrt{q}}{\zeta-\sqrt{q}}\right) \tag{41}$$

where R is some real constant. It is clear that we have taken $\beta=\sqrt{q}$ and $\chi=0$ in this case. It can be verified that such a conformal mapping always maps the annulus $q<|\zeta|<1$ to two equal-radius cylinders symmetrically disposed about the origin and both centred along the real-axis. The choice of the real parameters R and q dictates the radius of each of the cylinders and the distance between them.

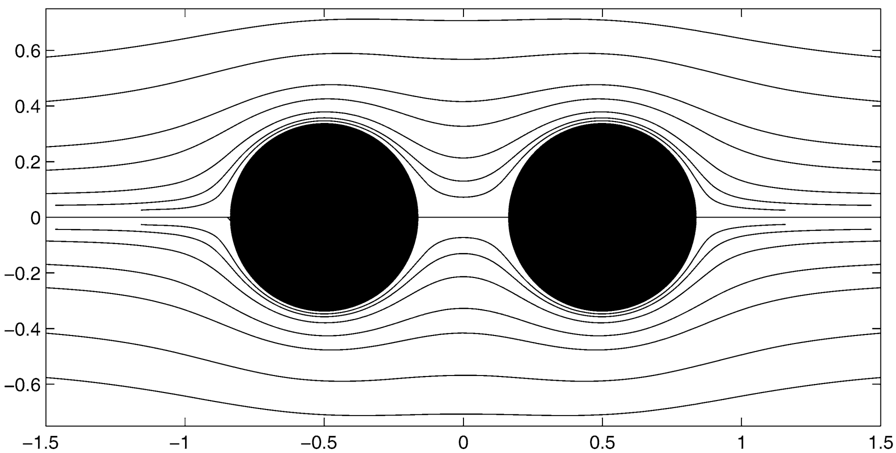

Fig. 2. Streamlines for uniform potential flow, with $\chi=0$, past two equal cylinders aligned horizontally.

4. Uniform flow around multiple cylinders

By way of example for the higher-connected case, consider the case of uniform flow around $M+1$ circular cylinders where $M \geqslant 2$. Each of the $M+1$ circles $\{C_j \mid j = 0, 1, \ldots, M\}$ will be taken to map to the boundary of one of the cylinders.

The conformal map to be used in the following examples is

$$z(\zeta) = \frac{a}{\zeta} + b \tag{42}$$

where the real parameter a is chosen to fix the radius of the cylinder corresponding to the image of $|\zeta| = 1$ and the (generally complex) parameter b is chosen to appropriately locate its centre. Clearly, we have chosen $\beta = 0$. On a minor technical point, note that in the particular case when $\beta = 0$, the function $\omega(\zeta, \infty)$ as given in (25) is not well-defined. In this case, we instead use the modified formula

$$W_0(\zeta, 0) = \log\left(\zeta \frac{\omega'(\zeta, 0)}{\omega'(\zeta, \infty)}\right) \tag{43}$$

to derive the expression for $W_1(\zeta, \beta)$.

Let Q_j be the radius and D_j the position of the centre of the j-th cylindrical obstacle ($j = 1, \ldots, M$) in the physical plane. It remains to determine the parameters $\{q_j, \delta_j \mid j = 1, \ldots, M\}$ from the known parameters $a, b, \{Q_j, D_j \mid j = 1, \ldots, M\}$. It is straightforward to show that the equations relating these parameters are given by

$$D_j = b + \frac{a\bar{\delta}_j}{|\delta_j|^2 - q_j^2} \tag{44}$$

and

$$Q_j = \frac{q_j |a|}{|\delta_j|^2 - q_j^2}. \tag{45}$$

Note that because $\zeta = 0$ maps to infinity, and therefore cannot be inside any of the circles $\{C_j \mid j = 1, \ldots, M\}$, then necessarily $|\delta_j|^2 > q_j^2$ for all $j = 1, \ldots, M$ so that $Q_j > 0$ as required.

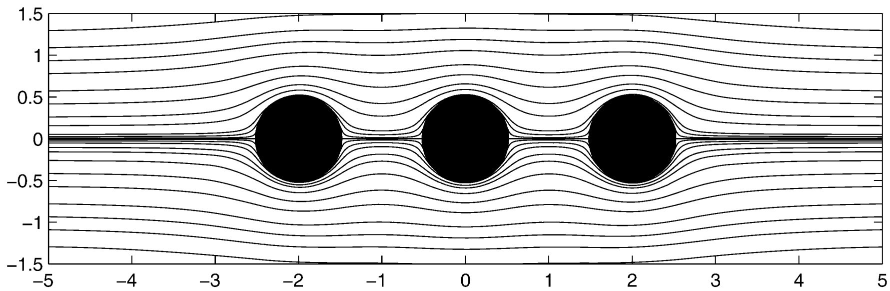

Fig. 3. Streamlines for uniform flow, with $\chi = 0$, past a line of three equal cylinders.

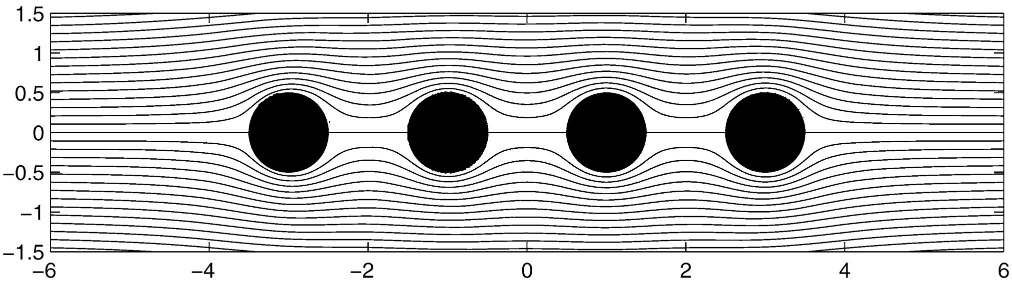

Fig. 4. Streamlines for uniform flow, with $\chi = 0$, past a line of four equal cylinders.

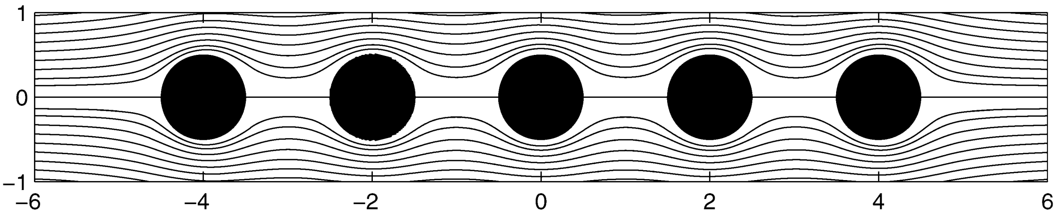

Fig. 5. Streamlines for uniform flow, with $\chi = 0$, past a line of five equal cylinders.

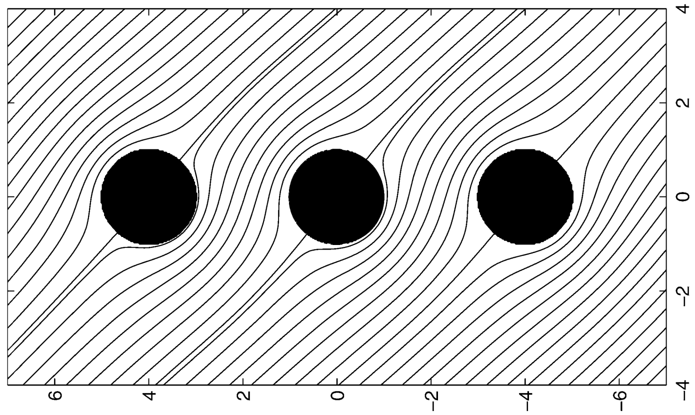

Fig. 6. Streamlines for uniform flow, with $\chi = \pi/4$, around three equal circular cylinders.

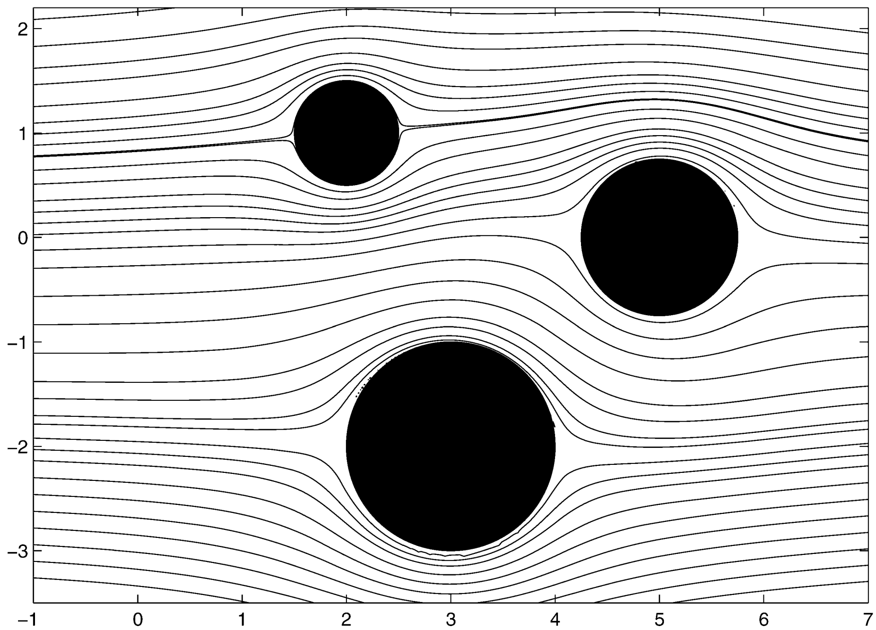

Fig. 7. Streamlines for uniform flow, with $\chi = 0$, around an arbitrary non-aligned distribution of three circular cylinders of varying radii.

To highlight the versatility of the construction, Figs. 3–5 show the steady streamline distributions associated with uniform flow, parallel to the x-axis so that $\chi = 0$, past gradually increasing numbers of identical cylinders equispaced along the x-axis. Fig. 6 shows the case of three equal cylinders with an oncoming streaming flow making an angle $\chi = \pi/4$ with the line-of-centres. Fig. 7 shows a case comprising uniform flow, with $\chi = 0$, past three non-equal cylinders arranged in a random configuration. All the figures are produced in a straightforward fashion by plotting the contours of $\mathrm{Im}[W_1(\zeta,\beta)]$ based on formulas (34) and (42).

As a final remark, it is worth pointing out – in order to make a connection between what we have done above and the third derivation of the simply connected result (1) presented in Section 2 – that the functions $W_1(\zeta,\beta)$ that we have constructed have a secondary interpretation as multiply connected mappings to parallel-slit domains. Indeed, as conformal maps, $W_1(\zeta,\beta)$ takes the circles $\{C_j \mid j=0,1,\ldots,M\}$ in the ζ-plane to a set of $M+1$ parallel slits in some image plane. They can therefore be interpreted as multiply connected generalizations of the classical (simply connected) Joukowski mapping. Note also that the approach to the two-cylinder problem due to Ferrari [10] is to consider the construction of the conformal mapping of the discs to two parallel slits and is therefore closest in spirit to the third derivation presented in Section 2.

5. Discussion

The analytical formulae presented in this paper are very general and, up to conformal mapping, give the solution for steady irrotational uniform flow past any finite distribution of obstacles. The example of flow past multiple cylindrical obstacles has been presented in detail because the conformal maps then have the simple form (42). To compute the flow past a more general configuration of obstacles, it is necessary to find the conformal mapping $z(\zeta)$ to the flow region from *some* canonical circular domain D_ζ. The parameters $\{q_j,\delta_j \mid j=1,\ldots,M\}$ determining D_ζ must be found from the given domain D_z (they are known as the *conformal moduli* [15] of the domain D_z). Then, the relevant complex potential for the flow is given, as a function of ζ, by (34). If the analytical form for $z(\zeta)$ is not known, it can nevertheless be computed numerically (using general methods described in [18] for example), and use can then be made of (34). The complex velocity field then follows from

$$u-\mathrm{i}v=\frac{W_{1\zeta}(\zeta,\beta)}{z_\zeta(\zeta)}. \tag{46}$$

An immediate extension of the results herein is that higher-order background flows, such as irrotational straining flows where the complex potential must be quadratic in z in the far-field (so that $w(z)\sim\epsilon z^2$ as $z\to\infty$ for some ϵ), can be obtained by considering higher-order derivatives of (34) with respect to the parameter β. For example the complex potential, $W_2(\zeta,\beta)$ say, for a straining flow past the collection of obstacles is given by taking some multiple of the derivative of (34) with respect to β. Note that, to generate complex potentials for higher-order background flows, parametric derivatives (with respect to the parameter β) of (34) must be taken and not derivatives with respect to the variable ζ.

A further mathematical extension, which is not obvious to perform in general, is the incorporation of non-zero circulations around the obstacles. This is important for aerodynamical applications since aerofoils with non-zero circulations in uniform flows are known to experience lift. The analysis herein has recently been extended [19] to include the effects of non-zero circulations around the obstacles sitting in the uniform flows. This means that an analytical framework now exists for computing, for example, the lift on a stack of aerofoils. Note that in the special doubly connected case (i.e. just two obstacles) it has been known for a long time how to incorporate non-zero circulation around the two obstacles (see, for example, Lagally [9] or Johnson and McDonald [11]).

Crowdy and Marshall [20] have recently presented a general analytical framework in which to study the problem of computing the motion of point vortices around an arbitrary distribution of obstacles. Such a dynamical system for the point vortex motion is Hamiltonian and, in [20], analytical formulae for the governing Hamiltonians are found in terms of the same function $\omega(\zeta,\alpha)$ used in the present analysis (this transcendental function has been dubbed the *Schottky–Klein prime function* by Baker [21]). Furthermore, the effects of any additional background flows in the point-vortex problems of [20] can be incorporated additively into the Hamiltonians once the relevant complex potential for the background flow is known. However, it is precisely such complex potentials which have been found here. The results herein can therefore be directly combined with the analysis of [20] to find the total Hamiltonians (and, hence, the equations of motion) for the dynamics of point vortices moving around obstacles in the presence of an ambient background flow.

On a technical note, it should be noted that the infinite product (25) and (26) defining the function $\omega(\zeta,\gamma)$ does not *always* converge. Its convergence depends on the distribution of circles determining the domain D_ζ (i.e., the values of $\{q_j,\delta_j \mid j=1,\ldots,M\}$). Roughly speaking, the convergence is good provided the circles in D_ζ are sufficiently small and well-separated [21]. Nevertheless, the infinite product *does* converge for a wide range of physically interesting examples such as the ones illustrated in Figs. 2–7. Moreover, the formula (20) is still valid even if the infinite product in (25) does not give a well-defined representation of the function. Work is currently in progress to identify alternative representations of $\omega(\zeta,\gamma)$ valid in those cases when the infinite product (25) is not convergent.

The mathematical problem solved here is a basic one in potential theory and the solutions can be applied to many other branches of physics. For example, the solutions also have relevance in electrostatics and give the distribution of electric field-lines around a finite array of conductors sitting in a uniform external field.

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Appendix A. The doubly connected case

We now indicate the analytical connection between the approach presented in this paper and the known solutions in the doubly connected case that have already appeared in the literature. Both Lagally [9] and Johnson and McDonald [11] employ the machinery of elliptic function theory. Here we elect to show the relation between our solution (39) and that of Johnson and McDonald [11] who write the complex potential corresponding to uniform background flow in terms of the Weierstrass zeta-function $Z(\tau)$ [22] having half periods equal to π and $i\gamma$ respectively. Indeed, the complex potential $w_B(\tau)$ given in [11] (in the first term of Eq. (2.13) of [11] with $m=0$) is

$$w_B(\tau)=\mathrm{i}\widehat{U}\left[\mathrm{e}^{-\mathrm{i}\hat{\alpha}}Z\big(\mathrm{i}(\tau-\hat{\beta})\big)-\mathrm{e}^{\mathrm{i}\hat{\alpha}}Z\big(\mathrm{i}(\tau+\hat{\beta})\big)\right]-2\mathrm{i}\widehat{U}\frac{\sin\hat{\alpha}Z(\pi)}{\pi}\tau \tag{47}$$

where $\hat{\alpha}$, $\hat{\beta}$ and $\widehat{U}$ are actually just called α, β and U in [11], but hats have been included here to distinguish from our own usage of the latter symbols.

To see how (47) relates to (39) we make the following identifications between the notation here and those used in [11]:

$$\tau=-\log\zeta,\quad \hat{\beta}=-\log\beta,\quad \gamma=-\log q,\quad \hat{\alpha}=\chi. \tag{48}$$

Note then that $\mathrm{i}\tau\mapsto\mathrm{i}\tau+2\pi$ corresponds to the transformation $\zeta\mapsto\mathrm{e}^{2\pi\mathrm{i}}\zeta=\zeta$ while $\mathrm{i}\tau\mapsto\mathrm{i}\tau+2\mathrm{i}\gamma$ corresponds to the transformation $\zeta\mapsto q^2\zeta$.

We now make use of the connection between $P(\zeta,q)$ and the first Jacobi theta function Θ_1 (see Whittaker and Watson [22]) given by

$$P(\zeta,q)=-\frac{\mathrm{i}C^{-1}\mathrm{e}^{-\tau/2}}{q^{1/4}}\Theta_1\left(\frac{\mathrm{i}\tau}{2},q\right) \tag{49}$$

where C is defined in (37). On use of (48) it follows that

$$P\big(\zeta\beta^{-1},q\big)=-\mathrm{i}C^{-1}q^{-1/4}\sqrt{\frac{\zeta}{\beta}}\,\Theta_1\left(\frac{\mathrm{i}(\tau-\hat{\beta})}{2},q\right),$$

$$P\big(\zeta\beta,q\big)=-\mathrm{i}C^{-1}q^{-1/4}\sqrt{\zeta\hat{\beta}}\,\Theta_1\left(\frac{\mathrm{i}(\tau+\hat{\beta})}{2},q\right). \tag{50}$$

By taking logarithmic derivatives of both of these expressions with respect to ζ, and then multiplying by ζ, we obtain the expressions

$$K\big(\zeta\beta^{-1},q\big)=\frac{1}{2}-\frac{\mathrm{i}}{2}\frac{\Theta_1'(\mathrm{i}(\tau-\hat{\beta})/2,q)}{\Theta_1(\mathrm{i}(\tau-\hat{\beta})/2,q)},\qquad K(\zeta\beta,q)=\frac{1}{2}-\frac{\mathrm{i}}{2}\frac{\Theta_1'(\mathrm{i}(\tau+\hat{\beta})/2,q)}{\Theta_1(\mathrm{i}(\tau+\hat{\beta})/2,q)}. \tag{51}$$

Now, on use of the identity (see Abramowitz and Stegun [23], page 650)

$$Z(z)=\frac{Z(\pi)z}{\pi}+\frac{1}{2}\frac{\Theta_1'(z/2,q)}{\Theta_1(z/2,q)}, \tag{52}$$

it follows that

$$\frac{1}{2}\frac{\Theta_1'(\mathrm{i}(\tau-\hat\beta)/2,q)}{\Theta_1(\mathrm{i}(\tau-\hat\beta)/2,q)} = Z\big(\mathrm{i}(\tau-\hat\beta)\big) - \frac{\mathrm{i}Z(\pi)}{\pi}(\tau-\hat\beta),$$
$$\frac{1}{2}\frac{\Theta_1'(\mathrm{i}(\tau+\hat\beta)/2,q)}{\Theta_1(\mathrm{i}(\tau+\hat\beta)/2,q)} = Z\big(\mathrm{i}(\tau+\hat\beta)\big) - \frac{\mathrm{i}Z(\pi)}{\pi}(\tau+\hat\beta), \quad (53)$$

from which, on use of (53) in (51), it can be deduced that

$$K\big(\zeta\beta^{-1},q\big) = \frac{1}{2} - \mathrm{i}\bigg(Z\big(\mathrm{i}(\tau-\hat\beta)\big) - \frac{\mathrm{i}Z(\pi)}{\pi}(\tau-\hat\beta)\bigg),$$
$$K(\zeta\beta,q) = \frac{1}{2} - \mathrm{i}\bigg(Z\big(\mathrm{i}(\tau+\hat\beta)\big) - \frac{\mathrm{i}Z(\pi)}{\pi}(\tau+\hat\beta)\bigg). \quad (54)$$

Substitution of (54) into (39) yields

$$W_1(\zeta,\beta) = -\mathrm{i}\frac{Ua}{\beta}\big(\mathrm{e}^{-\mathrm{i}\chi}Z\big(\mathrm{i}(\tau-\hat\beta)\big) - \mathrm{e}^{\mathrm{i}\chi}Z\big(\mathrm{i}(\tau+\hat\beta)\big)\big) + \frac{2\mathrm{i}Ua}{\beta}\frac{Z(\pi)\sin\chi}{\pi}\tau + \text{constant} \quad (55)$$

which is exactly the same as (47) to within a constant of proportionality (they are identical under the correspondences (48) if we also take $\widehat{U} = -Ua/\beta$) and an unimportant additive constant.

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