The irrotational motion generated by two planar stirrers in inviscid fluid

D. G. Crowdy, A. Surana, and K-Y. Yick
Department of Mathematics, Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge, Massachusetts 02139 and Department of Mechanical Engineering, Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge, Massachusetts 02139

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The irrotational motion of an inviscid incompressible fluid driven by two objects, of arbitrary shape, moving at specified velocities in a two-dimensional fluid region is determined. The problem is shown to be equivalent to a standard mathematical problem in potential theory known as the modified Schwarz problem. The solution is given, up to conformal mapping, by the classical Villat formula. © *2007 American Institute of Physics*. [DOI: 10.1063/1.2432155]

There has been a revival of interest in computing the irrotational flow generated by two objects moving in a planar incompressible inviscid fluid. Recently, in this journal, Wang[1] considered the problem of the irrotational motion generated by two moving cylindrical objects. At about the same time, Burton, Gratus, and Tucker[2] studied the same problem from a different mathematical viewpoint. An independent resurgence of interest in this basic problem has also occurred since Boyland, Aref, and Stremler[3] proposed the use of a circular disk containing ideal fluid and a finite number of circular-disk "stirrers" as a basic model (a "batch stirring device") for understanding the topological fluid mechanics associated with mixing/stirring. Other studies on such systems have been performed subsequently (e.g., Ref. 4).

Both Wang[1] and Burton, Gratus, and Tucker[2] have given different mathematical solutions to this problem. The purpose of this paper is to point out the fact, not mentioned in any of the above references, that this physical problem can be reduced to a standard mathematical problem of potential theory known as the *modified Schwarz problem*.[5] Moreover, in the case of two cylinders (so that the fluid region is an unbounded, doubly connected region), this problem admits a celebrated integral formula solution known as the *Villat formula*.[6] The form of this solution is not only compact, but also readily implementable in practice. It also yields a formula that is valid for stirrers of arbitrary shape (not just circular disks).

We seek an incompressible, irrotational flow in the unbounded fluid region D exterior to two solid objects D_0 and D_1. Figure 1 shows a schematic. There exists a velocity potential ϕ such that $\mathbf{u}=\nabla\phi$. The flow is assumed to decay at infinity (background flows can be incorporated additively if required) and be such that

$$\mathbf{u}\,.\,\mathbf{n}=\mathbf{U}_0\,.\,\mathbf{n}, \quad \text{on } \partial D_0,$$
$$\mathbf{u}\,.\,\mathbf{n}=\mathbf{U}_1\,.\,\mathbf{n}, \quad \text{on } \partial D_1, \tag{1}$$

where $\mathbf{U}_j$, $j=0,1$ denote the imposed velocities of the two moving objects. Since the flow is incompressible, $\nabla^2\phi=0$ in D. It will further be supposed that the flow circulation around each of the moving objects is zero.

Suppose $\mathbf{u}=(u,v)$ and consider a complex variable formulation of the problem. In what follows, if $\mathbf{a}=(a_x,a_y)$ denotes a vector, then the notation a will be used to denote its natural complexification $a=a_x+ia_y$. It is well known that the solution of the mathematical problem just stated is equivalent to finding a complex potential $w(z)$, analytic in D and decaying at infinity, satisfying the boundary conditions (1) where $dw/dz=u-iv$. The complex form of the unit tangent vector is z_s, where s denotes arclength, while the complex form of the unit normal is $-iz_s$. Note that, if $\mathbf{a}$ and $\mathbf{b}$ denote two vectors (with complex counterparts a and b), then $\mathbf{a}\,.\,\mathbf{b}=\mathrm{Re}[\overline{a}b]$. It follows that conditions (1) become

$$\mathrm{Re}\left[\frac{dw(z)}{dz}(-iz_s)\right]=\mathrm{Re}[\overline{U_0}(-iz_s)], \quad \text{on } \partial D_0,$$
$$\mathrm{Re}\left[\frac{dw(z)}{dz}(-iz_s)\right]=\mathrm{Re}[\overline{U_1}(-iz_s)], \quad \text{on } \partial D_1. \tag{2}$$

A crucial observation is that these conditions can be integrated with respect to s to give

$$\mathrm{Re}[-iw(z)]=\mathrm{Re}[-i\overline{U_0}z]+d_0, \quad \text{on } \partial D_0,$$
$$\mathrm{Re}[-iw(z)]=\mathrm{Re}[-i\overline{U_1}z]+d_1, \quad \text{on } \partial D_1, \tag{3}$$

where d_0 and d_1 are two real constants of integration (in an unsteady flow, d_0 and d_1 will be functions of time, but not space). Without loss of generality, d_0 can always be set to zero.

It is a consequence of the Riemann mapping theorem that any doubly connected domain D in a complex z plane can be mapped, via some conformal map $z(\zeta)$, from an annulus in the ζ plane given by $\rho<|\zeta|<1$, where ρ is the conformal modulus (determined by the choice of domain D). Define

$$W(\zeta)\equiv w(z(\zeta)). \tag{4}$$

Equation (3) then becomes the following boundary-value problem for $W(\zeta)$ analytic in the annulus $\rho<|\zeta|<1$:

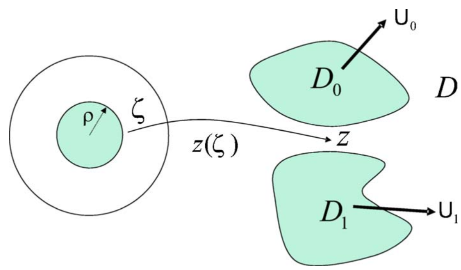

FIG. 1. Schematic of the problem. Two objects, D_0 and D_1, moving at prescribed velocities $\mathbf{U_0}$ and $\mathbf{U_1}$, generate an irrotational flow in the fluid region D. $z(\zeta)$ is a conformal mapping to D from a preimage annulus $\rho<|\zeta|<1$.

$$\mathrm{Re}[-iW(\zeta)] = \mathrm{Re}[-i\overline{U_0}z(\zeta)], \quad \text{on } |\zeta|=1,$$
$$\mathrm{Re}[-iW(\zeta)] = \mathrm{Re}[-i\overline{U_1}z(\zeta)] + d_1, \quad \text{on } |\zeta|=\rho. \tag{5}$$

However, this is the modified Schwarz problem[5] for an analytic function $W(\zeta)$ in an annulus given its real part on the boundary. The solution for a single-valued analytic function $W(\zeta)$ exists provided d_1 satisfies the compatibility condition

$$\frac{1}{2\pi i}\oint_{|\zeta|=1} \frac{d\zeta}{\zeta}\,\mathrm{Re}[-i\overline{U_0}z(\zeta)] = \frac{1}{2\pi i}\oint_{|\zeta|=\rho} \frac{d\zeta}{\zeta}\{\mathrm{Re}[-i\overline{U_1}z(\zeta)] + d_1\}. \tag{6}$$

If (6) is satisfied, $W(\zeta)$ is given explicitly by the *Villat formula*[6]

$$-iW(\zeta) = I_+(\zeta) - I_-(\zeta) + I_c + iC, \tag{7}$$

where

$$I_+(\zeta) = \frac{1}{2\pi i}\oint_{|\zeta'|=1} \frac{d\zeta'}{\zeta'} K(\zeta/\zeta')\mathrm{Re}[-i\overline{U_0}z(\zeta')],$$

$$I_-(\zeta) = \frac{1}{2\pi i}\oint_{|\zeta'|=\rho} \frac{d\zeta'}{\zeta'} K(\zeta/\zeta')\{\mathrm{Re}[-i\overline{U_1}z(\zeta')] + d_1\}, \tag{8}$$

$$I_c = -\frac{1}{2\pi i}\oint_{|\zeta'|=\rho} \frac{d\zeta'}{\zeta'}\{\mathrm{Re}[-i\overline{U_1}z(\zeta')] + d_1\},$$

and where

$$K(\zeta) \equiv 1 - 2\zeta\frac{P'(\zeta)}{P(\zeta)} \tag{9}$$

with

$$P(\zeta) \equiv (1-\zeta)\prod_{k=1}^{\infty}(1-\rho^{2k}\zeta)(1-\rho^{2k}\zeta^{-1}). \tag{10}$$

C is an arbitrary real constant whose value is inconsequential for determining the flow. The solution is now complete. Equation (7) is, in fact, more general than the solution given by Wang[1] since it is the expression for the relevant complex potential for *any* doubly connected fluid domain D given the conformal mapping $z(\zeta)$ from the annulus $\rho<|\zeta|<1$ to the region D. To illustrate this, Fig. 2 shows the instantaneous streamlines generated by two moving straight "paddles" in an irrotational fluid; the left paddle has speed $(0,1)$ and the right paddle has speed $(0,-1)$. The conformal mapping needed here (mapping an annulus to the exterior of two straight line slits) was derived by Crowdy and Marshall (see Fig. 8 of Ref. 11). The map used to produce Fig. 2 is given by

$$z(\zeta) = \mathcal{A}\left(\frac{P^2(-\zeta) + R^2P^2(\zeta)}{P^2(-\zeta) - R^2P^2(\zeta)}\right) \tag{11}$$

with $\mathcal{A}=1/5$, $R=1/2$, and $\rho=1/10$. For a given ζ, the integrals in (8) are best computed by using a trapezoidal rule for the numerical integration since this method gives superalgebraic convergence for the integration of periodic functions on periodic domains.

As a second illustrative example of the usefulness of formula (7), we consider modeling the wake behind a cylinder moving at constant speed toward a straight plane wall. This example is motivated by recent experiments by Leweke, Thompson, and Hourigan,[7] who studied the behavior of the wake behind a sphere colliding with a wall. They observed a pronounced rebound of the primary wake vortices, which is understood to be associated with viscous effects, and the secondary wake structure formed during the evolution. Saffman[9] has shown that no rebound is possible as a vortex pair approaches a solid wall in an inviscid fluid, but it is conceivable that image vorticity associated with the presence of the cylinder might affect this result. To examine this possibility, we considered a circular cylinder of unit radius traveling at constant speed toward a straight wall with a trailing Föppl point vortex wake. The flow is assumed to be inviscid, incompressible, and irrotational (apart from the trailing point vortex pair). When the cylinder and wake are far from the

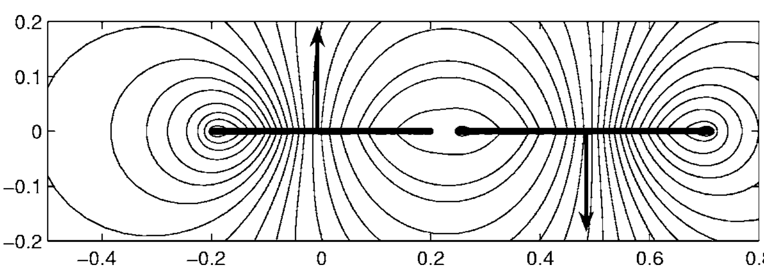

FIG. 2. Instantaneous streamlines generated by two moving paddles in an irrotational fluid. The paddle on the left has speed $(0,1)$, the paddle on the right has speed $(0,-1)$.

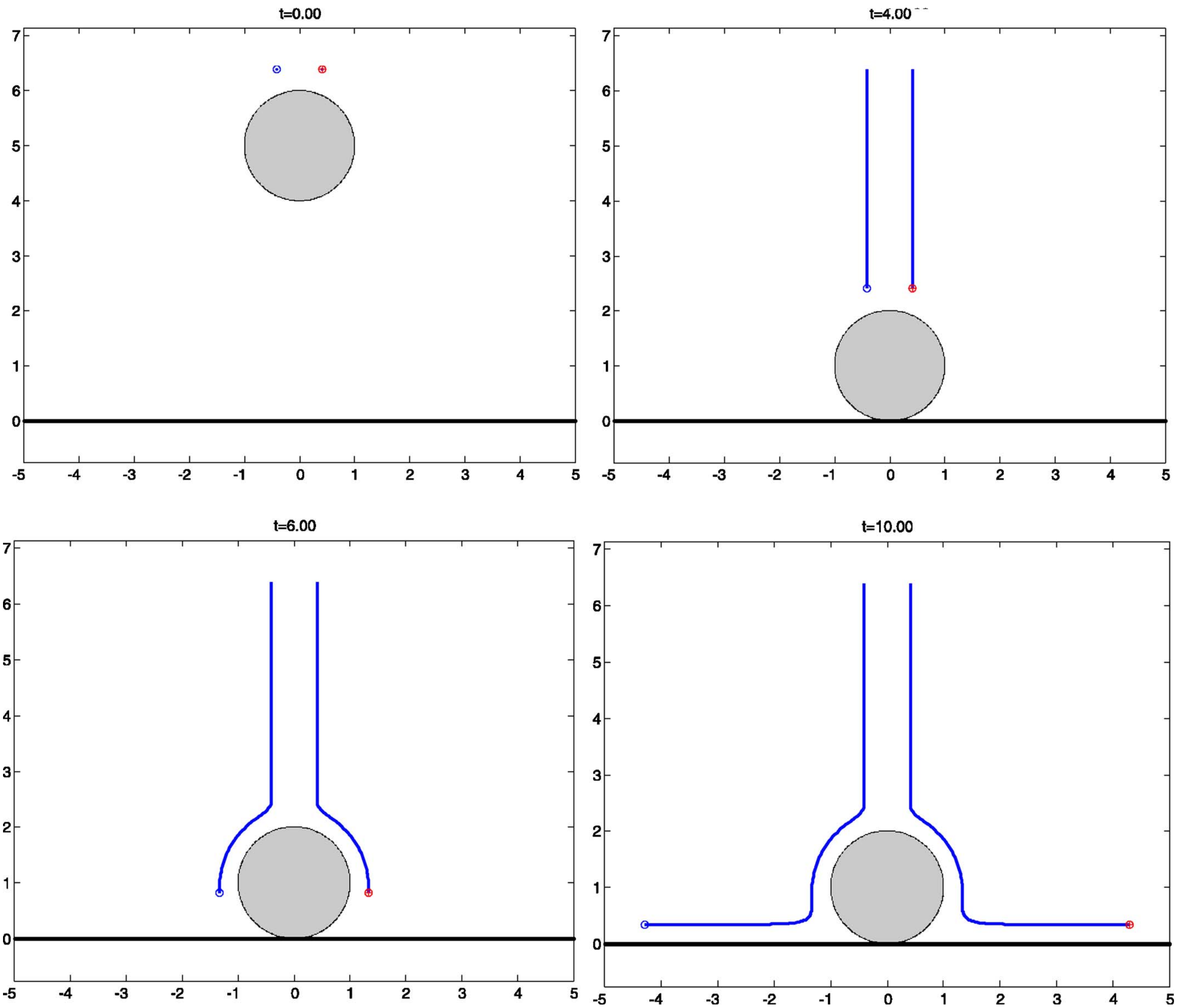

FIG. 3. Trajectories of a Föppl vortex pair behind a cylinder approaching a wall at constant speed and coming impulsively to rest just before contact. In inviscid theory, no rebound of the wake vortices is observed.

wall, the configuration should be given by the usual Föppl vortex solution.[8] As the wall is approached, however, the dynamics will become more complicated. However, it can be calculated in an analytic formulation using classical Kirchhoff-Routh theory[8] for the point vortex motion (generalized to the doubly connected case[10]), together with an imposed "background flow" contribution arising from the externally imposed motion of the cylinder toward the wall. It is precisely this background flow contribution that is given by formulas (7) and (8). Figure 3 shows the vortex trajectories for a cylinder approaching a wall at constant speed and coming impulsively to rest just before it touches the wall. Here, if $h(t)$ denotes the (externally specified) height of the cylinder above the wall, the relevant time-dependent conformal mapping from $\rho(t)<|\zeta|<1$ to the (doubly connected) fluid region is given by

$$z(\zeta,t) = iA(t)\left(\frac{1-\zeta}{1+\zeta}\right), \quad A(t) = \sqrt{h^2(t)-1},$$
$$\rho(t) = \sqrt{\frac{h(t)-A(t)}{h(t)+A(t)}}. \tag{12}$$

This mapping can be used both to construct the Kirchhoff-Routh path function and also used in (7) and (8) to compute the background flow contribution. The details follow from a direct application of the theory given in Ref. 10. No rebound of the vortex pair is observed in this purely inviscid calculation.

In summary, we have pointed out a connection between the problem of finding the potential flow induced by two moving objects (or arbitrary shape) in inviscid fluid and the classical modified Schwarz problem of potential theory.

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