EXACT SOLUTIONS TO THE UNSTEADY TWO-PHASE HELE-SHAW PROBLEM

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Summary

While many explicit solutions to the single-phase Hele-Shaw problem are known, solutions to the two-phase problem (also known as the 'Muskat problem') are scarce. This paper presents a new class of exact time-dependent solutions to the two-phase Hele-Shaw problem. It is demonstrated that an elliptical inclusion of one phase remains elliptical under evolution when immersed in any unsteady far-field linear flow of a second ambient phase. On the basis of this solution class, an 'elliptical inclusion model' for interactions in inhomogeneous porous media is outlined.

1. Introduction

The Hele-Shaw problem, in which fluid of one viscosity displaces a fluid of a different viscosity in the space between two glass plates, has been an important paradigm for free-surface dynamics for over a century. The literature on this, and related problems, is vast. A bibliography containing references up to the late nineties has been compiled by Howison (**1**). One reason for its importance is that the governing equations are identical to those governing the motion of interfaces in porous media. Thus, in one respect, the Hele-Shaw cell can be considered a simple apparatus for realizing flows in porous media.

It is well known that the one-phase Hele-Shaw problem, in which the less viscous fluid is assumed to be simply a region of constant pressure, is amenable to a number of analytical solution techniques. This has led to a large variety of exact mathematical solutions for both the steady and the unsteady evolution of the interface. For example, the form of a steady viscous finger in a Hele-Shaw channel was studied by Saffman and Taylor (**2**); Saffman (**3**) went on to show that an exact time-dependent solution for the finger can be found. Galin (**4**), Polubarinova-Kochina (**5**) and Richardson (**6**) have all found exact time-dependent solutions to the one-phase problem in other geometries using a variety of mathematical techniques, a key component of which is the description of the evolving interface in terms of a time-dependent conformal map. Shraiman and Bensimon (**7**) later formulated a theory of 'pole dynamics' for such free boundary problems, which is intimately related to these earlier theories of exact solutions.

By contrast, in the two-phase Hele-Shaw problem where the dynamics of each of the two fluids is now resolved equally, almost all of the exact solution techniques relevant to the single-phase problem simply fail. This problem is also known as the 'Muskat problem' (**8**). It is a well-known fact that many two-phase systems are generally not amenable to straightforward analysis by means

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of conformal mapping techniques. The reason is that while some simple preimage region, such as the interior of a unit disk, in a parametric plane might correspond, under a one-to-one conformal mapping, to the region occupied by one phase, it is generally *not* true that the exterior of the unit disk corresponds to the region occupied by the second phase. However, as will be shown here, there are certain special scenarios involving two-phase flow where conformal mapping techniques can be usefully employed.

A very small number of exact mathematical solutions of this type *are* known. It is clear that in the steadily-translating Saffman–Taylor finger solution, the constant pressure region can be trivially replaced by a region of viscous fluid in uniform translation—a fact pointed out by the original authors themselves (**2**). Howison (**9**) has listed several other simple solutions including travelling wave solutions in a channel, radially symmetric solutions and a stagnation point flow solution. To the best of this author's knowledge, the only non-trivial time-dependent exact solutions to the two-phase problem are due to Jacquard and Séguier (**10**). It turns out that the shape of the interface in their solution can be described by the same conformal mapping relevant to Saffman's (**3**) solution of the one-phase problem (although the time evolution of the parameters in the map is different). The solution scheme of (**10**) appears to be somewhat serendipitous. Howison (**9**) has reappraised the Jacquard–Séguier solution and proposed a more general context, an 'inverse method', in which to derive it, and possibly other solutions, in a more systematic manner. The more general mathematical question of the global existence and well-posedness of the Muskat problem has also been the focus of recent investigations by Siegel, Caflisch and Howison (**11**) and Ambrose (**12**).

This paper derives a class of exact time-dependent solutions of the two-phase Hele-Shaw problem that does not appear to have been reported previously. The solutions involve a 'radial geometry' (as opposed to the channel geometry of (**10**)) in which an unbounded region of one fluid evolves while containing a bounded elliptical inhomogeneity (or inclusion) of a second fluid. As for the single-phase problem in which a less viscous inclusion is modelled as a 'bubble' of constant pressure surrounded by a more viscous fluid, Howison (**13**) has discussed similarities between the problem in a channel and a radial geometry, especially in the context of the fingering phenomenon. Paterson (**14**) contributed some earlier work on fingering, for the single-phase problem, in a radial geometry.

2. Problem formulation

Consider the two-phase Hele-Shaw problem in an unbounded planar region. Let a bounded inclusion of one fluid (called fluid 2) be embedded in an unbounded region of another fluid (called fluid 1). Let the mobility of fluid 1 be k_1 and that of the fluid 2 be k_2; in the Hele-Shaw problem, $k_i = h^2/12\mu_i$ where μ_i denote the viscosities of the two fluids and h is the gap width of the cell. Let $\mathbf{u}_{1,2}$ and $p_{1,2}$ be, respectively, fluid velocities and pressures in each phase. Then

$$\mathbf{u}_1 = \nabla\phi_1, \quad \mathbf{u}_2 = \nabla\phi_2, \tag{2.1}$$

where

$$\phi_1 = -k_1 p_1, \quad \phi_2 = -k_2 p_2. \tag{2.2}$$

Under the assumption that there is no surface tension on the interface, one condition that must hold there is that the fluid pressures must be continuous so that

$$p_1 = p_2, \quad \text{on the interface.} \tag{2.3}$$

It is also necessary that the normal fluid velocities must be equal at the interface, and further, must give the normal velocity V_n of the interface. Thus, at the interface,

$$\frac{\partial \phi_1}{\partial n} = \frac{\partial \phi_2}{\partial n} = V_n. \tag{2.4}$$

It is condition (2.4) that governs the time evolution of the interface.

3. The solution method

Let the complex potentials in the two fluid regions be denoted by $w_1(z,t)$ and $w_2(z,t)$ so that

$$\phi_1 = \mathrm{Re}[w_1(z,t)], \quad \phi_2 = \mathrm{Re}[w_2(z,t)]. \tag{3.1}$$

The condition that the pressures are equal on the interface implies that

$$w_1(z,t) + \overline{w_1(z,t)} = \Lambda\left(w_2(z,t) + \overline{w_2(z,t)}\right), \tag{3.2}$$

where $\Lambda = k_1/k_2$. Equivalently, on differentiating this with respect to z, the condition is

$$w_1'(z,t) + \overline{w_1'(z,t)}\frac{d\bar{z}}{dz} = \Lambda\left(w_2'(z,t) + \overline{w_2'(z,t)}\frac{d\bar{z}}{dz}\right), \tag{3.3}$$

where the prime notation denotes the derivative with respect to the first argument of the function. The condition that the normal velocities of the two fluids are equal at the interface is equivalent to

$$\mathrm{Im}[w_1'(z,t)dz] = \mathrm{Im}[w_2'(z,t)dz], \tag{3.4}$$

or

$$w_1'(z,t) - \overline{w_1'(z,t)}\frac{d\bar{z}}{dz} = w_2'(z,t) - \overline{w_2'(z,t)}\frac{d\bar{z}}{dz}. \tag{3.5}$$

Adding (3.3) and (3.5) yields the relation

$$w_1'(z,t) = \frac{1}{2}\left((\Lambda+1)w_2'(z,t) + (\Lambda-1)\overline{w_2'(z,t)}\frac{d\bar{z}}{dz}\right). \tag{3.6}$$

We now restrict attention to a particular class of time-dependent flows which will be shown to admit exact solutions. First, we suppose that the flow in fluid region 2 is given by a complex potential having a derivative of the linear form

$$w_2'(z,t) = \epsilon(t)z + \delta(t) \tag{3.7}$$

for some complex-valued functions $\epsilon(t)$ and $\delta(t)$. Further, it will be assumed that the enclosed region 2 has an elliptical boundary at all times in the evolution, with its geometrical centre positioned at some (complex) point $\gamma(t)$. It will be shown that (3.7) is consistent with the preservation of the elliptical shape of the inclusion.

3.1 *Conformal mapping*

To proceed with the analysis it is convenient to introduce a conformal mapping from the unit ζ-disk in a parametric ζ-plane to fluid region 1. The fact that fluid region 2 is an elliptical inclusion implies that such a conformal mapping has the functional form

$$z(\zeta, t) = \gamma(t) + \frac{\alpha(t)}{\zeta} + \beta(t)\zeta, \tag{3.8}$$

where $\alpha(t)$ is a function of time that can be assumed to be real (by a rotational degree of freedom of the Riemann mapping theorem) while $\beta(t)$ is some generally complex function. Henceforth, for convenience, we suppress the explicit dependence of the parameters on time.

On the interface between the fluids where $\bar{\zeta} = \zeta^{-1}$, one can write

$$\bar{z} = \overline{\gamma} + \alpha\zeta + \frac{\bar{\beta}}{\zeta}. \tag{3.9}$$

However, it is also true, from (3.8), that

$$\frac{1}{\zeta} = \frac{z}{\alpha} - \frac{\gamma}{\alpha} - \frac{\beta}{\alpha}\zeta, \tag{3.10}$$

which is a relation valid everywhere inside the unit ζ-disk and, in particular, on its boundary. It follows that, on the interface,

$$\bar{z} = \overline{\gamma} - \frac{\bar{\beta}\gamma}{\alpha} + \frac{\bar{\beta}z}{\alpha} + \left(\alpha - \frac{|\beta|^2}{\alpha}\right)\zeta. \tag{3.11}$$

The right-hand side of (3.11), which is purely an analytic function of the complex variable ζ, can now be used to provided the analytic continuation of $\bar{z}$ off the unit ζ-circle (where relation (3.11) is originally valid) and into the interior of the unit ζ-disk (that is, into fluid region 1). It might be mentioned that, considered as a function of z (or ζ) the analytic continuation of $\bar{z}(z)$ off the interface is sometimes referred to as the *Schwarz function* (**15**) of the interface. Since $\zeta \to 0$ as $z \to \infty$, observe that the right-hand side is analytic everywhere in fluid region 1, becoming singular only as $z \to \infty$ where it tends to a linear function of z. Note also that, on differentiation with respect to z, we also have that

$$\frac{d\bar{z}}{dz} = \frac{\bar{\beta}}{\alpha} + \left(\alpha - \frac{|\beta|^2}{\alpha}\right)\left(\frac{dz}{d\zeta}\right)^{-1}. \tag{3.12}$$

Similarly, (3.12) provides the analytic continuation of $d\bar{z}/dz$ into fluid region 1 and, since $z_\zeta(\zeta)$ admits no zeros and is analytic inside the unit ζ-disk, it follows that $d\bar{z}/dz$ is analytic everywhere inside region 1, tending to a constant as $z \to \infty$.

Making use of (3.11), (3.12) and (3.7) in (3.6) yields the following explicit expression for the (derivative of the) complex potential governing the motion in fluid 1:

$$w_1'(z, t) = \frac{\Lambda + 1}{2}(\epsilon z + \delta) + \frac{\Lambda - 1}{2}\left[\bar{\delta} + \bar{\epsilon}\left(\bar{\gamma} - \frac{\bar{\beta}\gamma}{\alpha}\right) + \frac{\bar{\epsilon}\bar{\beta}}{\alpha}z + \bar{\epsilon}\left(\alpha - \frac{|\beta|^2}{\alpha}\right)\zeta\right]$$
$$\times\left(\frac{\bar{\beta}}{\alpha} - \frac{(\alpha^2 - |\beta|^2)\zeta^2}{\alpha(\alpha - \beta\zeta^2)}\right). \tag{3.13}$$

In contrast to the simple linear flow taking place in fluid 2, the flow in region 1 is clearly more complicated. A crucial observation is that it is analytic everywhere in fluid region 1 and, as $|z| \to \infty$,

$$w_1'(z,t) \to \left[\frac{(\Lambda+1)\epsilon}{2} + \frac{(\Lambda-1)\bar{\beta}^2\bar{\epsilon}}{2\alpha^2}\right] z + \frac{(\Lambda+1)\delta}{2} + \frac{(\Lambda-1)\bar{\beta}}{2\alpha} \times \left[\bar{\delta} + \bar{\epsilon}\left(\bar{\gamma} - \frac{\bar{\beta}\gamma}{\alpha}\right)\right] + \mathcal{O}(z^{-1}). \tag{3.14}$$

In the far field the flow is linear, comprising an irrotational strain superposed with a uniform flow.

While we have found the instantaneous complex potentials in both fluid regions, it remains to compute the evolution of the boundary to examine whether the elliptical shape of fluid region 2 is preserved under the dynamics. The kinematic condition governing the interface motion is equivalent to

$$-\mathrm{Im}\left[\frac{\partial z}{\partial t}\frac{\partial \bar{z}}{\partial s}\right] = \mathrm{Im}\left[w_2'(z,t)\frac{\partial z}{\partial s}\right]. \tag{3.15}$$

It is convenient to introduce the notation

$$z_t \equiv \frac{\partial z}{\partial t}, \quad z_\zeta \equiv \frac{\partial z}{\partial \zeta}. \tag{3.16}$$

On use of the condition that

$$\frac{\partial z}{\partial s} = -\frac{i\zeta z_\zeta}{|z_\zeta|}, \tag{3.17}$$

while

$$z_t(\zeta,t) = \dot{\gamma} + \frac{\dot{\alpha}}{\zeta} + \dot{\beta}\zeta, \quad \zeta z_\zeta(\zeta,t) = -\frac{\alpha}{\zeta} + \beta\zeta, \tag{3.18}$$

where, for convenience, we use dots to denote time derivatives, (3.15) takes the form

$$\mathrm{Re}\left[\left(-\alpha\zeta + \frac{\bar{\beta}}{\zeta}\right)\left(\dot{\gamma} + \frac{\dot{\alpha}}{\zeta} + \dot{\beta}\zeta\right)\right] = \mathrm{Re}\left[\left(\epsilon\left(\gamma + \frac{\alpha}{\zeta} + \beta\zeta\right) + \delta\right)\left(-\frac{\alpha}{\zeta} + \beta\zeta\right)\right]. \tag{3.19}$$

On use of the fact that $\bar{\zeta} = \zeta^{-1}$ on $|\zeta| = 1$, algebraic manipulations reveal that (3.19) is equivalent to the system of three nonlinear ordinary differential equations given by

$$\begin{aligned} &2\dot{\alpha}\alpha - \beta\overline{\dot{\beta}} - \dot{\beta}\bar{\beta} = 0, \\ &-\alpha\dot{\gamma} + \beta\overline{\dot{\gamma}} = \beta(\epsilon\gamma + \delta) - \alpha(\bar{\epsilon}\bar{\gamma} + \bar{\delta}), \\ &-\alpha\dot{\beta} + \dot{\alpha}\beta = \epsilon\beta^2 - \bar{\epsilon}\alpha^2. \end{aligned} \tag{3.20}$$

This system can be simplified. Equation $(3.20)_1$ can be integrated immediately to give

$$\alpha^2 - |\beta|^2 = \text{constant} \tag{3.21}$$

which, reassuringly, is just a statement that the area of ellipse-shaped fluid region 2 is constant in time. Taking the complex conjugate of $(3.20)_2$ and eliminating $\overline{\dot{\gamma}}$ yields

$$\dot{\gamma} = \bar{\epsilon}\bar{\gamma} + \bar{\delta}, \tag{3.22}$$

which is natural, since comparing with (3.7), shows that the centre of the ellipse moves with the local fluid velocity. It can also be shown, after some algebra and by making use of $(3.20)_1$ to eliminate $\dot{\alpha}$, that $(3.20)_3$ is equivalent to the nonlinear ordinary differential equation

$$\dot{e} = \overline{\epsilon(t)} - \epsilon(t)e^2, \tag{3.23}$$

where

$$e \equiv \frac{\beta}{\alpha}. \tag{3.24}$$

The complex parameter $e(t)$ encodes the eccentricity and orientation of the elliptical inclusion. Note that in order for the mapping (3.8) to represent a bounded ellipse with non-zero area, it is necessary that $|e(t)| < 1$.

It is important to remark that the special combination of a linear flow in fluid region 2 together with the elliptical shape of the interface conspires in such a way as to admit the class of exact solutions found here. These solutions seem, however, to be rather special and it does not seem a straightforward matter to extend this analysis, for example, to other bubble shapes.

4. Dependence on far-field parameters

The ordinary differential equations governing the evolution of the ellipse are (3.21), (3.22) and (3.23). These depend on the parameters $\epsilon(t)$ and $\delta(t)$ determining the linear flow inside the inclusion (fluid 2). In most applications, however, it is not expected that these will be controllable flow parameters. Rather, we expect the evolution of the elliptical inclusion to be determined by some imposed ambient far-field flow in fluid region 1. From (3.14), as $|z| \to \infty$, the linear ambient flow of fluid 1 is given by

$$w_1'(z,t) \sim E(t)z + D(t), \tag{4.1}$$

where

$$\begin{aligned} E(t) &= \frac{(\Lambda+1)\epsilon}{2} + \frac{(\Lambda-1)\bar{e}^2\bar{\epsilon}}{2}, \\ D(t) &= \frac{(\Lambda+1)\delta}{2} + \frac{(\Lambda-1)\bar{e}}{2}\left[\bar{\delta} + \bar{\epsilon}\,(\bar{\gamma} - \bar{e}\gamma)\right]. \end{aligned} \tag{4.2}$$

Clearly, $E(t)$ corresponds to some ambient strain rate while $D(t)$ is some uniform ambient flow. Given externally specified functions $E(t)$ and $D(t)$, the system (4.2) (which is linear in $\epsilon(t)$ and $\delta(t)$) can be solved for $\epsilon(t)$ and $\delta(t)$ as functions of $E(t)$, $D(t)$ and the instantaneous parameter $e(t)$.

5. Equilibria and dynamics

It is natural to first seek equilibrium solutions of system (3.20) in the case where the linear flow in the far field is steady. Without loss of generality, let $E(t) = E_0$, where E_0 is a real constant, and set $D(t) = 0$. For equilibrium, we must then have $\gamma(t) = 0$, so that $\delta(t) = 0$, together with

$$\bar{\epsilon} = e^2\epsilon, \qquad E_0 = \left(\frac{\Lambda+1}{2}\right)\epsilon + \left(\frac{\Lambda-1}{2}\right)\bar{\epsilon}\bar{e}^2. \tag{5.1}$$

However, setting $\epsilon = re^{i\phi}$ and seeking solutions of (5.1) yields only the two solutions $r = 1, \phi = 0$ and $r = \sqrt{\lambda}, \phi = 0$, where $\lambda = (\Lambda+1)/(\Lambda-1)$. Neither of these solutions is physically interesting: the first corresponds to a zero-area 'flat-plate' inclusion aligned along the real axis; the second solution is inadmissible since we require $|e| \leqslant 1$ while $\sqrt{\lambda} > 1$ for all $\Lambda > 0$. It does not therefore appear that the system (3.20) admits any interesting equilibrium solutions.

To explore the more general dynamics, note from (4.2) that it is consistent to set $D(t) = \delta(t) = \gamma(t) = 0$. Physically this means that there is a pure straining flow in the far field (with no uniform flow) and the inclusion remains centred at $z = 0$ for all times. As initial conditions, we take the inclusion to be circular with area π at $t = 0$. This corresponds to $e(0) = 0$. Once $e(t)$ is found, $\alpha(t)$ and $\beta(t)$ follow from

$$\alpha(t) = \frac{1}{\sqrt{1 - |e(t)|^2}}, \quad \beta(t) = e(t)\alpha(t). \tag{5.2}$$

In this case, it can be shown that the evolution equation for $e(t)$ is

$$\dot{e} = -\frac{2(\Lambda - 1)e^2\bar{e}^2\overline{E} - 4\Lambda E e^2 + 2(\Lambda + 1)\overline{E}}{(\Lambda - 1)^2 e^2\bar{e}^2 - (\Lambda + 1)^2}. \tag{5.3}$$

When the principal axes of the ambient straining flow are perpendicular and remain fixed for all times, an analytical solution of the differential equation (5.3) is possible. Without loss of generality, let $E(t)$ be a real function of time so that the principal axes of strain are aligned with the real and imaginary axes. It is clear from (5.3) that, since $E(t)$ is real, we expect $\overline{e(t)} = e(t)$ at all times.

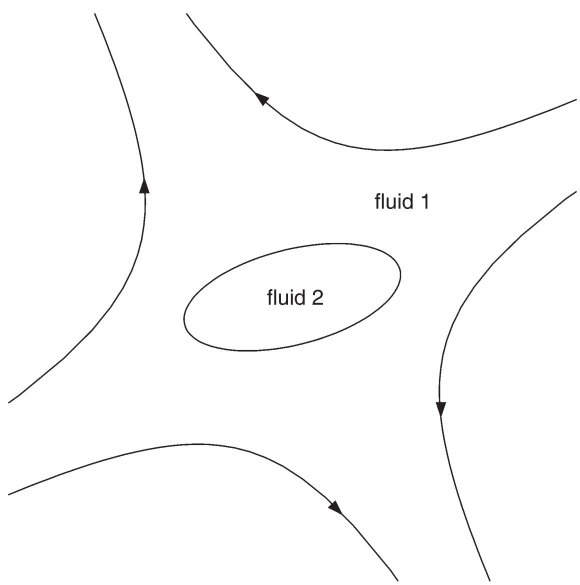

Fig. 1 A schematic of the flow configuration giving rise to exact mathematical solutions. A time-evolving elliptical inclusion of fluid 2 is embedded in a (possibly time-dependent) ambient linear straining flow of fluid 1

Then (5.3) assumes the form

$$\dot{e} = -\frac{2E(t)}{\Lambda - 1}\left(\frac{e^4 - \mu e^2 + \lambda}{e^4 - \lambda^2}\right), \tag{5.4}$$

where $\lambda = (\Lambda + 1)/(\Lambda - 1)$ and $\mu \equiv 2\Lambda/(\Lambda - 1)$. It is easy to check that this equation simplifies to

$$\dot{e} = -\frac{2E(t)}{\Lambda - 1}\left(\frac{e^2 - 1}{e^2 + \lambda}\right), \tag{5.5}$$

and so the solution of this separable ordinary differential equation is given by

$$\int_0^e \frac{e'^2 + \lambda}{e'^2 - 1} de' = -\int_0^t \frac{2E(t')dt'}{\Lambda - 1}. \tag{5.6}$$

After performing the integration on the left-hand side, the final solution is

$$\Lambda \tanh^{-1}(e) - \left(\frac{\Lambda - 1}{2}\right) e = \int_0^t E(t')dt'. \tag{5.7}$$

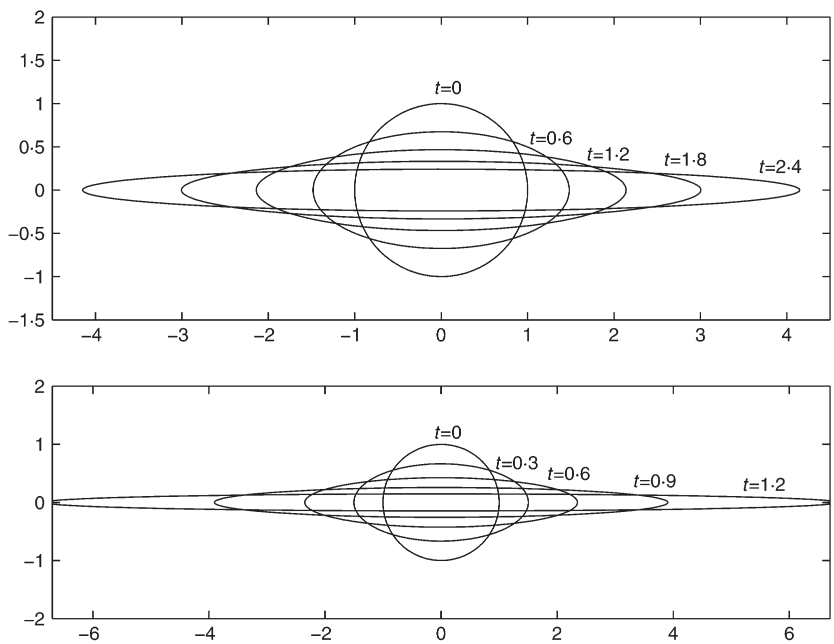

Fig. 2 Evolution of an initially circular inclusion of unit radius. $E = 1$ and $\Lambda = 2$ (above) and $\Lambda = 0{\cdot}5$ (below)

This explicit formula can be useful. Suppose we take $E(t) = 1$ so that the inclusion is in a constant ambient straining flow, then (5.7) becomes

$$\Lambda \tanh^{-1}(e) - \left(\frac{\Lambda - 1}{2}\right) e = t, \tag{5.8}$$

which shows that if $e(t)$ tends to unity from below then necessarily $t \to \infty$. This implies that the inclusion becomes infinitely elongated along the real axis as $t \to \infty$, but its boundary remains analytic for all finite times. It can also be deduced that if $E(t)$ is periodic with some period ω then so is its primitive with respect to t and hence, by (5.7), so is $e(t)$. Further, it can also be seen that, for fixed amplitude and frequency of the driving straining flow in the far field, the amplitude of the oscillations of $e(t)$ decreases as Λ increases. As illustrative examples, Fig. 2 shows the evolution of an initially circular inclusion of unit radius when subjected to a constant straining flow with $E(t) = 1$ and for $\Lambda = 2, 0{\cdot}5$. Finally, Fig. 3 shows the typical evolution of an initially circular inclusion in an oscillatory straining flow with $E(t) = \cos t$.

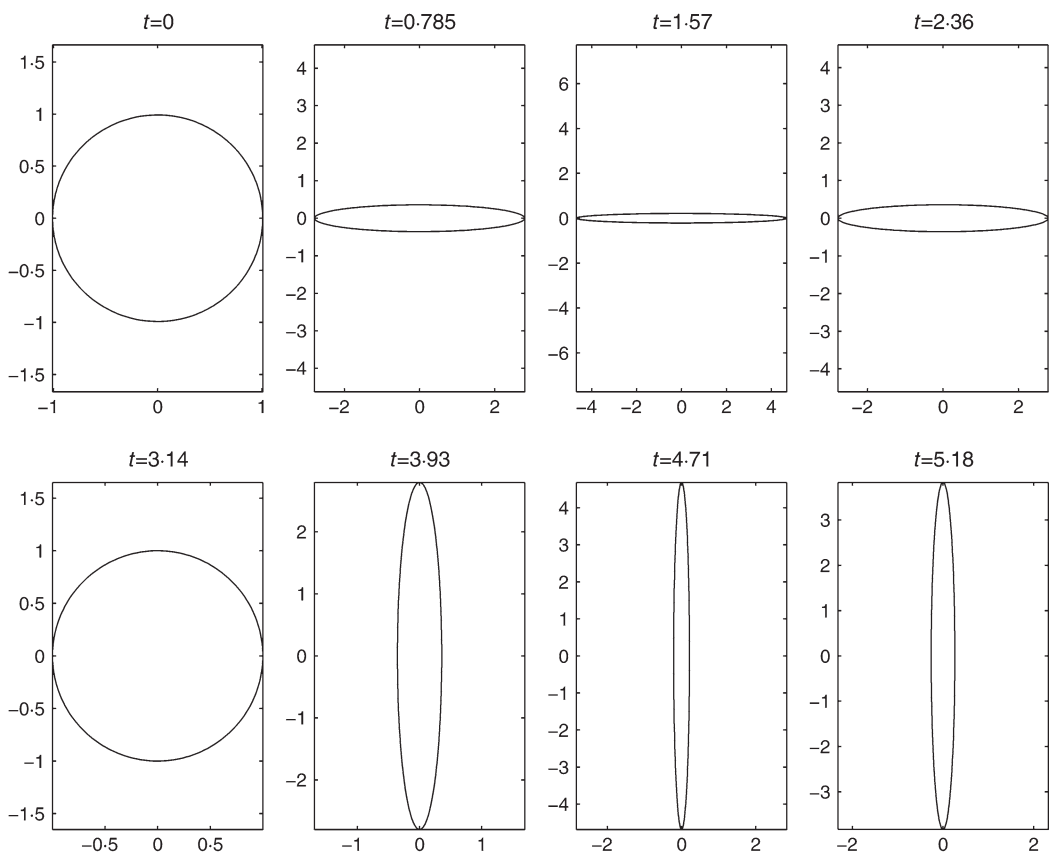

Fig. 3 Evolution of an initially circular inclusion of unit radius. $\Lambda = 0{\cdot}5$ and $E = \cos t$. By $t = 2\pi$, the inclusion returns to its original state

In more general cases, when $E(t)$ is a generally complex-valued function, the complex nonlinear ordinary differential equation (5.3) can easily be integrated numerically.

6. An elliptical inclusion model

There are a number of physical problems where an elliptical inhomogeneity evolves, and remains elliptical, in a linear ambient field. For example, an elliptical patch of uniform vorticity is known to remain elliptical when placed in any linear ambient straining and/or shear flow (**16**). Similarly, a compressible elliptical inclusion in an ambient linear slow viscous Stokes flow is also known to remain elliptical, in this case, even changing its area as it evolves (**17**). In the vortex dynamics problem, this has led naturally to the so-called 'elliptical vortex approximation' of two-dimensional vortex interactions. In this approximation, each vortical region is assumed to be an elliptical vortex patch that is sufficiently far from all its neighbouring vortical regions. Then, the flow induced in the neighbourhood of any chosen elliptical patch by all the other patches is expanded to linear order and this linear expansion is used as the 'far-field' flow in which the chosen ellipse is evolving. Similarly, the preservation of isolated, shrinking elliptical pores in Stokes flow has formed the basis of a related 'elliptical pore model' for late-stage viscous sintering (**18**). This paper has shown that, in the two-phase Hele-Shaw problem, the same phenomenon occurs. That is, an isolated elliptical inhomogeneity of one fluid remains elliptical when placed in a linear ambient flow of another fluid. It is clear that this fact might similarly form the basis of an 'elliptical inclusion model' of interaction of inhomogeneities in porous media. Given a distribution of inhomogeneities evolving in an ambient fluid due to some driving mechanism (for example, a far-field strain), each inhomogeneity can be assumed to be elliptical with some centre, area, orientation and eccentricity. Expanding the local flow (due to the forcing and other inhomogeneities) in the neighbourhood of any chosen inhomogeneity will then provide the local values of $E(t)$ and $D(t)$ (in the notation of this paper) with which the evolution of the inhomogeneity can be computed using the ordinary differential equations derived here.

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