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Green's functions for Laplace's equation in multiply connected domains

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Analytical formulae for the first-type Green's function for Laplace's equation in multiply connected circular domains are presented. The method is constructive and relies on the use of a special function known as the Schottky–Klein prime function associated with multiply connected circular domains. It is shown that all the important functions of potential theory, including the first-type Green's function, the modified Green's functions and the harmonic measures of a domain, can be written in terms of this prime function. A broad range of representative examples are given to demonstrate the efficacy of the method as well as a quantitative comparison, in special cases, with the results obtained using other independent methods.

Keywords: Green's function; Laplace equation; multiply connected.

1. Introduction

Of interest in this paper is the practical construction of a function well-known to every undergraduate in engineering, the physical sciences and applied mathematics: the first-type Green's function for Laplace's equation. It arises in areas of physics such as electrostatics, gravitation, fluid dynamics and transport theory all the way through to digital filter design and the more mathematical areas of potential theory, numerical analysis and approximation theory. Basically, the first-type Green's function is a function that is harmonic everywhere in the given domain except for an isolated unit-strength singularity and which vanishes on all the domain boundaries.

The challenge embraced in this paper is the following: to find an effective constructive technique, based on analytical formulae dependent on a special transcendental function known as the 'Schottky–Klein prime function' (Baker, 1995; Hejhal, 1972), for the Green's function in multiply connected circular domains.

There are several results concerning the construction of the first-type Green's function in multiply connected domains but the approach presented here, based on the use of the Schottky–Klein prime function, appears to be new. Mityushev & Rogosin (2000) describe a construction of the (complex) Green's function in arbitrary multiply connected circular domains consisting of the unbounded region exterior to a finite collection of circular discs. They employ the theory of functional equations and the method of successive approximations to construct representations of the Green's function (and find the solution of the Schwarz problem) in such multiply connected domains. They also survey a number of other constructions and the reader should consult their monograph (Mityushev & Rogosin, 2000) for additional background and references. Recently, by strategic use of Schwarz–Christoffel mappings, Embree & Trefethen (1999) have shown how to construct first-type Green's functions for the restricted

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class of multiply connected polygonal domains that are reflectionally symmetric with respect to some specified axis of symmetry.

Our method is based on the fact that the well-known functions of potential theory (the first-type Green's function, modified Green's functions and harmonic measures) have underlying connections with the theory of conformal mapping of multiply connected domains. Both Nehari (1952) and Schiffer give treatments of these theoretical connections from different, but related, perspectives. Schiffer bases his presentation from the outset on the existence and properties of the first-type Green's function and systematically develops important connections with conformal mapping theory. Nehari (1952) proceeds in the reverse direction. His point of departure is the theory of conformal mapping of multiply connected domains and he works towards the construction of the first-type Green's function. Both authors rely heavily on seminal contributions to the theory of conformal mapping of multiply connected domains made by Koebe (1914). Julia (1934) treats similar material and gives explicit formulae for the important functions from potential theory. His perspective is, however, different to that considered here and we return to this point in Section 12.

Although our construction is for multiply connected circular domains, the results are also relevant to a general (i.e. not necessarily circular) multiply connected domain. This is because circular domains constitute a set of canonical multiply connected domains (Goluzin, 1969; Nehari, 1952) so that any given domain is conformally equivalent to 'some' circular domain of the kind considered here. Second, the boundary-value problem for finding the first-type Green's function is a conformally invariant one; so once the Green's function is known in some canonical set of multiply connected domains, it is, in principle, determined in all conformally equivalent domains via conformal mapping.

2. Modified Green's functions

Let D be an arbitrary bounded and M-connected planar domain. Suppose D is bounded by $M+1$ smooth Jordan curves called $\{C_j \mid j = 0, 1, \ldots, M\}$. C_0 is taken as the outermost boundary so that $\{C_k \mid k = 1, \ldots, M\}$ denote the M enclosed boundaries (or the boundaries of the M 'holes' in the domain). Let ζ and α be complex variables denoting two distinct points in D. The 'modified Green's function' is defined as the function $G_0(\zeta; \alpha)$ satisfying the following properties:

(i) The function

$$G_0(\zeta;\alpha) + \frac{1}{2\pi}\log|\zeta-\alpha| \tag{1}$$

is harmonic, with respect to (ζ_x, ζ_y) where $\zeta = \zeta_x + \mathrm{i}\zeta_y$, throughout the region D including the point α.

(ii) If $\partial G_0/\partial n$ is the normal derivative of G_0 on a boundary curve, then

$$\begin{aligned} &G_0(\zeta;\alpha) = 0 \quad \text{on } C_0,\\ &G_0(\zeta;\alpha) = \gamma_{0k}(\alpha) \quad \text{on } C_k, \quad k = 1,\ldots,M,\\ &\oint_{C_k} \frac{\partial G_0}{\partial n}\,\mathrm{d}s = 0, \quad k = 1,\ldots,M, \end{aligned} \tag{2}$$

where ds denotes the arclength and the elements of the set $\{\gamma_{0k}(\alpha) \mid k = 1,\ldots,M\}$ are some functions of α but not ζ.

The function $G_0(\zeta;\alpha)$ defined by conditions (i) and (ii) above exists uniquely (Koebe, 1914) and satisfies the reciprocity condition

$$G_0(\zeta;\alpha) = G_0(\alpha;\zeta). \tag{3}$$

It is clear from the definition that the boundary C_0 has a special significance with respect to the function $G_0(\zeta;\alpha)$ defined above. It is the boundary on which $G_0(\zeta;\alpha)$ is normalized to vanish and is the only choice of boundary for D for which the quantity

$$\oint_{C_0} \frac{\partial G_0}{\partial n} \mathrm{d}s \tag{4}$$

does not vanish. The subscript on G_0 has been chosen to reflect this special significance of C_0. But it should be clear that there are M alternative modified Green's functions that can also be defined analogously: one simply makes the boundary component C_j the one which has the special significance afforded to C_0 in the definition of G_0. Extending the subscript notation, these additional modified Green's functions will be denoted $\{G_j(\zeta;\alpha) | j = 1, \ldots, M\}$. $G_j(\zeta;\alpha)$ will have a logarithmic singularity at α and satisfy the modified boundary conditions

$$\begin{aligned} G_j(\zeta;\alpha) &= 0 \quad \text{on } C_j, \\ G_j(\zeta;\alpha) &= \gamma_{jk}(\alpha) \quad \text{on } C_k, \quad k \neq j, \\ \oint_{C_k} \frac{\partial G_j}{\partial n} \mathrm{d}s &= 0, \quad k \neq j, \end{aligned} \tag{5}$$

where the elements of the set $\{\gamma_{jk}(\alpha) | j = 1, \ldots, M; k = 1, \ldots, M\}$ are functions of α but not ζ.

3. The first-type Green's function

The first-type Green's function, here denoted $\mathcal{G}(\zeta;\alpha)$, is defined differently. It is the real-valued function, defined with respect to a given point α in the domain D, satisfying the following conditions:

(i) The function

$$\mathcal{G}(\zeta;\alpha) + \frac{1}{2\pi} \log |\zeta - \alpha| \tag{6}$$

is harmonic throughout the region D including at the point α.

(ii) $\mathcal{G}(\zeta,\alpha)$ is such that

$$\mathcal{G}(\zeta;\alpha) = 0 \quad \text{on } C_k, \quad k = 0, 1, \ldots, M. \tag{7}$$

Note that the quantities

$$\oint_{C_k} \frac{\partial \mathcal{G}}{\partial n} \mathrm{d}s, \quad k = 0, 1, \ldots, M, \tag{8}$$

will then generally be nonzero.

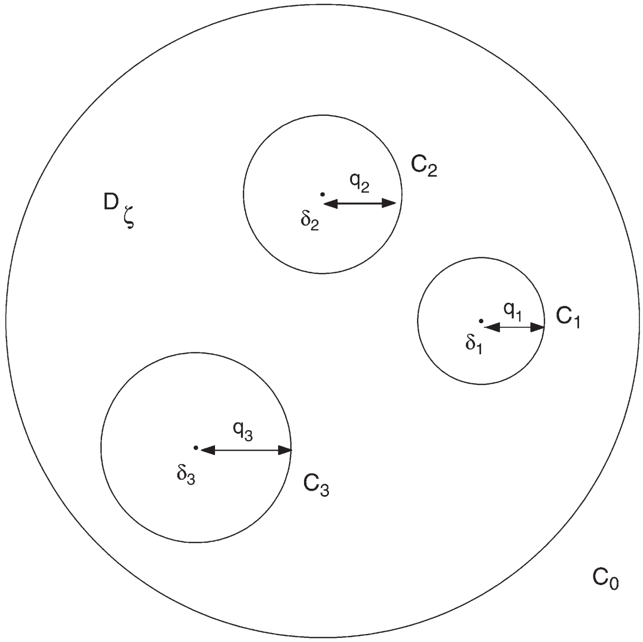

FIG. 1. Schematic of a typical multiply connected bounded circular region consisting of the unit disc with some interior circular discs excised. The case shown is quadruply connected. The centres of the enclosed circular discs are $\{\delta_j \,|\, j = 1, \ldots, M\}$ and their radii are $\{q_j \,|\, j = 1, \ldots, M\}$.

4. Circular domains

Our aim is to construct $\mathcal{G}(\zeta;\alpha)$ in multiply connected circular domains. Henceforth, D_ζ will denote such a circular domain in the ζ-plane. Specifically, let D_ζ be the interior of the unit ζ-disc with M smaller circular discs excised. $M = 0$ is the simply connected case. Consistent with the notations of Sections 2 and 3, the boundaries of the smaller excised circular discs will be denoted $\{C_j \,|\, j = 1, \ldots, M\}$. C_0 will denote the outer unit circle $|\zeta| = 1$.

To uniquely specify an M-connected D_ζ, the centres and radii of $\{C_j \,|\, j = 1, \ldots, M\}$ are needed. Let $\{\delta_j \in \mathbb{C} \,|\, j = 1, \ldots, M\}$ be the centres of these circles and let $\{q_j \in \mathbb{R} \,|\, j = 1, \ldots, M\}$ be their radii. A definition sketch of a quadruply connected case is shown in Fig. 1.

5. Schottky groups

To proceed, first define M Möbius maps $\{\phi_j \,|\, j = 1, \ldots, M\}$ corresponding to the conjugation map for points on the circle C_j. That is, if C_j has equation

$$|\zeta - \delta_j|^2 = (\zeta - \delta_j)(\bar{\zeta} - \bar{\delta}_j) = q_j^2, \tag{9}$$

then

$$\bar{\zeta} = \bar{\delta}_j + \frac{q_j^2}{\zeta - \delta_j} \tag{10}$$

and so

$$\phi_j(\zeta) \equiv \bar{\delta}_j + \frac{q_j^2}{\zeta - \delta_j}. \tag{11}$$

If ζ is a point on C_j, its complex conjugate is $\bar{\zeta} = \phi_j(\zeta)$. We will also define

$$\phi_0(\zeta) = \zeta^{-1}. \tag{12}$$

Next, introduce the Möbius maps

$$\theta_j(\zeta) \equiv \overline{\phi}_j(\zeta^{-1}) = \delta_j + \frac{q_j^2\zeta}{1 - \bar{\delta}_j\zeta}. \tag{13}$$

Let C_j' be the circle obtained by the reflection of C_j in the unit circle C_0, i.e. the circle obtained by the transformation $\zeta \mapsto \bar{\zeta}^{-1}$. It is easy to verify that the image of the circle C_j' under the transformation $\theta_j(\zeta)$ is C_j. Since the M circles $\{C_j | j = 1, \ldots, M\}$ are nonoverlapping, so are the M circles $\{C_j' | j = 1, \ldots, M\}$. The classical 'Schottky group' Θ is defined to be the infinite free group of mappings generated by compositions of the $2M$ basic Möbius maps $\{\theta_j | j = 1, \ldots, M\}$ and their inverses $\{\theta_j^{-1} | j = 1, \ldots, M\}$, including the identity map. Beardon (1984) gives a general discussion of such groups.

Consider the (generally unbounded) region of the plane exterior to the $2M$ circles $\{C_j | j = 1, \ldots, M\}$ and $\{C_j' | j = 1, \ldots, M\}$. A schematic is shown in Fig. 2. This region is known as the 'fundamental region' associated with the Schottky group generated by the Möbius maps $\{\theta_j | j = 1, \ldots, M\}$ and their inverses. This fundamental region can be understood as having two 'halves'—the half that is inside the unit circle but exterior to the circles $\{C_j | j = 1, \ldots, M\}$ is D_ζ, the region that is outside the unit circle and exterior to the circles $\{C_j' | j = 1, \ldots, M\}$ is the 'other' half. The reason it is called fundamental is that the rest of the ζ-plane is a tessellation of an infinite number of 'equivalent' regions which are obtained by transformation of the fundamental region under the elements of the Schottky group. Any point in the ζ-plane that can be reached by an element of the group Θ acting on a point inside F is known as a 'regular point' of the group.

The Möbius maps introduced above have two important properties that can easily be established. The first is that

$$\theta_j^{-1}(\zeta) = \frac{1}{\phi_j(\zeta)}, \quad \forall\, \zeta. \tag{14}$$

This can be verified using the definitions (11) and (13) (or, alternatively, by considering the geometrical effect of each map). The second property, which follows from the first, is that

$$\theta_j^{-1}(\zeta^{-1}) = \frac{1}{\phi_j(\zeta^{-1})} = \frac{1}{\overline{\bar{\phi}_j(\bar{\zeta}^{-1})}} = \frac{1}{\overline{\theta_j(\bar{\zeta})}} = \frac{1}{\bar{\theta}_j(\zeta)}, \quad \forall\, \zeta. \tag{15}$$

Some special infinite subsets of mappings in a given Schottky group will be needed in what follows. A special notation is now introduced. This notation is not standard but is introduced here to clarify the presentation. The full Schottky group is denoted Θ. The notation ${}_i\Theta_j$ is used to denote all mappings in the full group which do not have a power of θ_i or θ_i^{-1} on the left-hand end or a power of θ_j or θ_j^{-1} on the right-hand end. As a special case of this, the notation Θ_j simply means all mappings in the group

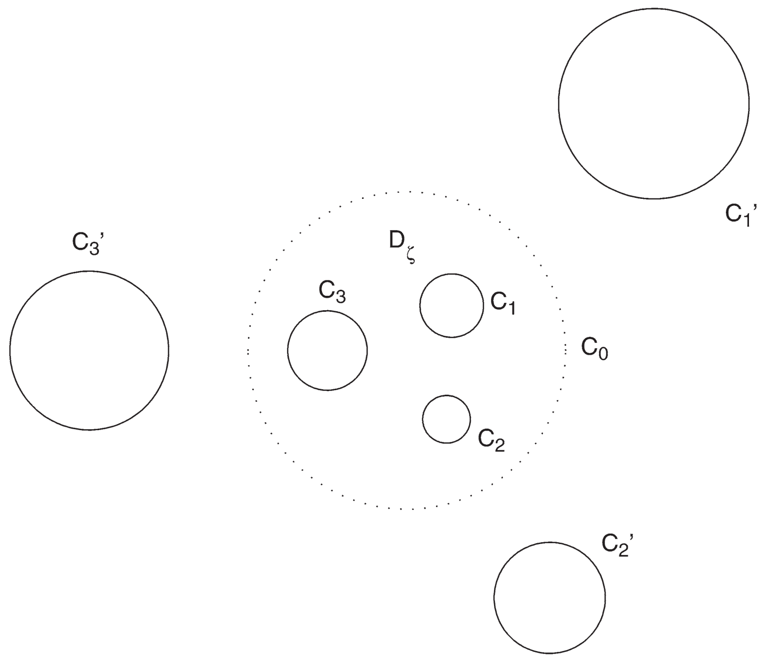

FIG. 2. Schematic of a typical fundamental region exterior to the set of circles $C_1, C_1', C_2, C_2', C_3$ and C_3'.

which do not have any positive or negative power of θ_j at the right-hand end (but with no stipulation about what appears on the left-hand end). Similarly, ${}_j\Theta$ means all mappings which do not have any positive or negative power of θ_j at the left-hand end (but with no stipulation about what appears on the right-hand end). In addition, the single-prime notation will be used to denote a subset where the identity is excluded from the set; thus Θ_1' denotes all mappings, excluding the identity and all transformations with a positive or negative power of θ_1 at the right-hand end. The double-prime notation will be used to denote a subset where the identity and all inverse mappings are excluded from the set. This means, e.g. that if $\theta_1\theta_2^{-1}$ is included in the set, the inverse mapping $\theta_2\theta_1^{-1}$ must be excluded. Thus, Θ'' means all mappings excluding the identity and all inverses. Similarly, the notation ${}_1\Theta_2''$ denotes all mappings, excluding inverses and the identity, which do not have any power of θ_1 or θ_1^{-1} on the left-hand end or any power of θ_2 or θ_2^{-1} on the right-hand end. In the same way, Θ_j'' denotes all mappings, excluding the identity and all inverses, which do not have any positive or negative power of θ_j at the right-hand end.

6. The Schottky–Klein prime function

In this section, we introduce an important transcendental function—the Schottky–Klein prime function—in terms of which the analytical formulae for $\mathcal{G}(\zeta;\alpha)$ will be expressed.

It is important to emphasize that this function is a well-defined (and uniquely defined) function that can be associated with any given multiply connected domain. How to compute it effectively is another matter. Since our emphasis here is on analytical formulae, we present our results in terms of a classical infinite product formula over the elements of the Schottky group introduced in Section 5. This product does not converge for all Schottky groups (although there are a number of known sufficiency conditions

(Baker, 1995) on the group that guarantee the convergence of the product). However, it is crucial to point out that our final formulae expressed in terms of the prime function remain valid even when the infinite product representation of the prime function does not converge. Crowdy & Marshall (2007) have recently presented a novel numerical scheme for the computation of the Schottky–Klein prime function which does not rely on a product (or sum) over elements of a Schottky group. The new scheme is based on understanding the prime function from the point of view of Riemann surface theory (there is a compact Riemann surface, known as the Schottky double, that can be associated with any multiply connected planar domain (Hejhal, 1972)). This alternative algorithm can be used to compute the prime function when the infinite product used in this paper does not converge (or is too slowly convergent to be useful in practice).

Following Baker (1995), the Schottky–Klein prime function is defined as

$$\omega(\zeta,\gamma) = (\zeta-\gamma)\omega'(\zeta,\gamma), \tag{16}$$

where the function $\omega'(\zeta,\gamma)$ is given by

$$\omega'(\zeta,\gamma) = \prod_{\theta_i\in\Theta''} \frac{(\theta_i(\zeta)-\gamma)(\theta_i(\gamma)-\zeta)}{(\theta_i(\zeta)-\zeta)(\theta_i(\gamma)-\gamma)} \tag{17}$$

and where the product is over all mappings θ_i in the set Θ''. ω' can also be written as

$$\omega'(\zeta,\gamma) = \prod_{\theta_i\in\Theta''} \{\zeta,\theta_i(\zeta),\gamma,\theta_i(\gamma)\}, \tag{18}$$

where the brace notation denotes a cross-ratio of the four arguments. This will be useful later. The function $\omega(\zeta,\gamma)$ is single valued on the whole ζ-plane, has a zero at γ and all points equivalent to γ under the mappings of the group Θ. The prime notation is not used here to denote differentiation.

The Schottky–Klein prime function has some important transformation properties. One such property is that it is antisymmetric in its arguments, i.e.

$$\omega(\zeta,\gamma) = -\omega(\gamma,\zeta). \tag{19}$$

This is clear from the inspection of (16) and (17). A second important property is given by

$$\frac{\omega(\theta_j(\zeta),\gamma_1)}{\omega(\theta_j(\zeta),\gamma_2)} = \beta_j(\gamma_1,\gamma_2)\frac{\omega(\zeta,\gamma_1)}{\omega(\zeta,\gamma_2)}, \tag{20}$$

where θ_j is any one of the basic maps of the Schottky group. A detailed derivation of this result is given in Chapter 12 of Baker (1995). A formula for $\beta_j(\gamma_1,\gamma_2)$ is

$$\beta_j(\gamma_1,\gamma_2) = \prod_{\theta_k\in\Theta_j} \frac{(\gamma_1-\theta_k(B_j))(\gamma_2-\theta_k(A_j))}{(\gamma_1-\theta_k(A_j))(\gamma_2-\theta_k(B_j))}, \tag{21}$$

where A_j and B_j are the two fixed points of the mapping θ_j satisfying

$$\theta_j(A_j) = A_j, \quad \theta_j(B_j) = B_j. \tag{22}$$

They are therefore the two solutions of a quadratic equation. It follows that

$$\frac{\theta_j(\zeta)-B_j}{\theta_j(\zeta)-A_j} = \mu_j\, \mathrm{e}^{\mathrm{i}\kappa_j}\frac{\zeta-B_j}{\zeta-A_j}, \tag{23}$$

for some real parameters μ_j and κ_j. The roots A_j and B_j are ordered such that $|\mu_j| < 1$ (in our case, A_j lies in C_j' while B_j lies in C_j, so we can distinguish between these fixed points by the simple criteria that $|A_j| > 1$ and $|B_j| < 1$). It is also known (Baker, 1995) that, for any regular point ζ,

$$\theta_j^{-\infty}(\zeta) = A_j, \quad \theta_j^{\infty}(\zeta) = B_j. \tag{24}$$

7. Explicit expressions for $\{G_j(\zeta;\alpha)\}$

In this section, explicit formulae for the modified Green's functions associated with D_ζ are found.

Given a circular domain D_ζ, the associated Schottky–Klein prime function $\omega(\zeta,\gamma)$ can be constructed. As discussed in Section 2, there are $M+1$ modified Green's functions $\{G_j(\zeta;\alpha) \mid j = 0, 1, \ldots, M\}$ that can be defined.

It will now be argued that an explicit expression for $G_j(\zeta;\alpha)$ is

$$G_j(\zeta;\alpha) = -\frac{1}{4\pi}\log|R_j(\zeta;\alpha)|, \tag{25}$$

where we define the $M+1$ functions

$$R_j(\zeta;\alpha) = \frac{\omega(\zeta,\alpha)\overline{\omega}(\phi_j(\zeta),\phi_j(\alpha))}{\omega(\zeta,\overline{\phi}_j(\bar{\alpha}))\overline{\omega}(\phi_j(\zeta),\bar{\alpha})}, \quad j = 0, 1, \ldots, M. \tag{26}$$

Consider the fundamental region generated by D_ζ and the reflection of D_ζ in the jth circle. Then, since α is in the half of this fundamental region corresponding to D_ζ, $\overline{\phi_j}(\bar{\alpha})$ will be in the other half. From (25), it follows that $G_j(\zeta;\alpha)$ has a single isolated logarithmic singularity in D_ζ at $\zeta = \alpha$. Since the zero of R_j is second order, locally $G_j(\zeta;\alpha)$ has the expansion

$$G_j(\zeta;\alpha) = -\frac{1}{2\pi}\log|\zeta-\alpha| + \mathrm{O}(1), \tag{27}$$

which is what is required.

It remains to verify that (25) satisfies the appropriate boundary conditions. It can be shown that, on the circle C_k,

$$|R_j(\zeta;\alpha)| = \left|\frac{\beta_j(\overline{\phi_j}(\bar{\alpha}),\alpha)}{\beta_k(\overline{\phi_j}(\bar{\alpha}),\alpha)}\right|. \tag{28}$$

This formula is established in the appendix to Crowdy & Marshall (2006) (there, the function $R_j(\zeta;\alpha)$ is denoted $\tilde{R}_j(\zeta;\alpha)$ but all other notations are the same as in the present paper). Equation (28) holds for all integers j and k (between 0 and M) provided we adopt the convention that $\beta_0(\zeta,\alpha) \equiv 1$.

It is immediate that, on C_j, $|R_j(\zeta,\alpha)| = 1$ so $G_j(\zeta;\alpha) = 0$ there. On C_k with $k \neq j$, we have

$$G_j(\zeta;\alpha) = -\frac{1}{4\pi}\log\left|\frac{\beta_j(\overline{\phi_j}(\bar{\alpha}),\alpha)}{\beta_k(\overline{\phi_j}(\bar{\alpha}),\alpha)}\right|. \tag{29}$$

This means that the functions $\{\gamma_{jk}(\alpha)\}$ defined in (2) and (5) are given by the formulae

$$\gamma_{jk}(\alpha) = -\frac{1}{4\pi}\log\left|\frac{\beta_j(\overline{\phi_j}(\bar{\alpha}),\alpha)}{\beta_k(\overline{\phi_j}(\bar{\alpha}),\alpha)}\right|, \tag{30}$$

for $k \neq j$.

Finally, note that in any fundamental region associated with the Schottky group, $G_j(\zeta;\alpha)$ has exactly two logarithmic singularities of equal and opposite strength: one at α in D_ζ and the other at $\overline{\phi}_j(\overline{\alpha})$ in the other half of the fundamental region. A natural way to define a branch of $G_j(\zeta;\alpha)$ is therefore to join each such pair of logarithmic singularities in the fundamental region, and in each region equivalent to it under the elements of the group, by a branch cut. It follows that

$$\mathrm{Im}\left[\oint_{C_j} \mathrm{d}[G_k(\zeta;\alpha)]\right] = 0 \tag{31}$$

unless $k = j$ which, after a little algebra, can be shown to be equivalent to the conditions that the quantities

$$\oint_{C_j} \frac{\partial G_k}{\partial n}\mathrm{d}s \tag{32}$$

will be nonzero only for $k = j$.

Having identified a function satisfying all the conditions required of a modified Green's function, we now exploit the result that the latter function is unique (Koebe, 1914). We have therefore constructed expressions for the required set of $M + 1$ modified Green's functions in terms of the prime function.

Before proceeding, it is worth mentioning that there are alternative (equivalent) ways of expressing the functions $\{G_j(\zeta;\alpha)|j = 0, 1, \ldots, M\}$ in terms of the prime function. Indeed, by making use of the properties of the prime function already outlined, it can be also be shown (we omit the details) that

$$G_j(\zeta;\alpha) = -\frac{1}{2\pi}\log\left|\frac{q_j}{(\alpha-\delta_j)}\frac{\omega(\zeta,\alpha)}{\omega(\zeta,\overline{\phi_j(\alpha)})}\right|. \tag{33}$$

If preferred, all the results to follow can be rederived using formula (33) instead of (25) and (26).

8. Harmonic measures

In this section, explicit formulae for the 'harmonic measures' associated with the circular domain D_ζ are found.

We define the M harmonic measures, $\{\sigma_j|j = 1, \ldots, M\}$, to be the set of functions, harmonic in D, that satisfy the conditions

$$\sigma_j(\zeta) = \begin{cases} 1 & \text{on } C_j, \\ 0 & \text{on } C_k, \quad k \neq j. \end{cases} \tag{34}$$

These functions are known to constitute a basis of an M-dimensional vector space of functions which are harmonic in D and constant on the boundary of D. It turns out that we can construct this basis now that explicit formulae for the $M + 1$ modified Green's functions are known. Nehari (1952) and Schiffer discuss the general theory from different, but related, perspectives. First, define a set of M nontrivial functions given by

$$S_{0j}(\zeta;\alpha) = \mathcal{A}_j(\alpha)\left(\frac{R_0(\zeta;\alpha)}{R_j(\zeta;\alpha)}\right)^{1/2} \tag{35}$$

for $j = 1, \ldots, M$ (note that the choice $j = 0$ yields a trivial constant function), where $\mathcal{A}_j(\alpha)$ is a normalization constant chosen so that $S_{0j}(\zeta;\alpha) = 1$ on C_0. Specifically, it is easy to show that

$$\mathcal{A}_j(\alpha) = |\beta_j(\bar{\phi}_j(\bar{\alpha}), \alpha)|^{1/2}. \tag{36}$$

Then, define $\tilde{\sigma}_j(\zeta)$ for $j = 1, \ldots, M$ by

$$\tilde{\sigma}_j(\zeta) \equiv \log |S_{0j}(\zeta;\alpha)|. \tag{37}$$

These M functions are known explicitly since formulae for $R_j(\zeta;\alpha)$ are given in (26). It is also known that, on C_k,

$$\begin{aligned} |\tilde{\sigma}_j(\zeta)| &= \frac{1}{2}\log\left|\mathcal{A}_j(\alpha)^2 \frac{R_0(\zeta;\alpha)}{R_j(\zeta;\alpha)}\right| \\ &= \frac{1}{2}\log\left|\beta_j(\bar{\phi}_j(\bar{\alpha}),\alpha)\frac{\beta_0(\overline{\phi_0}(\bar{\alpha}),\alpha)\beta_k(\overline{\phi_j}(\bar{\alpha}),\alpha)}{\beta_k(\overline{\phi_0}(\bar{\alpha}),\alpha)\beta_j(\overline{\phi_j}(\bar{\alpha}),\alpha)}\right| \\ &= \frac{1}{2}\log\left|\frac{\beta_k(\overline{\phi_j}(\bar{\alpha}),\alpha)}{\beta_k(\bar{\alpha}^{-1},\alpha)}\right| \\ &= \frac{1}{2}\log|\beta_k(\bar{\phi}_j(\bar{\alpha}),\bar{\alpha}^{-1})|, \end{aligned} \tag{38}$$

where we have used (28). Then, it follows from the general theory that σ_j is some linear combination of the M harmonic measures $\{\tilde{\sigma}_j | j = 1, \ldots, M\}$. Let

$$\sigma_j(\zeta) = \sum_{k=1}^{M} \alpha_{jk}\tilde{\sigma}_k(\zeta), \tag{39}$$

where α_{jk} denote the elements of some matrix to be determined. Imposing conditions (34) on each of the M circles, it is easily found that

$$\boldsymbol{\alpha}_{ij} = [\mathbf{A}^{-1}]_{ij}, \tag{40}$$

where the matrix $\mathbf{A}$ has elements

$$\mathbf{A}_{ij} = \frac{1}{2}\log|\beta_j(\overline{\phi_i}(\bar{\alpha}),\bar{\alpha}^{-1})|. \tag{41}$$

It is shown in Appendix A that the quantities $\beta_j(\overline{\phi_i}(\bar{\alpha}),\bar{\alpha}^{-1})$ are all independent of α. Indeed, it is also noted that

$$\beta_j(\overline{\phi_i}(\bar{\alpha}),\bar{\alpha}^{-1}) = \exp(2\pi \mathrm{i}\tau_{j,i}), \tag{42}$$

where the parameters $\tau_{j,k}$ are defined in Chapter 12 of Baker (1995) and depend only on the parameters $\{q_j, \delta_j | j = 1, \ldots, M\}$. Indeed, when $k \neq j$,

$$\tau_{j,k} = \frac{1}{2\pi \mathrm{i}}\log\left[\prod_{\psi\in_k\Theta_j}\left(\frac{\psi(B_j)-B_k}{\psi(B_j)-A_k}\right)\left(\frac{\psi(A_j)-A_k}{\psi(A_j)-B_k}\right)\right], \tag{43}$$

where A_k and B_k are the two fixed points of the mapping θ_k, while

$$\tau_{k,k} = \frac{1}{2\pi \mathrm{i}} \log \left[\mu_k \, \mathrm{e}^{\mathrm{i}\kappa_k} \prod_{\psi \in_k \Theta_k''} \left[\left(\frac{\psi(B_k) - B_k}{\psi(B_k) - A_k} \right) \left(\frac{\psi(A_k) - A_k}{\psi(A_k) - B_k} \right) \right]^2 \right], \tag{44}$$

where μ_k and κ_k are defined in (23).

With the coefficients α_{ij} known, the harmonic measures $\{\sigma_j | j = 1, \ldots, M\}$ are now determined from (39). Nehari (1952) gives details of a general proof that the matrix $\mathbf{A}$ is always invertible.

9. Construction of $\mathcal{G}(\zeta; \alpha)$

Given expressions for the modified Green's functions and the M harmonic measures of the domain D_ζ in terms of the prime function, it is now possible to construct a formula for the first-type Green's function $\mathcal{G}(\zeta; \alpha)$ for the circular domain D_ζ. It is given by

$$\mathcal{G}(\zeta; \alpha) = G_0(\zeta; \alpha) - \sum_{j=1}^{M} \gamma_{0j}(\alpha)\sigma_j(\zeta). \tag{45}$$

This function has the same logarithmic singularity at $\zeta = \alpha$ as the modified Green's function $G_0(\zeta; \alpha)$. It is harmonic everywhere else in D_ζ. On C_0, we have $\mathcal{G}(\zeta; \alpha) = 0$ since both $G_0(\zeta; \alpha)$ and all the harmonic measures $\{\sigma_j(\zeta) | j = 1, \ldots, M\}$ vanish there. It is also clear that $\mathcal{G}(\zeta; \alpha)$ vanishes on all the interior circles $\{C_j | j = 1, \ldots, M\}$ since $G_0(\zeta; \alpha) = \gamma_{0j}(\alpha)$ on C_j and this is precisely cancelled by the term $-\gamma_{0j}(\alpha)\sigma_j(\zeta)$ which equals $-\gamma_{0j}(\alpha)$ for ζ on C_j.

10. The doubly connected case

It is instructive to see how the general theory reduces to familiar formulae in the doubly connected case.

Any doubly connected domain is conformally equivalent to an annular region $q < |\zeta| < 1$ for some value of the conformal modulus q (Nehari, 1952). In this case, $\delta_1 = 0$ and $q_1 = q$ so that the relevant Möbius maps are

$$\phi_1(\zeta) = \frac{q^2}{\zeta}, \quad \theta_1(\zeta) = q^2\zeta. \tag{46}$$

The associated Schottky group has just one generator. Its elements are $\{\theta_1^j | j \in \mathbb{Z}\}$. The associated Schottky–Klein prime function can be shown to be

$$\omega(\zeta, \gamma) = -\frac{\gamma}{C^2} P(\zeta/\gamma, q), \tag{47}$$

where

$$P(\zeta, q) \equiv (1 - \zeta) \prod_{k=1}^{\infty} (1 - q^{2k}\zeta)(1 - q^{2k}\zeta^{-1}), \quad C \equiv \prod_{k=1}^{\infty} (1 - q^{2k}). \tag{48}$$

It is straightforward to verify, directly from the definition (48), that

$$P(\zeta^{-1}, q) = -\zeta^{-1} P(\zeta, q), \quad P(q^2\zeta, q) = -\zeta^{-1} P(\zeta, q). \tag{49}$$

It follows that

$$R_0(\zeta,\alpha) = \frac{\omega(\zeta,\alpha)\overline{\omega}(\zeta^{-1},\alpha^{-1})}{\omega(\zeta,\bar{\alpha}^{-1})\overline{\omega}(\zeta^{-1},\bar{\alpha})} = \frac{P(\zeta\alpha^{-1},q)P(\alpha\zeta^{-1},q)}{P(\zeta\bar{\alpha},q)P(\zeta^{-1}\bar{\alpha}^{-1},q)},$$

$$R_1(\zeta,\alpha) = \frac{\omega(\zeta,\alpha)\overline{\omega}(q^2\zeta^{-1},q^2\alpha^{-1})}{\omega(\zeta,q^2\bar{\alpha}^{-1})\overline{\omega}(q^2\zeta^{-1},\bar{\alpha})} = \frac{P(\zeta\alpha^{-1},q)P(\alpha\zeta^{-1},q)}{P(\zeta\bar{\alpha}q^{-2},q)P(q^2\zeta^{-1}\bar{\alpha}^{-1},q)}. \tag{50}$$

Now, define

$$S_{01}(\zeta;\alpha) = \mathcal{A}_1(\alpha)\left(\frac{R_0(\zeta;\alpha)}{R_1(\zeta;\alpha)}\right)^{1/2}. \tag{51}$$

It follows that

$$\mathcal{A}_1(\alpha) = \frac{q}{|\alpha|} \tag{52}$$

and

$$\tilde{\sigma}_1(\zeta) = \log|S_{01}(\zeta,\alpha)| = \log|\zeta|, \tag{53}$$

where we arrive at the final equality after some algebraic manipulation which makes use of the formulae (49). Therefore $\sigma_1(\zeta) = \alpha_{11}\tilde{\sigma}_1(\zeta)$, where α_{11} is some constant chosen so that $\sigma_1(\zeta) = 1$ on C_1. There is no matrix inversion to be performed here and it follows immediately that

$$\sigma_1(\zeta) = \frac{\log|\zeta|}{\log q} = \begin{cases} 0 & \text{on } C_0, \\ 1 & \text{on } C_1. \end{cases} \tag{54}$$

On use of (54) and (45), we deduce that the first-type Green's function in the annulus is given by

$$\mathcal{G}(\zeta;\alpha) = -\frac{1}{2\pi}\left[\log\left|\frac{\alpha P(\zeta\alpha^{-1},q)}{P(\zeta\bar{\alpha},q)}\right| - \log|\alpha|\frac{\log|\zeta|}{\log q}\right], \tag{55}$$

where we have used the fact that

$$G_0(\zeta;\alpha) = -\frac{1}{4\pi}\log\left|\frac{P(\zeta\alpha^{-1},q)P(\alpha\zeta^{-1},q)}{P(\zeta\bar{\alpha},q)P(\zeta^{-1}\bar{\alpha}^{-1},q)}\right| = -\frac{1}{2\pi}\log\left|\frac{\alpha P(\zeta\alpha^{-1},q)}{P(\zeta\bar{\alpha},q)}\right|. \tag{56}$$

11. Higher connected examples

Having constructed explicit formulae for the first-type Green's function in multiply connected circular domains, the Green's function in conformally equivalent domains follows by conformal transplantation. This renders the methods here relevant to a rather broad class of domains. We now include representative examples illustrating the efficacy of our construction in various domains of connectivity greater than two.

To truncate the infinite products, it is convenient to categorize all possible compositions of the basic maps according to their 'level'. As an illustration, consider the case in which there are four basic maps $\{\theta_j \mid j = 1,2,3,4\}$. The identity map is considered to be the 'level-zero map'. The four basic maps, together with their inverses, $\{\theta_j^{-1} \mid j = 1,2,3,4\}$ constitute the eight 'level-one maps'. All possible combinations of any 'two' of these eight level-one maps which do not reduce to the identity, e.g.

$$\theta_1(\theta_1(\zeta)), \theta_1(\theta_2(\zeta)), \theta_1(\theta_3(\zeta)), \theta_1(\theta_4(\zeta)), \theta_2(\theta_1(\zeta)), \theta_2(\theta_2(\zeta)), \ldots \tag{57}$$

will be called the 'level-two maps', all possible combinations of any 'three' of the eight level-one maps that do not reduce to a lower-level map will be called the 'level-three maps' and so on.

In all the calculations to follow, the Schottky–Klein prime function is computed to level-three accuracy, meaning that all levels up to level three are included in the product while all higher level terms are truncated. It is known from comparison studies with other (more accurate) methods of computing the prime function—see Crowdy & Marshall (2007)—that one can typically expect between 4 and 6 digits of accuracy with this truncation. This is more than adequate for many applications. Greater accuracy can be obtained by using higher levels of truncation.

When finding the matrix $\mathbf{A}_{ij}$, note that one can either use the definition (41) directly together with (21) or, alternatively, the equivalent formulae given in (42)–(44). Indeed, a comparison of the matrices obtained by both methods can be used as a numerical check.

11.1 *Green's functions of circular domains*

A basic problem in electrostatics is to compute the electric field generated by a point charge in the presence of a finite distribution of conductors. Let there be $M + 1$ such conductors and let them all be circular and let the point charge be at z_α. Since the conductors are equipotentials, the mathematical problem for the electric field potential is that of finding the first-type Green's function, with singularity at z_α, in the unbounded domain, call it D_z, exterior to the finite set of circular conductors.

Figures 3 and 4 depict examples of bounded triply connected circular domains with the singularity placed at different points in the domain. In each figure, to the left, just the critical contours are shown (the critical contour values are given in the caption), while to the right, a global distribution of contours is shown. A common feature is the presence of exactly two 'critical points' of the Green's function at which two 'separatrix' contours separate the region into qualitatively distinct contour types. These critical points are points in the domain at which both first partial derivatives of the Green's function

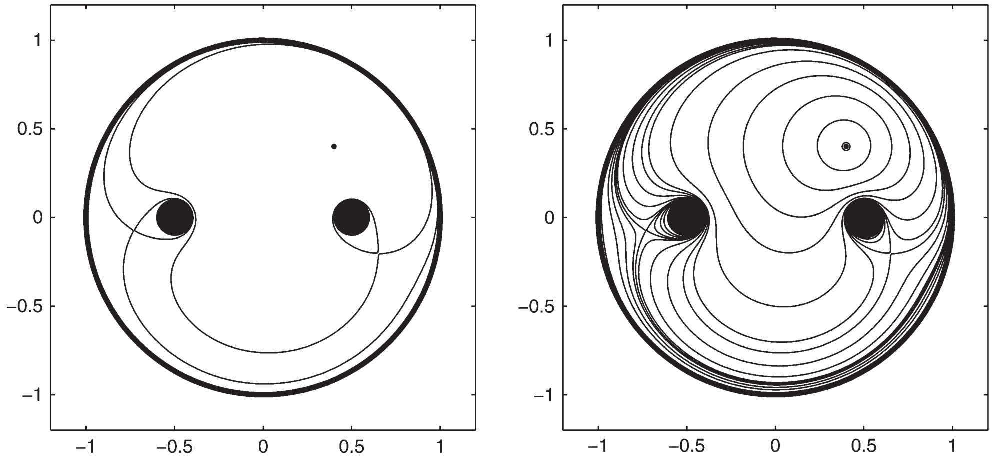

FIG. 3. Unit disc with two equipotential conductors of radius 0.1 centred at ±0.5. The singularity is at 0.4(1 + i). The critical contours are shown on the left and correspond to level-line values 0.0038 and 0.0009.

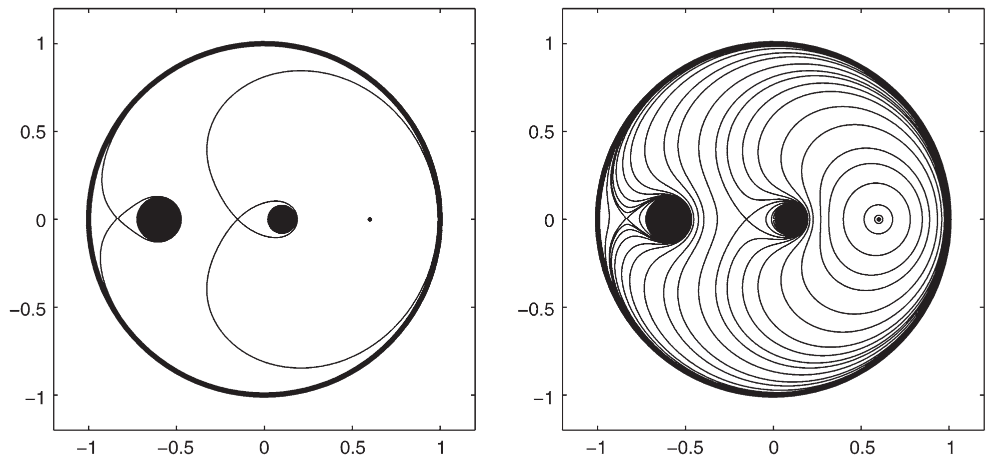

FIG. 4. Unit disc with two equipotential conductors, one of radius 0.125 centred at -0.6 and another of radius 0.08 centred at 0.1. The singularity is at 0.6. The critical contours are shown on the left and correspond to level-line values 0.0072 and 0.0001.

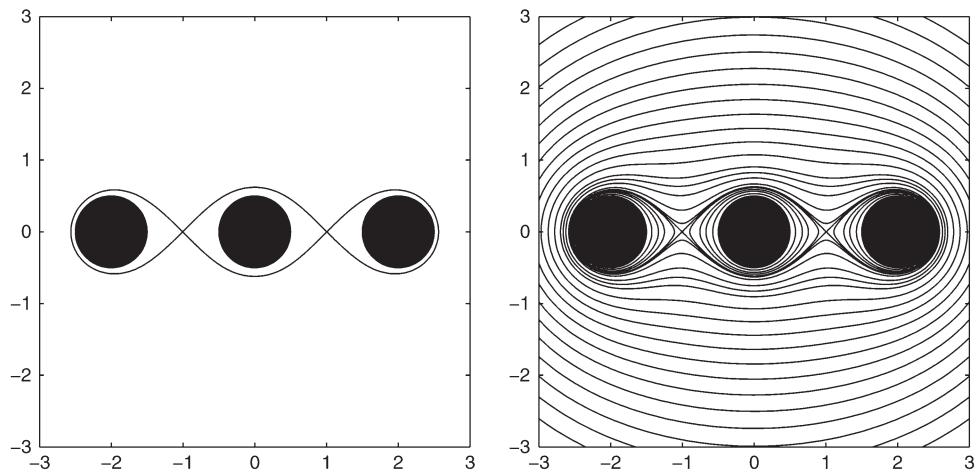

FIG. 5. Three equal equipotential conductors each of radius 0.5 centred at ± 2 and 0. The singularity is at infinity. The critical contour is shown on the left and corresponds to level-line value 0.0105.

vanish simultaneously. The fact that there are just two such points is consistent with a general result (Nevanlinna, 1970) on Green's functions in multiply connected domains which says that the Green's function of an $(M+1)$-connected domain has precisely M critical points.

Möbius mappings map circles to circles. Such a map will therefore map a bounded circular preimage region D_ζ to the unbounded circular region D_z exterior to some circular discs. Figures 5 and 6 show a series of illustrative examples of unbounded triply connected circular domains obtained by such a conformal map from a bounded preimage region. Figure 5 shows the case where the singularity of the Green's function (or point charge) is at infinity. In this case, the symmetry of the configuration is such that the two critical points are on the real axis between the circular conductors. Figure 6 shows how complex the geometry of the critical contours can become for a general position of the singularity.

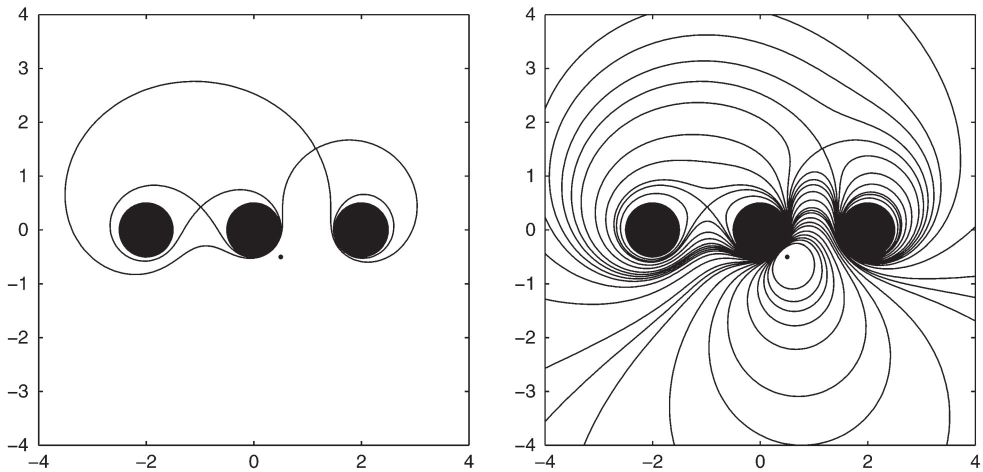

FIG. 6. Three equal equipotential conductors each of radius 0.5 centred at ± 2 and 0. The singularity is at $0.5(1 - \mathrm{i})$. The critical contours are shown on the left and correspond to level-line values 0.0016 and 0.0055.

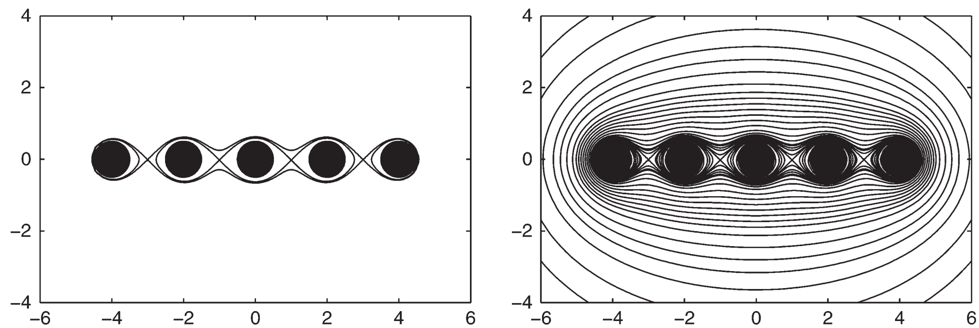

FIG. 7. Five equal equipotential conductors each of radius 0.5 centred at ± 4, ± 2 and 0. The singularity is at infinity. The critical contours are shown on the left and correspond to level-line values 0.0060 and 0.0076.

Higher connected regions can be tackled with no extra difficulty. Figure 7 shows a quintuply connected unbounded domain with the point charge out at infinity. Figure 8 shows the same configuration of conductors but with the point charge now at a finite position.

It is important to give some quantitative validation of the new construction. A useful diagnostic is furnished by the critical points. Table 1 gives a quantitative comparison of the position, on the positive real axis, of the critical point of the Green's function exterior to a left–right symmetric distribution of three circular conductors where the central cylinder intersects the real axis at $\pm a$, the left-most cylinder at $-b$ and $-c$ and the right-most at b and c, where $a < b < c$. The singularity is at infinity. By the symmetry, the critical points will be at $\pm e_1$ for some real e_1 reported in the table. Shown in Table 1 is a comparison of the values determined by the formulae derived above and those given by a numerical method, due to Trefethen (2005), based on Laurent expansions of the (complex) Green's function about the centres of the conductors and use of a least squares method to ensure that the boundary conditions are satisfied (the MATLAB code is called many_disks.m (Trefethen, 2005)). Twenty terms in each of the Laurent expansions are used and the errors in the least squares fit are found to be of the order of 10^{-8} to 10^{-10}. The table shows that the results agree within 4–6 digits of accuracy in all cases. This

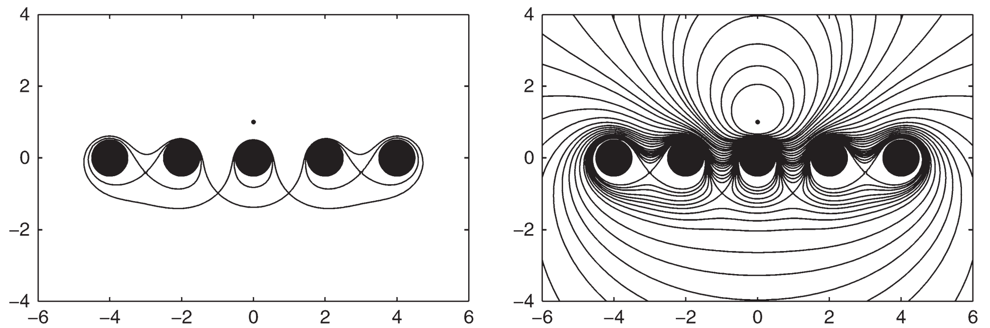

FIG. 8. Five equal equipotential conductors each of radius 0.5 centred at ±4, ±2 and 0. The singularity is at (0, 1). The critical contours are shown on the left and correspond to level-line values 0.0016 and 0.0033.

TABLE 1 *Comparison of critical point positions of differing right–left symmetric distributions of three conductors on the real axis*

Geometry (a, b, c)	New method e_1	Least squares e_1
(0.1, 0.3, 0.5)	0.2009235	0.2009345
(0.1, 0.5, 0.7)	0.3047789	0.3047020
(0.1, 0.7, 0.9)	0.4104804	0.4104831
(0.3, 0.7, 0.9)	0.5250642	0.5250685
(0.3, 0.5, 0.7)	0.4093054	0.4093686

is as expected given the level-three truncation of the prime function. Greater accuracy can be obtained by using a higher level of truncation or by employing alternative means (Crowdy & Marshall, 2007) of computing the prime function.

11.2 *Potential theory of several intervals*

The mapping formula from a bounded circular domain D_ζ to the unbounded region exterior to a symmetric arrangement of (an odd number) intervals on the real axis in a z-plane is given by

$$z(\zeta) = C\left(\frac{\omega(\zeta,-1)^2 + \omega(\zeta,1)^2}{\omega(\zeta,-1)^2 - \omega(\zeta,1)^2}\right), \tag{58}$$

where C is some real constant. The preimage region D_ζ must have a distribution of circles $\{C_j \,|\, j = 1, \ldots, M\}$ that are centred on the real ζ-axis and reflectionally symmetric about the imaginary axis. The present authors have not seen this result reported in the literature, but to see how to construct it, consider the sequence of conformal mappings given by

$$\begin{aligned}
\zeta_1(\zeta) &= -\frac{\omega(\zeta,1)}{\omega(\zeta,-1)},\\
\zeta_2(\zeta_1) &= \frac{1-\zeta_1}{1+\zeta_1},\\
z(\zeta_2) &= \frac{C}{2}\left(\zeta_2 + \frac{1}{\zeta_2}\right),
\end{aligned} \tag{59}$$

where C is any real constant. A schematic illustrating this sequence of mappings in the triply connected case (i.e. three intervals) is shown in Fig. 9. The first mapping takes the circular region D_ζ to the right-half ζ_1-plane with a number of finite-length slits on the real axis. To see this, note that $\zeta = 1$ maps to $\zeta_1 = 0$, while $\zeta = -1$ maps to $\zeta_1 = \infty$. To see that the rest of the unit circle maps to the imaginary ζ_1-axis, note that on C_0 we have

$$\overline{\zeta_1(\zeta)} = -\frac{\overline{\omega}(\zeta^{-1}, 1)}{\overline{\omega}(\zeta^{-1}, -1)} = \frac{\omega(\zeta, 1)}{\omega(\zeta, -1)} = -\zeta_1(\zeta), \tag{60}$$

where we have used

$$\overline{\omega}(\zeta^{-1}, \gamma^{-1}) = -\zeta^{-1}\gamma^{-1}\omega(\zeta, \gamma) \tag{61}$$

which is an identity whose proof is given in an appendix to Crowdy & Marshall (2005). By similar manipulations, it is possible to show that the image of any interior circle C_j centred on the real ζ-axis under the mapping $\zeta_1(\zeta)$ has constant argument equal to zero. Hence, the boundary of any interior circular disc

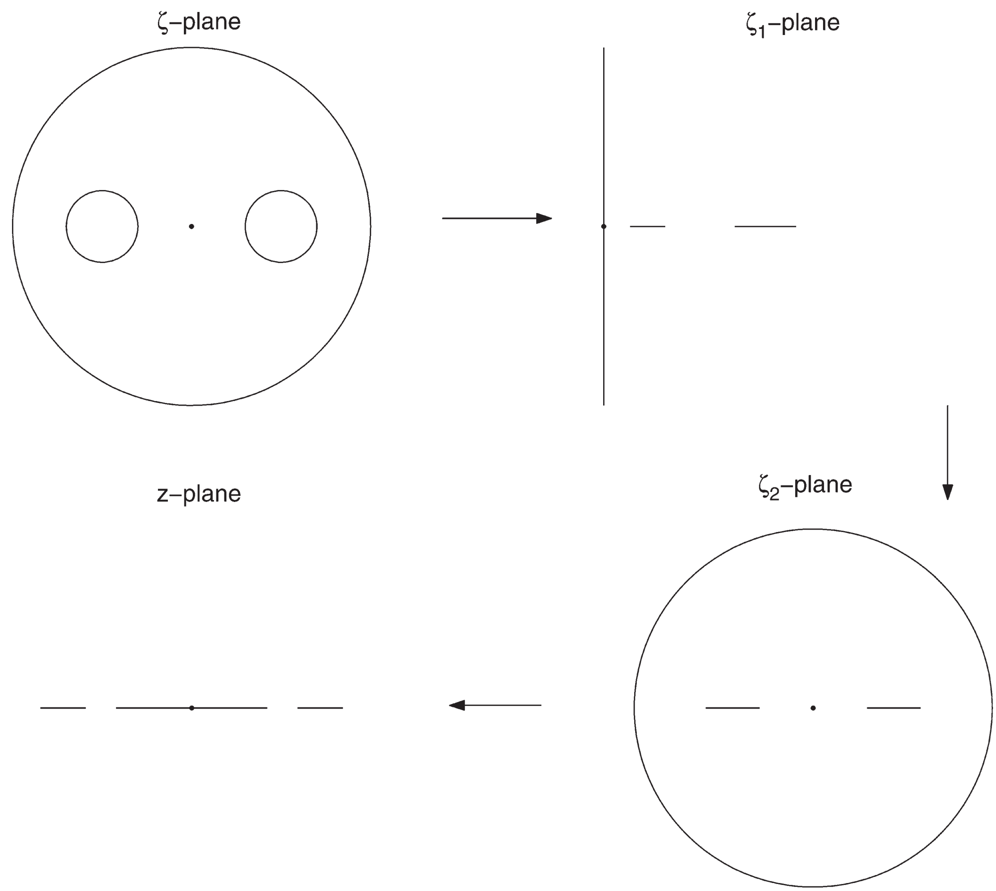

FIG. 9. Schematic illustrating the composition of conformal mappings producing the formula for the triply connected slit mapping. The origin in each conformal mapping plane is shown as a dot.

centred on the real ζ-axis maps to a slit on the positive real ζ_1-axis. The second Möbius mapping maps this slit half-plane region in the ζ_1-plane to the unit ζ_2-disc similarly cut along its diameter on the real axis by finite-length slits. The third Joukowski mapping maps the interior of the unit disc in the ζ_2-plane to the whole of the z-plane exterior to a series of finite-length slits on the real axis. A composition of these maps yields (58).

Figures 10 and 11 show the case of three equipotential slits along the real axis with the singularity at infinity (Fig. 10) and at a finite point (Fig. 11). Figures 12 and 13 show two additional slit configurations with the singularity at infinity. Note, from Fig. 13, how close to the slits the critical contour becomes as the gap intervals close up.

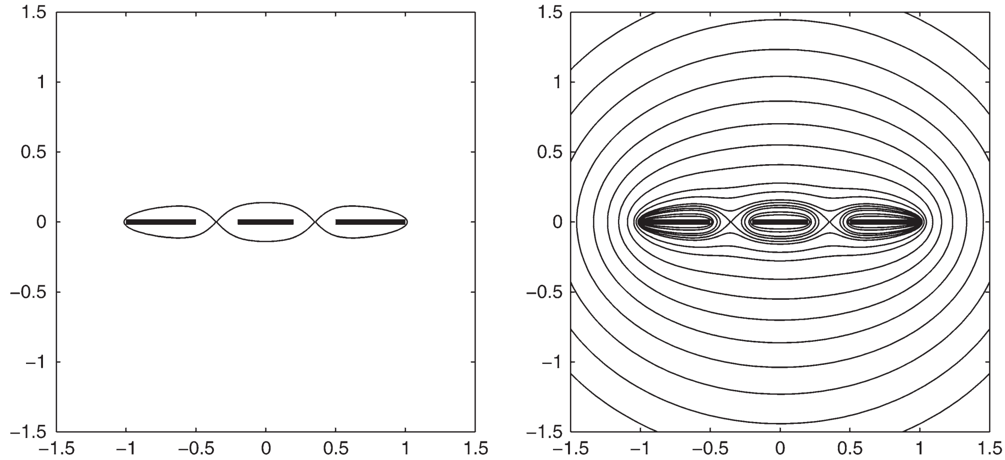

FIG. 10. Three equipotential intervals between $[-1, -0.5]$, $[-0.2, 0.2]$ and $[0.5, 1]$. The singularity is at infinity (the preimage is $\alpha = 0$). The critical contour is shown on the left and corresponds to the level-line value 0.0266.

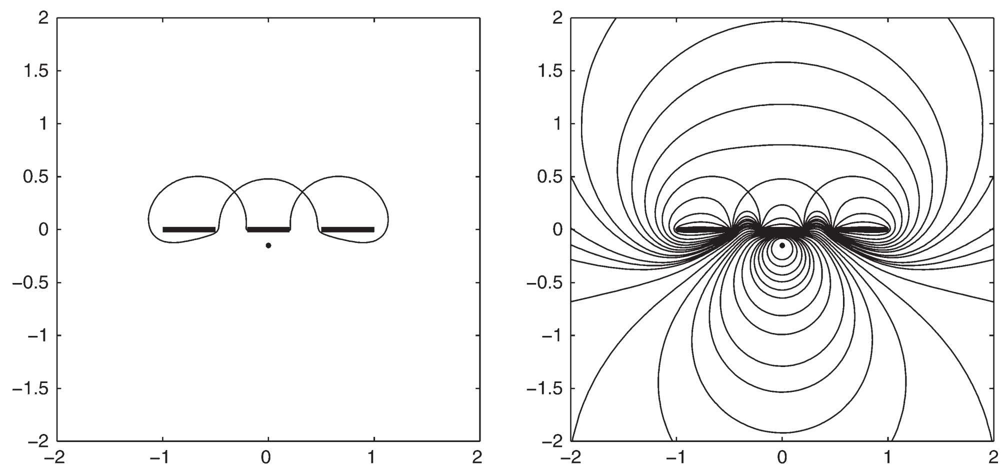

FIG. 11. Three equipotential intervals between $[-1, -0.5]$, $[-0.2, 0.2]$ and $[0.5, 1]$. The singularity is at $-0.1513\mathrm{i}$ (the preimage is $\alpha = 0.5\mathrm{i}$). The critical contour is shown on the left and corresponds to the level-line value 0.0130.

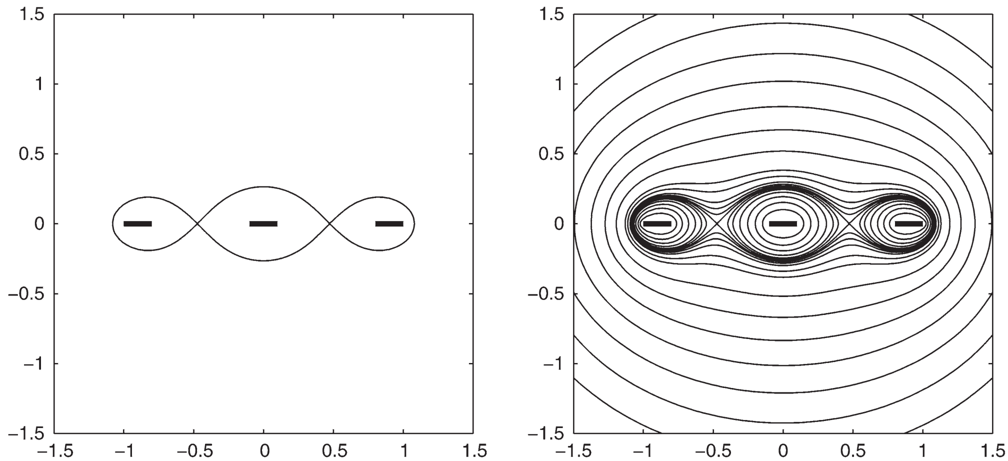

FIG. 12. Three equipotential intervals between $[-1, -0.8]$, $[-0.1, 0.1]$ and $[0.8, 1]$. The singularity is at infinity (the preimage is $\alpha = 0$). The critical contour is shown on the left and corresponds to the level-line value 0.0798.

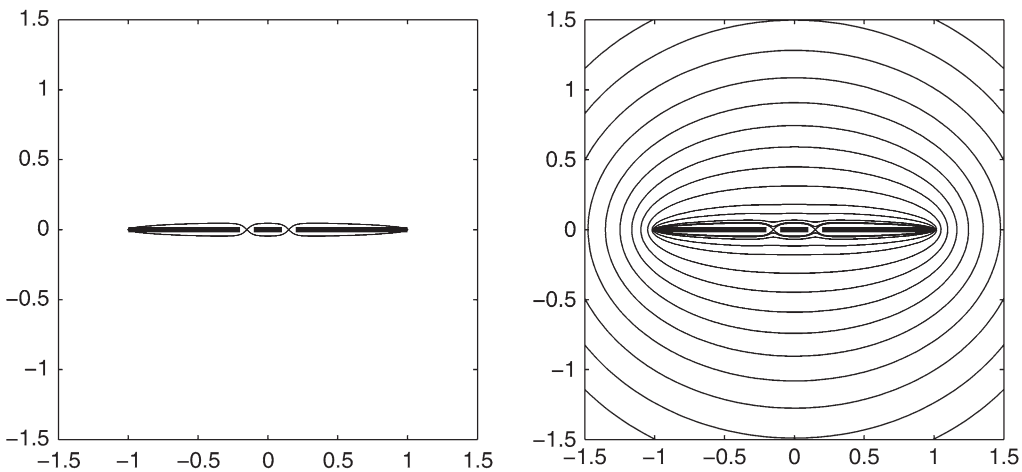

FIG. 13. Three equipotential intervals between $[-1, -0.2]$, $[-0.1, 0.1]$ and $[0.2, 1]$. The singularity is at infinity (the preimage is $\alpha = 0$). The critical contour is shown on the left and corresponds to the level-line value 0.0082.

For quantitative validation, we exploit the fact that the potential theory of several intervals on the real line can be addressed using alternative arguments. Shen *et al.* (2001) have studied this problem (and its applications to digital filter design and polynomial-based matrix iteration methods) based on the formulation of Embree & Trefethen (1999). In Shen *et al.* (2001), an independent means is given for determining the positions of the critical points of the Green's function based on quadratures and the solution of a linear system. This method is now briefly described. It is then used to compare with the values for the critical points obtained from the new formulae of this paper.

Consider a set of n disjoint intervals $[-1, a_1], [b_1, a_2], \ldots, [a_{n-1}, b_n], [a_n, 1]$. Let $\{e_k | k = 1, \ldots, n\}$ be the critical points of the Green's function associated with the domain exterior to these intervals.

Define the polynomials

$$Q(z) = (z^2 - 1)\prod_{k=1}^{n}(z - a_k)(z - b_k),$$

$$P(z) = \prod_{k=1}^{n}(z - e_k) \equiv z^n - c_1 z^{n-1} - c_2 z^{n-2} - \cdots - c_n. \tag{62}$$

Let $\{I_k | k = 1, \ldots, n\}$ denote the n gap intervals. Then, according to Theorem 3 of Shen *et al.* (2001), if

$$M_{jk} = \int_{I_j} \frac{t^{n-k}\,\mathrm{d}t}{\sqrt{Q(t)}}, \quad \mathbf{b}_j = \int_{I_j} \frac{t^n\,\mathrm{d}t}{\sqrt{Q(t)}}, \tag{63}$$

then the vector $\mathbf{c} \equiv (c_1, c_2, \ldots, c_n)^{\top}$ solves the linear system $M\mathbf{c} = \mathbf{b}$. Having solved for $\mathbf{c}$, the values $\{e_k | k = 1, \ldots, n\}$ can then be determined.

Table 2 gives data for a symmetric distribution of three intervals $[-1, -b]$, $[-a, a]$ and $[b, 1]$ along the real axis together with the position, e_1, of the critical point between these intervals as calculated by the formulae of this paper and Shen *et al.* (2001). The values of the second column of the table are computed using the formulae of this paper (with level-three truncation) together with Newton's method to find the zero of the derivative of the Green's function in the gap interval. In all cases, agreement to several digits of accuracy is found.

Table 3 gives the data for a symmetric distribution of five intervals $[-1, -d]$, $[-c, -b]$, $[-a, a]$, $[b, c]$ and $[d, 1]$ along the real axis and the positions e_1 (between a and b) and e_2 (between c and d) of the critical points in the gap intervals as calculated by the formulae of this paper (to level-three accuracy) and Shen *et al.* (2001). Again, there is a good agreement to several decimal places.

TABLE 2 *Comparison of critical point positions of differing right–left symmetric distributions of three equipotential slits on the real axis*

Geometry (a, b)	New method e_1	Shen *et al.* e_1
(0.1, 0.2)	0.150077	0.150067
(0.1, 0.3)	0.200123	0.200119
(0.1, 0.4)	0.250416	0.250414
(0.1, 0.5)	0.301591	0.301591
(0.1, 0.6)	0.473644	0.473644
(0.1, 0.7)	0.550757	0.550757

TABLE 3 *Comparison of critical point positions of differing right–left symmetric distributions of five equipotential slits on the real axis*

Geometry (a, b, c, d)	New method (e_1, e_2)	Shen *et al.* (e_1, e_2)
(0.05, 0.15, 0.25, 0.35)	(0.100161, 0.299993)	(0.100071, 0.299885)
(0.1, 0.35, 0.55, 0.8)	(0.226884, 0.683378)	(0.226896, 0.683390)
(0.18, 0.23, 0.59, 0.64)	(0.205137, 0.613791)	(0.205067, 0.615306)

12. Discussion

This paper has described a new constructive method, based on analytical formulae involving the Schottky–Klein prime function, for finding the first-type Green's function in multiply connected bounded circular domains. By conformal transplantation, the first-type Green's functions in more general multiply connected domains can also be found. The method exploits connections between potential theory and conformal mapping theory and, to render the ideas concrete and constructive, combines these with the results from classical function theory to build the relevant expressions. The efficacy of the construction has been validated by comparison, in a number of special example cases, to two independent (numerical and analytical) means of computing the Green's function.

Our emphasis has been on analytical formulae and we have employed an infinite product representation of the Schottky–Klein prime function (and we have assumed that the domain D_ζ is such that this product is convergent). The prime function is a well-defined function even when the infinite product representation is not convergent. In such cases, other schemes for computing the prime function—e.g. that presented recently by Crowdy & Marshall (2007)—can be used in conjunction with the new formulae (for the modified Green's functions, harmonic measures and the first-type Green's function) presented in this paper.

There is a conceptually different way to compute the first-type Green's function in a multiply connected domain that is based on a 'dual' view of the theory (this alternative approach lies at the heart of the treatment of Julia (1934)). Instead of performing the analysis in a multiply connected domain as we have done, one can alternatively introduce a series of 'cross-cuts' in the domain to render it simply connected. Then, it is known that one can uniformize the boundaries of the resulting simply connected domain by means of a conformal mapping from an upper half-plane with $2M$ half-discs, all centred on the real axis, excised. The $2M$ semicircular boundaries of such a domain correspond to the $2M$ cross-cuts (each of the M cross-cuts has two sides), while the remainder of the real axis corresponds to the original boundaries of the multiply connected domain. This is sometimes referred to as the Fuchsian uniformization of the domain and, in a manner akin to that employed here, it is possible to employ function-theoretic results in this alternative uniformization domain to find the relevant formulae for the first-type Green's function.

Finally, one of the many uses of the first-type Green's function is as an analytical tool for finding the solution of the classical Schwarz problem (and modified Schwarz problem) in multiply connected domains (Mityushev & Rogosin, 2000). In recent work, Crowdy (2007) has presented novel integral formulae, again expressed in terms of the Schottky–Klein prime function, for the general solution of the Schwarz problem in multiply connected circular domains. The approach in Crowdy (2007) has close connections to the results presented here.

Acknowledgements

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Appendix A. Properties of $\{\beta_k(\overline{\phi}_j(\bar{\alpha}), \bar{\alpha}^{-1})\}$

In this appendix, the properties of $\{\beta_k(\overline{\phi}_j(\bar{\alpha}), \bar{\alpha}^{-1})\}$ will be derived. By definition,

$$\beta_k(\overline{\phi}_j(\bar{\alpha}), \bar{\alpha}^{-1}) = \prod_{\theta \in \Theta_k} \left(\frac{\overline{\phi}_j(\bar{\alpha}) - \theta(B_k)}{\overline{\phi}_j(\bar{\alpha}) - \theta(A_k)} \right) \left(\frac{\bar{\alpha}^{-1} - \theta(A_k)}{\bar{\alpha}^{-1} - \theta(B_k)} \right). \tag{A.1}$$

Applying θ^{-1} to each term in this cross-ratio, we can also write

$$\begin{aligned} \beta_k(\overline{\phi}_j(\bar{\alpha}), \bar{\alpha}^{-1}) &= \prod_{\theta \in \Theta_k} \left(\frac{\theta^{-1}(\overline{\phi}_j(\bar{\alpha})) - B_k}{\theta^{-1}(\overline{\phi}_j(\bar{\alpha})) - A_k} \right) \left(\frac{\theta^{-1}(\bar{\alpha}^{-1}) - A_k}{\theta^{-1}(\bar{\alpha}^{-1}) - B_k} \right) \\ &= \prod_{\theta \in \Theta_k} \left(\frac{\theta^{-1}(\theta_j(\bar{\alpha}^{-1})) - B_k}{\theta^{-1}(\theta_j(\bar{\alpha}^{-1})) - A_k} \right) \left(\frac{\theta^{-1}(\bar{\alpha}^{-1}) - A_k}{\theta^{-1}(\bar{\alpha}^{-1}) - B_k} \right). \end{aligned} \tag{A.2}$$

Now observe that if $\theta \in \Theta_k$, then $\theta^{-1} \in {}_k\Theta$. So $\{\theta^{-1} | \theta \in \Theta_k\} \subseteq {}_k\Theta$. But, if $\tilde{\theta} \in {}_k\Theta$ then $\tilde{\theta}^{-1} \in \Theta_k$, so $\tilde{\theta}^{-1} = \theta$ for some $\theta \in \Theta_k$. So $\tilde{\theta} = \theta^{-1}$ and ${}_k\Theta \subseteq \{\theta^{-1} | \theta \in \Theta_k\}$. We therefore conclude that $\{\theta^{-1} | \theta \in \Theta_k\} = {}_k\Theta$. It follows that

$$\beta_k(\overline{\phi}_j(\bar{\alpha}), \bar{\alpha}^{-1}) = \prod_{\theta \in_k \Theta} \left(\frac{\theta(\theta_j(\bar{\alpha}^{-1})) - B_k}{\theta(\theta_j(\bar{\alpha}^{-1})) - A_k} \right) \left(\frac{\theta(\bar{\alpha}^{-1}) - A_k}{\theta(\bar{\alpha}^{-1}) - B_k} \right). \tag{A.3}$$

We now consider the cases $j \neq k$ and $j = k$ separately.

Case $k \neq j$: For every $\theta \in {}_k\Theta$, we can write $\theta = \psi\theta_j^r$, where $\psi \in {}_k\Theta_j$ and $r \in \mathbb{Z}$. So ${}_k\Theta \in \{\psi\theta_j^r | \psi \subseteq {}_k\Theta_j,\ r \in \mathbb{Z}\}$ but it is also obvious that $\{\psi\theta_j^r | \psi \in {}_k\Theta_j,\ r \in \mathbb{Z}\} \subseteq {}_k\Theta$. Therefore, ${}_k\Theta = \{\psi\theta_j^r | \psi \in_k \Theta_j, r \in \mathbb{Z}\}$. Equation (A.3) can be rewritten as

$$\begin{aligned}
\beta_k(\overline{\phi}_j(\bar{\alpha}), \bar{\alpha}^{-1}) &= \prod_{\psi \in_k \Theta_j} \prod_{r \in \mathbb{Z}} \left(\frac{\psi(\theta_j^{r+1}(\bar{\alpha}^{-1})) - B_k}{\psi(\theta_j^{r+1}(\bar{\alpha}^{-1})) - A_k} \right) \left(\frac{\psi(\theta_j^r(\bar{\alpha}^{-1})) - A_k}{\psi(\theta_j^r(\bar{\alpha}^{-1})) - B_k} \right) \\
&= \prod_{\psi \in_k \Theta_j} \left(\frac{\psi(\theta_j^{\infty}(\bar{\alpha}^{-1})) - B_k}{\psi(\theta_j^{\infty}(\bar{\alpha}^{-1})) - A_k} \right) \left(\frac{\psi(\theta_j^{-\infty}(\bar{\alpha}^{-1})) - A_k}{\psi(\theta_j^{-\infty}(\bar{\alpha}^{-1})) - B_k} \right) \\
&= \prod_{\psi \in_k \Theta_j} \left(\frac{\psi(B_j) - B_k}{\psi(B_j) - A_k} \right) \left(\frac{\psi(A_j) - A_k}{\psi(A_j) - B_k} \right),
\end{aligned} \tag{A.4}$$

where the second equality follows because of cancellations between most of the terms in the second product and the third equality follows from (24). First, note that the final expression of (A.4) is independent of α. Second, note that the right-hand side of (A.4) corresponds precisely to the definition of $\exp(2\pi \mathrm{i}\tau_{j,k})$ given in Chapter 12 of Baker (1995).

Case $k = j$: We can consider the set $\{\psi\theta_k^r | \psi \in_k \Theta_k', r \in \mathbb{Z}\}$ which is contained in the set ${}_k\Theta'$. But if $\theta \in_k \Theta'$, we can write $\theta = \psi\theta_k^r$ for some $\psi \in_k \Theta_k'$. So ${}_k\Theta' \subseteq \{\psi\theta_k^r | \psi \in_k \Theta_k', r \in \mathbb{Z}\}$. Thus, ${}_k\Theta'$ is the same as $\{\psi\theta_k^r | \psi \in_k \Theta_k', r \in \mathbb{Z}\}$. Thus, it follows that

$${}_k\Theta = \{\text{identity map}\} \cup \{\psi\theta_k^r | \psi \in_k \Theta_k',\ r \in \mathbb{Z}\}. \tag{A.5}$$

So, from (A.3), separating off the identity term yields

$$\begin{aligned}
\beta_k(\overline{\phi}_k(\bar{\alpha}), \bar{\alpha}^{-1}) = &\left(\frac{\theta_k(\bar{\alpha}^{-1}) - B_k}{\theta_k(\bar{\alpha}^{-1}) - A_k} \right) \left(\frac{\bar{\alpha}^{-1} - A_k}{\bar{\alpha}^{-1} - B_k} \right) \\
&\times \prod_{\psi \in_k \Theta_k'} \prod_{r \in \mathbb{Z}} \left(\frac{\psi(\theta_k^{r+1}(\bar{\alpha}^{-1})) - B_k}{\psi(\theta_k^{r+1}(\bar{\alpha}^{-1})) - A_k} \right) \left(\frac{\psi(\theta_k^r(\bar{\alpha}^{-1})) - A_k}{\psi(\theta_k^r(\bar{\alpha}^{-1})) - B_k} \right) \\
= &\left(\frac{\theta_k(\bar{\alpha}^{-1}) - B_k}{\theta_k(\bar{\alpha}^{-1}) - A_k} \right) \left(\frac{\bar{\alpha}^{-1} - A_k}{\bar{\alpha}^{-1} - B_k} \right) \\
&\times \prod_{\psi \in_k \Theta_k'} \left(\frac{\psi(\theta_k^{\infty}(\bar{\alpha}^{-1})) - B_k}{\psi(\theta_k^{\infty}(\bar{\alpha}^{-1})) - A_k} \right) \left(\frac{\psi(\theta_k^{-\infty}(\bar{\alpha}^{-1})) - A_k}{\psi(\theta_k^{-\infty}(\bar{\alpha}^{-1})) - B_k} \right),
\end{aligned} \tag{A.6}$$

where the second equality follows from cancellations in most of the terms of the second product. But, we know that

$$\left(\frac{\theta_k(\bar{\alpha}^{-1}) - B_k}{\theta_k(\bar{\alpha}^{-1}) - A_k}\right)\left(\frac{\bar{\alpha}^{-1} - A_k}{\bar{\alpha}^{-1} - B_k}\right) = \mu_k \, \mathrm{e}^{\mathrm{i}\kappa_k}, \tag{A.7}$$

so that, on use of (A.7) and (24) it follows from (A.6) that,

$$\begin{aligned}
\beta_k(\overline{\phi}_k(\bar{\alpha}), \bar{\alpha}^{-1}) &= \mu_k \, \mathrm{e}^{\mathrm{i}\kappa_k} \prod_{\psi \in_k \Theta'_k} \left(\frac{\psi(B_k) - B_k}{\psi(B_k) - A_k}\right)\left(\frac{\psi(A_k) - A_k}{\psi(A_k) - B_k}\right) \\
&= \mu_k \, \mathrm{e}^{\mathrm{i}\kappa_k} \prod_{\psi \in_k \Theta''_k} \left(\frac{\psi(B_k) - B_k}{\psi(B_k) - A_k}\right)\left(\frac{\psi(A_k) - A_k}{\psi(A_k) - B_k}\right) \\
&\quad \times \left(\frac{\psi^{-1}(B_k) - B_k}{\psi^{-1}(B_k) - A_k}\right)\left(\frac{\psi^{-1}(A_k) - A_k}{\psi^{-1}(A_k) - B_k}\right) \\
&= \mu_k \, \mathrm{e}^{\mathrm{i}\kappa_k} \prod_{\psi \in_k \Theta''_k} \left[\left(\frac{\psi(B_k) - B_k}{\psi(B_k) - A_k}\right)\left(\frac{\psi(A_k) - A_k}{\psi(A_k) - B_k}\right)\right]^2,
\end{aligned} \tag{A.8}$$

where the last line follows by applying ψ to each term in the second cross-ratio in the infinite product. First note that the final term in (A.8) is independent of α. Second, the right-hand side of (A.8) turns out to be precisely the quantity defined as $\exp(2\pi \mathrm{i}\tau_{k,k})$ in Chapter 12 of Baker (1995).

In summary, it has been shown that the quantities $\beta_k(\overline{\phi}_j(\bar{\alpha}), \bar{\alpha}^{-1})$ are independent of the value of α. Further, it is found that

$$\beta_k(\overline{\phi}_j(\bar{\alpha}), \bar{\alpha}^{-1}) = \exp(2\pi \mathrm{i}\tau_{j,k}), \quad \text{for } j, k = 1, \ldots, m, \tag{A.9}$$

where the quantities on the right-hand side of (A.9) are defined in Chapter 12 of Baker (1995).