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Analytical formulae for the Kirchhoff–Routh path function in multiply connected domains

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Explicit formulae for the Kirchhoff–Routh path functions (or Hamiltonians) governing the motion of N-point vortices in multiply connected domains are derived when all circulations around the holes in the domain are zero. The method uses the Schottky–Klein prime function to find representations of the hydrodynamic Green's function in multiply connected circular domains. The Green's function is then used to construct the associated Kirchhoff–Routh path function. The path function in more general multiply connected domains then follows from a transformation property of the path function under conformal mapping of the canonical circular domains. Illustrative examples are presented for the case of single vortex motion in multiply connected domains.

Keywords: point vortex; Kirchhoff–Routh; Hamiltonian

1. Introduction

The study of point vortex dynamics is an important area of fluid dynamics already commanding a vast literature. The review by Aref *et al.* (2002) provides a recent survey of results involving vortex equilibria (or vortex crystals), mainly in unbounded and periodic configurations, while the recent monograph by Newton (2002) gives a broader perspective of the general N-vortex problem, including discussions of vortex motion in unbounded and bounded planar domains, as well as on curved surfaces such as the surface of a sphere.

Although the motion of point vortices in unbounded domains has received much attention, the theory of point vortex motion in domains bounded by impenetrable walls is much less developed. The simplest example is a single-point vortex adjacent to an infinite straight wall. Such a vortex translates at constant speed, maintaining a constant distance from the wall. This motion is conveniently understood as being induced by an opposite circulation 'image' vortex behind the wall. This is perhaps the simplest example of the celebrated 'method of images' (Milne-Thomson 1968). Several more elaborate examples involving simply connected fluid regions are given in ch. 3 of Newton (2002) while others are described by Saffman (1992). Many of these examples rely on the transformation properties, under conformal mapping, of what is known as the Kirchhoff–Routh path function, which is essentially the Hamiltonian governing the vortex motion. The Hamiltonian formulation of point vortex dynamics and

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the Kirchhoff–Routh path function date back to the work of Kirchhoff & Routh (Routh 1881). It was reappraised much later by Lin (1941*a*,*b*), who considered multiply connected domains, and more recently by Flucher & Gustafsson (1997) (see also ch. 15 of Flucher 1999), who have analysed various aspects of the general boundary-value problem arising from the problem of point vortex motion in bounded domains.

The motion of a single vortex in bounded, simply connected domains is relatively well studied. Gustafsson (1979) and Richardson (1980) have shown that the Kirchhoff–Routh path function satisfies an elliptic Liouville equation in the bounded domain D and is infinite everywhere on the boundary. On the subject of N-vortex motion in multiply connected domains, the literature is sparse. Lin (1941*a*) established the existence and uniqueness of a generalized Kirchhoff–Routh path function in this case, but does not construct it explicitly or give any specific examples.

In this paper, an analytical formula for the hydrodynamic Green's function introduced by Lin (1941*a*) is found in the class of multiply connected circular domains. This is achieved using a special transcendental function called the Schottky–Klein prime function (Baker 1995). A *circular domain* is a planar domain all of whose boundary components are circles. By using this Green's function, formulae for the associated Kirchhoff–Routh path function for general N-vortex motion in such circular domains can be constructed. However, Lin (1941*b*) has also shown how to derive formulae for the Kirchhoff–Routh path function in conformally equivalent, multiply connected domains. Thus, if a formula for the conformal mapping from a given circular multiply connected domain to a more general domain is known, then the path function in the new domain can be constructed in an analytical form.

2. The hydrodynamic Green's function

Lin (1941*a*) introduced a special Green's function $G(x, y; x_0, y_0)$ with respect to the two points (x, y) and (x_0, y_0) in a fluid domain D in the following way. Three separate cases of domain D (cases 1–3 below) are considered depending on whether D is bounded or unbounded. Let $M \geq 0$ be an integer. Suppose D is bounded by $M+1$ impenetrable walls, and let these boundaries of D be $\{C_j | j=0,1,\ldots,M\}$. If D is bounded, then C_0 will be taken as the outer boundary with $\{C_k | k=1,\ldots,M\}$ denoting the M enclosed boundaries. If D is unbounded but has a boundary extending to infinity, then this infinite-length boundary will be denoted C_0. Lin's special hydrodynamic Green's function is the function $G(x, y; x_0, y_0)$ satisfying the following properties.

(i) The function

$$g(x, y; x_0, y_0) = -G(x, y; x_0, y_0) - \frac{1}{2\pi} \log r_0, \tag{2.1}$$

is harmonic with respect to (x, y) throughout the region D including at the point $(x_0,\ y_0)$. Here, r_0 is

$$r_0 = \sqrt{(x-x_0)^2+(y-y_0)^2}. \tag{2.2}$$

(ii) If $\partial G/\partial n$ is the normal derivative of G on a curve, then

$$\left.\begin{aligned} G(x,y;x_0,y_0) &= A_k, \quad \text{on } C_k,\ k=1,\ldots,M, \\ \oint_{C_k}\frac{\partial G}{\partial n}\mathrm{d}s &= 0, \qquad\qquad k=1,\ldots,M, \end{aligned}\right\} \tag{2.3}$$

where ds denotes an element of arc and $\{A_k|k=1,\ldots,M\}$ are constants.

(iii) *Case 1*. If D has a closed outer boundary C_0, then

$$G(x,y;x_0,y_0) = 0 \quad \text{on } C_0. \tag{2.4}$$

(iv) *Case 2*. If D is unbounded and extends to infinity in all directions, then over a very large circle of radius r_0, G behaves as follows:

$$\left.\begin{aligned} G(x,y;x_0,y_0) &= -\frac{1}{2\pi}\log r_0 + \mathcal{O}(1/r_0), \\ \frac{\partial G}{\partial s} &= \mathcal{O}(1/r_0^2), \\ \frac{\partial G}{\partial n} &= -\frac{1}{2\pi r_0} + \mathcal{O}(1/r_0^2), \end{aligned}\right\} \tag{2.5}$$

where $\partial G/\partial s$ is the tangential derivative along the circle.

(v) *Case 3*. If D is unbounded but has boundaries extending to infinity, then G behaves as follows:

$$\left.\begin{aligned} G(x,y;x_0,y_0) &= 0, \qquad \text{on } C_0, \\ G(x,y;x_0,y_0) &= o(1), \quad \text{on a very large circle of radius } r_0. \end{aligned}\right\} \tag{2.6}$$

Lin also established the following two lemmas.

Lemma 2.1. *The function $G(x,\ y;\ x_0,\ y_0)$ defined by conditions (i)–(v) above exists uniquely and is a generalized Green's function satisfying the reciprocity condition*

$$G(x,y;x_0,y_0) = G(x_0,y_0;x,y). \tag{2.7}$$

Lemma 2.2. *If N vortices of strengths $\{\Gamma_k|\quad k=1,\ldots,N\}$ are present in an incompressible fluid at the points $\{(x_k,\ \ y_k)|k=1,\ldots,N\}$ in a general region D bounded by fixed boundaries, the streamfunction of the fluid motion is given by*

$$\psi(x,y;x_1,y_1,\ldots,x_N,y_N) = \psi_0(x,y) + \sum_{k=1}^{N}\Gamma_k G(x,y;x_k,y_k), \tag{2.8}$$

where the properties of G are given in lemma 2.1 *and $\psi_0(x,y)$ is the streamfunction due to outside agencies and satisfying the boundary conditions* *of no flow through the domain boundaries.* ψ_0 *is independent of the point vortex positions.*

Finally, Lin establishes the following theorem.

Theorem 2.3. *For the motion of vortices of circulations* $\{\Gamma_k | k=1, \ldots, N\}$ *in a general region* D *bounded by fixed boundaries, there exists a Kirchhoff–Routh function* $H(x_1, y_1, \ldots, x_N, y_N)$ *such that*

$$\Gamma_k \frac{\mathrm{d}x_k}{\mathrm{d}t} = \frac{\partial H}{\partial y_k}, \quad \Gamma_k \frac{\mathrm{d}y_k}{\mathrm{d}t} = -\frac{\partial H}{\partial x_k}, \tag{2.9}$$

where $H(x_1, y_1, \ldots, x_N, y_N)$ *is given by*

$$H(x_1,y_1,\ldots,x_N,y_N) = \sum_{k=1}^{N} \Gamma_k \psi_0(x_k,y_k) + \sum_{\substack{k_1,k_2=1 \\ k_1>k_2}}^{N} \Gamma_{k_1}\Gamma_{k_2} G(x_{k_1},y_{k_1};x_{k_2},y_{k_2}) - \frac{1}{2}\sum_{k=1}^{N} \Gamma_k^2 g(x_k,y_k;x_k,y_k). \tag{2.10}$$

In rescaled coordinates $(\sqrt{\Gamma_k}x_k, \sqrt{\Gamma_k}y_k)$ equation (2.9) is a Hamiltonian system in canonical form.

Flucher & Gustafsson (1997) refer to Lin's special Green's function as the *hydrodynamic Green's function* and we will adopt this terminology. They also consider an associated function called the *Robin function*. It is the regular part of the above hydrodynamic Green's function evaluated at the singularity. If the hydrodynamic Green's function G is decomposed into a radially symmetric singular part and a regular part as in equation (2.1), then the Robin function $\mathcal{R}(x_0, y_0)$ is defined as

$$\mathcal{R}(x_0, y_0) \equiv g(x_0, y_0; x_0, y_0). \tag{2.11}$$

This implies that, near the singularity at (x_0, y_0), G can be expanded as

$$G(x, y; x_0, y_0) = -\frac{1}{2\pi}\log r_0 - \mathcal{R}(x_0, y_0) + \mathcal{O}(r_0). \tag{2.12}$$

It is more convenient for what follows to introduce complex coordinates $\zeta = x + \mathrm{i}y$ and $\bar{\zeta} = x - \mathrm{i}y$. Thus, if the complex number $\alpha = x_0 + \mathrm{i}y_0$ denotes the complex position of the singularity of the Green's function we will henceforth write $G(\zeta; \alpha)$ instead of $G(x, y; x_0, y_0)$.

3. Construction of G in circular domains

We will now show how to construct an explicit representation for G in a general, multiply connected circular domain of arbitrary finite connectivity. Let D_ζ be the interior of the unit ζ-disc with M smaller circular discs excised. $M=0$ is the simply connected case. Let the boundaries of these smaller circular discs be denoted $\{C_j | j=1, \ldots, M\}$. Let the unit circle $|\zeta|=1$ be denoted C_0. The complex numbers $\{\delta_j | j=1, \ldots, M\}$ are the centres of the enclosed circular discs while the real numbers $\{q_j | j=1, \ldots, M\}$ will denote their radii.

This special class of multiply connected domains is significant for two reasons. First, such circular domains are known to be canonical domains for conformal mapping to general multiply connected domains (Nehari 1952). That is, any given multiply connected domain can be obtained by conformal mapping of a circular domain of the same connectivity for some choice of the parameters $\{\delta_j | j=1, \ldots, M\}$ and $\{q_j | j=1, \ldots, M\}$. These parameters must be determined as part of the construction of the conformal mapping and, in the latter context, are referred to as the *conformal moduli* of the domain (Nehari 1952). Second, Lin (1941*b*) gives explicit formulae for the transformation properties of the Kirchhoff–Routh path function under conformal mapping. In particular, if a conformal map $z(\zeta)$ maps a given region D_ζ in a ζ-plane to a region D_z in a z-plane, and $H^{(\zeta)}$ and $H^{(z)}$, respectively, denote the Hamiltonians in the ζ and z-planes, then these Hamiltonians are related by the formula

$$H^{(z)}(z_1, \bar{z}_1, \ldots, z_N, \bar{z}_N) = H^{(\zeta)}(\zeta_1, \bar{\zeta}_1, \ldots, \zeta_N, \bar{\zeta}_N) + \sum_{k=1}^{N} \frac{\Gamma_k^2}{4\pi} \log|z_\zeta(\zeta_k)|, \tag{3.1}$$

where $\{\zeta_k | k=1, \ldots, N\}$ and $\{z_k = z(\zeta_k) | k=1, \ldots, N\}$ are the point vortex positions in the ζ- and z-planes, respectively.

In combination, these two facts mean that the formulae to be derived in this paper will theoretically yield formulae for the Kirchhoff–Routh path function for N vortices in any multiply connected domain for which a conformal mapping from a circular preimage region is known explicitly.

4. Schottky groups

First, define M Möbius maps $\{\phi_j | j=1, \ldots, M\}$ corresponding to the conjugation map for points on the circle C_j. That is, if C_j has equation

$$|\zeta - \delta_j|^2 = (\zeta - \delta_j)(\bar{\zeta} - \bar{\delta}_j) = q_j^2, \tag{4.1}$$

then

$$\bar{\zeta} = \bar{\delta}_j + \frac{q_j^2}{\zeta - \delta_j}, \tag{4.2}$$

and so

$$\phi_j(\zeta) \equiv \bar{\delta}_j + \frac{q_j^2}{\zeta - \delta_j}. \tag{4.3}$$

If ζ is a point on C_j, then its complex conjugate is given by $\bar{\zeta} = \phi_j(\zeta)$.

Next, introduce the Möbius maps

$$\theta_j(\zeta) \equiv \bar{\phi}_j(\zeta^{-1}) = \delta_j + \frac{q_j^2 \zeta}{1 - \bar{\delta}_j \zeta}. \tag{4.4}$$

Let C_j' be the circle obtained by reflection of the circle C_j in the unit circle $|\zeta| = 1$ (i.e. the circle obtained by the transformation $\zeta \mapsto 1/\bar{\zeta}$). It is easily verified that the image of the circle C_j' under the transformation θ_j is the circle C_j. Since the M

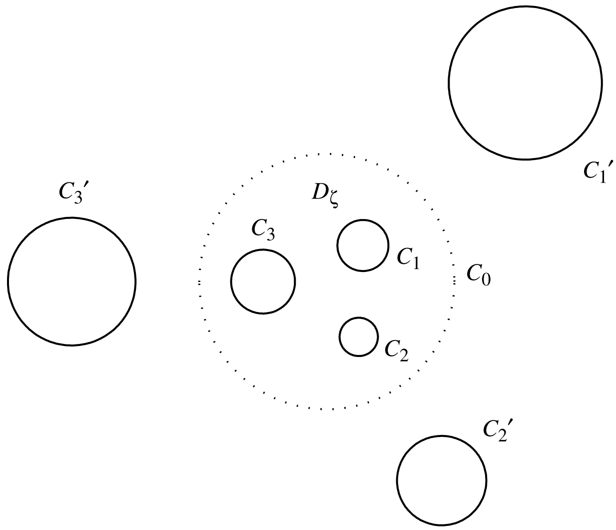

Figure 1. A typical circular region D_ζ is the region interior to the unit circle C_0 (shown as a dotted line) and exterior to the three circles C_1, C_2 and C_3. In the case shown, D_ζ is quadruply connected. The fundamental region is the unbounded region exterior to all six Schottky circles C_1, C_1', C_2, C_2', C_3, C_3'. The radius of circle C_j is denoted q_j while the position of its centre is δ_j.

circles $\{C_j\}$ are non-overlapping, so too are the M circles $\{C_j'\}$. The (classical) *Schottky group* Θ is defined to be the infinite free group of mappings generated by compositions of the M basic Möbius maps $\{\theta_j | j=1,\ldots,M\}$ and their inverses $\{\theta_j^{-1} | j=1,\ldots,M\}$ and including the identity map. The $2M$ circles $\{C_j, C_j' | j=1,\ldots,M\}$ are known as the *Schottky circles*. Beardon (1984) gives a general discussion of such groups. A very accessible discussion of Schottky groups and their mathematical properties can also be found in a recent monograph by Mumford *et al.* (2002).

Consider the (generally unbounded) region of the plane exterior to the $2M$ circles $\{C_j\}$ and $\{C_j'\}$. A schematic is shown in figure 1. This region is known as the *fundamental region* associated with the Schottky group. This fundamental region can be understood as having two 'halves'—the half that is inside the unit circle but exterior to the circles $\{C_j\}$ is the physical region (which we are calling D_ζ), and the region that is outside the unit circle and exterior to the circles $\{C_j'\}$ is the non-physical half.

There are two important properties of these Möbius maps that can easily be established. The first is that

$$\theta_j^{-1}(\zeta) = \frac{1}{\phi_j(\zeta)}, \quad \forall\, \zeta. \tag{4.5}$$

This can be verified using the definitions (4.3) and (4.4) (or, alternatively, by considering the geometrical effect of each map). The second property, which follows from the first, is that

$$\theta_j^{-1}(\zeta^{-1}) = \frac{1}{\phi_j(\zeta^{-1})} = \frac{1}{\overline{\bar{\phi}_j(\bar{\zeta}^{-1})}} = \frac{1}{\overline{\theta_j(\bar{\zeta})}} = \frac{1}{\bar{\theta}_j(\zeta)}, \quad \forall\, \zeta. \tag{4.6}$$

Some special infinite subsets of mappings in a given Schottky group will be needed in what follows. A special notation is now introduced. This notation is not standard but is introduced here to clarify the presentation.

The full Schottky group is denoted Θ. The notation ${}_i\Theta_j$ is used to denote all mappings in the full group, which do not have a power of θ_i or θ_i^{-1} on the left-hand end or a power of θ_j or θ_j^{-1} on the right-hand end. As a special case of this, the notation Θ_j simply means all mappings in the group that do not have any positive or negative power of θ_j at the right-hand end (but with no stipulation about what appears on the left-hand end). Similarly, ${}_j\Theta$ means all mappings that do not have any positive or negative power of θ_j at the left-hand end (but with no stipulation about what appears on the right-hand end). In addition, the single prime notation will be used to denote a subset where the identity is excluded from the set; thus, Θ_1' denotes all mappings, excluding the identity and all transformations with a positive or negative power of θ_1 at the right-hand end. The double prime notation will be used to denote a subset where the identity *and* all inverse mappings are excluded from the set. This means, for example, that if $\theta_1\theta_2$ is included in the set, then the mapping $\theta_2^{-1}\theta_1^{-1}$ must be excluded. Thus, Θ'' means all mappings excluding the identity and all inverses. Similarly, the notation ${}_1\Theta_2''$ denotes all mappings, excluding inverses and the identity, which do not have any power of θ_1 or θ_1^{-1} on the left-hand end or any power of θ_2 or θ_2^{-1} on the right-hand end. Likewise, Θ_j'' denotes all mappings, excluding the identity and all inverses, which do not have any positive or negative power of θ_j at the right-hand end.

5. The Schottky–Klein prime function

Following Baker (1995), the Schottky–Klein prime function is defined as

$$\omega(\zeta,\gamma) = (\zeta-\gamma)\omega'(\zeta,\gamma), \tag{5.1}$$

where the function $\omega'(\zeta, \gamma)$ is given by

$$\omega'(\zeta,\gamma) = \prod_{\theta_i\in\Theta''} \frac{(\theta_i(\zeta)-\gamma)(\theta_i(\gamma)-\zeta)}{(\theta_i(\zeta)-\zeta)(\theta_i(\gamma)-\gamma)}, \tag{5.2}$$

and where the product is over all mappings θ_i in the set Θ''; ω' can also be written as

$$\omega'(\zeta,\gamma) = \prod_{\theta_i\in\Theta''} \{\zeta,\theta_i(\zeta),\gamma,\theta_i(\gamma)\}, \tag{5.3}$$

where the brace notation denotes a cross-ratio of the four arguments. This will be useful later. The function $\omega(\zeta, \gamma)$ is single valued on the whole ζ-plane and has a zero at γ and all points equivalent to γ under the mappings of the group Θ. The prime notation is not used here to denote differentiation.

The Schottky–Klein prime function has some important transformation properties. One such property is that it is antisymmetric in its arguments, that is,

$$\omega(\zeta,\gamma) = -\omega(\gamma,\zeta). \tag{5.4}$$

This is clear from inspection of equations (5.1) and (5.2). A second important property is given by

$$\frac{\omega(\theta_j(\zeta),\gamma_1)}{\omega(\theta_j(\zeta),\gamma_2)}=\beta_j(\gamma_1,\gamma_2)\frac{\omega(\zeta,\gamma_1)}{\omega(\zeta,\gamma_2)}, \tag{5.5}$$

where θ_j is any one of the basic maps of the Schottky group. A detailed derivation of this result is given in ch. 12 of Baker (1995). A formula for $\beta_j(\gamma_1, \gamma_2)$ is

$$\beta_j(\gamma_1,\gamma_2)=\prod_{\theta_k\in\Theta_j}\frac{(\gamma_1-\theta_k(B_j))(\gamma_2-\theta_k(A_j))}{(\gamma_1-\theta_k(A_j))(\gamma_2-\theta_k(B_j))}, \tag{5.6}$$

where A_j and B_j are the two fixed points of the mapping θ_j satisfying

$$\theta_j(A_j)=A_j, \quad \theta_j(B_j)=B_j. \tag{5.7}$$

A_j and B_j satisfy an equation of the form

$$\frac{\theta_j(\zeta)-B_j}{\theta_j(\zeta)-A_j}=\mu_j\,\mathrm{e}^{\mathrm{i}k_j}\frac{\zeta-B_j}{\zeta-A_j}, \tag{5.8}$$

for some real constants μ_j, k_j, and are distinguished by the fact that $|\mu_j|<1$ in equation (5.8). For the distribution of Schottky circles $\{C_j, C_j'\}$ considered in §4, the prime function also has the property that

$$\bar{\omega}(\zeta^{-1},\gamma^{-1})=-\frac{1}{\zeta\gamma}\omega(\zeta,\gamma), \tag{5.9}$$

where the conjugate function $\bar{\omega}(\zeta,\gamma)$ is defined by

$$\bar{\omega}(\zeta,\gamma)=\overline{\omega(\bar{\zeta},\bar{\gamma})}. \tag{5.10}$$

A derivation of equation (5.9) is given in appendix A.

It is convenient to categorize all possible compositions of the basic maps according to their *level*. As an illustration, consider the case in which there are four basic maps $\{\theta_j|j=1,2,3,4\}$. The identity map is considered to be the *level-zero map*. The four basic maps, together with their inverses, $\{\theta_j^{-1}|j=1,2,3,4\}$ constitute the eight *level-one maps*. All possible combinations of any *two* of these eight level-one maps which do not reduce to the identity, for example,

$$\theta_1(\theta_1(\zeta)),\quad \theta_1(\theta_2(\zeta)),\quad \theta_1(\theta_3(\zeta)),\quad \theta_1(\theta_4(\zeta)),\quad \theta_2(\theta_1(\zeta)),\quad \theta_2(\theta_2(\zeta)),\ldots, \tag{5.11}$$

will be called the *level-two maps*; all possible combinations of any *three* of the eight level-one maps that do not reduce to a lower-level map will be called the *level-three maps*, and so on.

On a practical note, to write a function routine to calculate $\omega(\zeta,\gamma)$ numerically, it is necessary to truncate the infinite product in equation (5.1). This is achieved in a natural way by including all Möbius maps up to some chosen level and truncating the contribution to the product from all higher-level maps. The truncation, which includes all maps up to level three, has been used to compute the examples in this paper. The software programme MATLAB is particularly suited to construction of the Schottky–Klein prime function because the action of an element of the Schottky group on the point ζ can be written as multiplication by a 2×2 matrix on the vector $(\zeta, 1)^{\mathrm{T}}$—a linear algebra operation that is performed very efficiently in MATLAB.

6. Explicit solution for *G*

Given a circular domain D_ζ with given moduli (such as that shown in figure 1), the associated Schottky–Klein prime function $\omega(\zeta, \gamma)$ can be constructed. Let the singularity of the hydrodynamic Green's function G in this domain be at α. The complex potential $W(\zeta; \alpha)$ for the flow is such that

$$G(\zeta;\alpha) = \mathrm{Im}[W(\zeta;\alpha)], \tag{6.1}$$

and an explicit expression for it is

$$W(\zeta;\alpha) = -\frac{\mathrm{i}}{4\pi}\log\left(\frac{\omega(\zeta,\alpha)\bar{\omega}(\zeta^{-1},\alpha^{-1})}{\omega(\zeta,\bar{\alpha}^{-1})\bar{\omega}(\zeta^{-1},\bar{\alpha})}\right). \tag{6.2}$$

It is natural to choose the branch of the logarithm so that the branch points at α and $\bar{\alpha}^{-1}$ are joined by a branch cut, as are all image-pairs of these two points under the transformations of the group (i.e. in all regions 'equivalent' to the fundamental region). An explicit representation for $G(\zeta;\alpha)$ is

$$G(\zeta;\alpha) = \mathrm{Im}[W(\zeta;\alpha)] = -\frac{1}{4\pi}\log\left|\frac{\omega(\zeta,\alpha)\bar{\omega}(\zeta^{-1},\alpha^{-1})}{\omega(\zeta,\bar{\alpha}^{-1})\bar{\omega}(\zeta^{-1},\bar{\alpha})}\right|. \tag{6.3}$$

Formulae (6.2) and (6.3) are the principal new results of this paper.

In order to prove that equations (6.2) and (6.3) satisfy the conditions outlined above, consider the function

$$S(\zeta;\alpha) \equiv \frac{\omega(\zeta,\alpha)\bar{\omega}(\zeta^{-1},\alpha^{-1})}{\omega(\zeta,\bar{\alpha}^{-1})\bar{\omega}(\zeta^{-1},\bar{\alpha})}. \tag{6.4}$$

$S(\zeta;\alpha)$ has a second-order zero at $\zeta=\alpha$ (as well as at all points in the plane equivalent to α under the action of the group). $S(\zeta;\alpha)$ also has a second-order pole at the point $\bar{\alpha}^{-1}$ (and all equivalent points). Let α be a point in the physical half of the fundamental region. It follows that $\bar{\alpha}^{-1}$ will be in the non-physical half. Since

$$G(\zeta;\alpha) = -\frac{1}{4\pi}\log|S(\zeta;\alpha)|, \tag{6.5}$$

this means that, in the physical half of the fundamental region D_ζ, $G(\zeta;\alpha)$ has a single isolated logarithmic singularity at $\zeta=\alpha$, as required. Given that the zero of S at α is second order, locally, $G(\zeta;\alpha)$ has the expansion

$$G(\zeta;\alpha) = -\frac{1}{2\pi}\log|\zeta-\alpha| + \mathcal{O}(1), \tag{6.6}$$

again as required.

It has yet to be verified that equation (6.3) satisfies the required boundary conditions on all the circles $\{C_j | j=0,1,\ldots,M\}$. On C_0,

$$\overline{S(\zeta;\alpha)} = \frac{\bar{\omega}(\zeta^{-1},\bar{\alpha})\omega(\zeta,\bar{\alpha}^{-1})}{\bar{\omega}(\zeta^{-1},\alpha^{-1})\omega(\zeta,\alpha)} = \frac{1}{S(\zeta;\alpha)}, \tag{6.7}$$

where we have used the fact that $\bar{\zeta}=\zeta^{-1}$ on C_0. Since $|S(\zeta;\alpha)|=1$ on C_0, then it follows from equation (6.5) that

$$G(\zeta;\alpha) = 0, \quad \text{on } C_0. \tag{6.8}$$

This is the normalization condition stipulated in equation (2.4).

On the other hand, on any one of the interior circles $\{C_j | j=1,\ldots,M\}$,

$$\begin{aligned}\overline{S(\zeta;\alpha)} &= \frac{\bar{\omega}(\phi_j(\zeta),\bar{\alpha})\omega(\phi_j(\zeta)^{-1},\bar{\alpha}^{-1})}{\bar{\omega}(\phi_j(\zeta),\alpha^{-1})\omega(\phi_j(\zeta)^{-1},\alpha)} = \frac{\bar{\omega}(\bar{\theta}_j(\zeta^{-1}),\bar{\alpha})\omega(\bar{\theta}_j(\zeta^{-1})^{-1},\bar{\alpha}^{-1})}{\bar{\omega}(\bar{\theta}_j(\zeta^{-1}),\alpha^{-1})\omega(\bar{\theta}_j(\zeta^{-1})^{-1},\alpha)} \\ &= \frac{\bar{\omega}(\bar{\theta}_j(\zeta^{-1}),\bar{\alpha})\bar{\omega}(\bar{\theta}_j(\zeta^{-1}),\bar{\alpha})}{|\alpha|^2\bar{\omega}(\bar{\theta}_j(\zeta^{-1}),\alpha^{-1})\bar{\omega}(\bar{\theta}_j(\zeta^{-1}),\alpha^{-1})}.\end{aligned} \tag{6.9}$$

However, we can now use equation (5.5) to give

$$\begin{aligned}\overline{S(\zeta;\alpha)} &= \overline{\beta_j(\alpha,\bar{\alpha}^{-1})}^2 \frac{\bar{\omega}(\zeta^{-1},\bar{\alpha})\bar{\omega}(\zeta^{-1},\bar{\alpha})}{|\alpha|^2\bar{\omega}(\zeta^{-1},\alpha^{-1})\bar{\omega}(\zeta^{-1},\alpha^{-1})} \\ &= \overline{\beta_j(\alpha,\bar{\alpha}^{-1})}^2 \frac{\omega(\zeta,\bar{\alpha}^{-1})\bar{\omega}(\zeta^{-1},\bar{\alpha})}{\omega(\zeta,\alpha)\bar{\omega}(\zeta^{-1},\alpha^{-1})} = \frac{\overline{\beta_j(\alpha,\bar{\alpha}^{-1})}^2}{S(\zeta;\alpha)}.\end{aligned} \tag{6.10}$$

Formula (6.10) immediately implies that, on C_j,

$$|S(\zeta;\alpha)| = \overline{\beta_j(\alpha,\bar{\alpha}^{-1})}, \tag{6.11}$$

so that

$$G(\zeta;\alpha) = -\frac{1}{4\pi}\log|S(\zeta;\alpha)| = -\frac{1}{4\pi}\log\overline{\beta_j(\alpha,\bar{\alpha}^{-1})}, \quad \text{on } C_j. \tag{6.12}$$

This means the parameters $\{A_j | j=1,\ldots,M\}$ of equation (2.3) are

$$A_j = -\frac{1}{4\pi}\log\overline{\beta_j(\alpha,\bar{\alpha}^{-1})}. \tag{6.13}$$

Utilizing equation (5.6), a formula for $\beta_j(\alpha,\bar{\alpha}^{-1})$ is

$$\beta_j(\alpha,\bar{\alpha}^{-1}) = \prod_{\theta_k\in\Theta_j}\frac{(\alpha-\theta_k(B_j))(\bar{\alpha}^{-1}-\theta_k(A_j))}{(\alpha-\theta_k(A_j))(\bar{\alpha}^{-1}-\theta_k(B_j))}. \tag{6.14}$$

From equation (6.10), $\beta_j(\alpha,\bar{\alpha}^{-1})$ must be a real quantity, but it is not clear from inspection whether the right-hand side of equation (6.14) is always real. For completeness, a demonstration of this is given in appendix B. It turns out that for any $\alpha\in\mathbb{C}$, $\{\beta_j(\alpha,\bar{\alpha}^{-1})\}$ are all real positive quantities. Finally, some algebraic manipulations reveal that the associated Robin function is given by

$$\mathcal{R}(\alpha,\bar{\alpha}) = \frac{1}{4\pi}\log\left|\frac{\omega'(\alpha,\alpha)\bar{\omega}'(\alpha^{-1},\alpha^{-1})}{\alpha^2\omega(\alpha,\bar{\alpha}^{-1})\bar{\omega}(\alpha^{-1},\bar{\alpha})}\right|. \tag{6.15}$$

(a) Normalization and symmetry

Lemma 2.1 states that the hydrodynamic Green's function satisfies a reciprocity relation given by

$$G(\zeta;\alpha) = G(\alpha;\zeta). \tag{6.16}$$

It is appropriate to verify that the explicit formula given in equation (6.3) satisfies equation (6.16) because it is not obvious by inspection. To this end, consider

$$G(\alpha;\zeta) = -\frac{1}{4\pi}\log\left|\frac{\omega(\alpha,\zeta)\bar{\omega}(\alpha^{-1},\zeta^{-1})}{\omega(\alpha,\bar{\zeta}^{-1})\bar{\omega}(\alpha^{-1},\bar{\zeta})}\right|. \tag{6.17}$$

Note first that, by using equation (5.4),

$$|\omega(\alpha,\zeta)\bar{\omega}(\alpha^{-1},\zeta^{-1})| = |\omega(\zeta,\alpha)\bar{\omega}(\zeta^{-1},\alpha^{-1})|. \tag{6.18}$$

Next, note that

$$|\omega(\alpha,\bar{\zeta}^{-1})| = |\bar{\omega}(\bar{\alpha},\zeta^{-1})| = |\bar{\omega}(\zeta^{-1},\bar{\alpha})|, \tag{6.19}$$

where the first equality is simply a statement of the fact that the moduli of complex conjugate numbers are equal and the second equality follows from the use of equation (5.4). By similar manipulations

$$|\bar{\omega}(\alpha^{-1},\bar{\zeta})| = |\omega(\bar{\alpha}^{-1},\zeta)| = |\omega(\zeta,\bar{\alpha}^{-1})|. \tag{6.20}$$

By using equations (6.18)–(6.20) in equation (6.17), the reciprocity relation (6.16) is confirmed.

Utilizing equation (5.9), it is also possible to write $W(\zeta;\alpha)$, and hence $G(\zeta;\alpha)$, in the alternative equivalent forms

$$W(\zeta;\alpha) = -\frac{\mathrm{i}}{2\pi}\log\left(\frac{1}{|\alpha|}\frac{\omega(\zeta,\alpha)}{\omega(\zeta,\bar{\alpha}^{-1})}\right), \quad G(\zeta;\alpha) = -\frac{1}{2\pi}\log\left|\frac{1}{\alpha}\frac{\omega(\zeta,\alpha)}{\omega(\zeta,\bar{\alpha}^{-1})}\right|. \tag{6.21}$$

However, we prefer the representations given in equations (6.2) and (6.3), because it is easily seen from these formulae that the normalization $G=0$ on C_0 has been enforced. This normalization is crucial not only for the uniqueness of the hydrodynamic Green's function, but also so that the reciprocity condition (6.16) is satisfied (Flucher & Gustafsson 1997).

(b) Conditions on $\mathcal{R}$ on the boundaries

A result given in Flucher & Gustafsson (1997) is that the Robin function $\mathcal{R}(\alpha,\bar{\alpha})$ is singular on all boundaries of the domain. It is appropriate to verify this for the Robin function (6.15) found for the case of multiply connected circular domains.

First, note that as α tends to a point on C_0, it is clear that because α and $\bar{\alpha}^{-1}$ have the same argument, they will approach each other as $|\alpha| \to 1$. Thus, the denominator in the argument of the logarithm in equation (6.15) will tend to zero in this limit. This verifies that $\mathcal{R}(\alpha, \bar{\alpha})$ is singular on C_0.

Similarly, for a point α on $\{C_j | j=1, \ldots, M\}$, note that

$$|\omega(\alpha, \bar{\alpha}^{-1})| = |\omega(\overline{\phi_j(\alpha)}, \bar{\alpha}^{-1})| = |\omega(\overline{\phi_j}(\bar{\alpha}), \bar{\alpha}^{-1})| = |\omega(\theta_j(\bar{\alpha}^{-1}), \bar{\alpha}^{-1})|. \tag{6.22}$$

However, this final term is zero using the fact that $\omega(\zeta, \bar{\alpha}^{-1})$ has a zero at $\bar{\alpha}^{-1}$ and at all transformations of this point under the mappings of the group Θ. In particular, it will have a zero at $\theta_j(\bar{\alpha}^{-1})$. Given that $|\omega(\alpha, \bar{\alpha}^{-1})|$ appears as the denominator of the argument of the logarithm in equation (6.15), and because the numerator is easily seen not to vanish, it follows that $\mathcal{R}(\alpha, \bar{\alpha})$ is singular at all points on the boundaries $\{C_j | j=1, \ldots, M\}$.

(c) *Round-island circulations*

The circulation around the jth island is by definition

$$\mathrm{Re}\left[\oint_{C_j} \frac{\mathrm{d}W}{\mathrm{d}\zeta} \mathrm{d}\zeta\right] = \mathrm{Re}\left[\oint_{C_j} \mathrm{d}W\right] = \mathrm{Re}[W]_{C_j}, \tag{6.23}$$

where the notation $[W]_{C_j}$ denotes the change in value of W on making a single circuit around the closed curve C_j. By the choice of logarithmic branch cuts in the ζ-plane made earlier, none of the branch cuts across any of the circles $\{C_j | j=1, \ldots, M\}$; hence W does not change value on making a circuit of any of these circles. The round-island circulations are therefore all zero. However, the same is not true of the unit circle C_0, because a branch cut crosses C_0 in order to join α to $\bar{\alpha}^{-1}$. If it is also required to render the circulation around C_0 equal to zero (case 2 of the definition in §2), then another point vortex of opposite circulation $-\Gamma$ must be added in the physical half of the fundamental region. Let this additional vortex be at a point β. The complex potential for a vortex of circulation Γ will then become

$$W(\zeta; \alpha, \beta) = -\frac{\mathrm{i}\Gamma}{4\pi} \log\left(\frac{\omega(\zeta, \alpha)\bar{\omega}(\zeta^{-1}, \alpha^{-1})}{\omega(\zeta, \bar{\alpha}^{-1})\bar{\omega}(\zeta^{-1}, \bar{\alpha})}\right) + \frac{\mathrm{i}\Gamma}{4\pi} \log\left(\frac{\omega(\zeta, \beta)\bar{\omega}(\zeta^{-1}, \beta^{-1})}{\omega(\zeta, \bar{\beta}^{-1})\bar{\omega}(\zeta^{-1}, \bar{\beta})}\right), \tag{6.24}$$

where the branch of the function is chosen so that branch cuts join α to β inside the physical half of the fundamental region, while another branch cut joins $\bar{\alpha}^{-1}$ to $\bar{\beta}^{-1}$ in the non-physical half (with analogous choices of cuts being made in all other equivalent regions). This construction is particularly useful in the case of unbounded flows, where C_0 is conformally mapped to an $(M+1)$th island and there is a point ζ_∞ in D_ζ mapping to infinity. If it is required to make the circulation around this island zero (so that *all* round-island circulations are zero), then it is usual to add an additional point vortex of circulation $-\Gamma$ to the point at infinity, with β being taken to be ζ_∞. In their studies of vortex motion past two circular islands, Johnson & McDonald (2004*a*) also introduce a point vortex singularity at infinity to render the circulation around both islands equal to zero.

(*d*) *Inter-island fluxes*

It is well known that the difference between two values of G evaluated on two different islands gives the (time-dependent) 'inter-island flux' of fluid between the two islands. Let $\mathcal{F}_{ij}$ denote the flux between islands i and j, then by using equation (6.12), we can obtain explicit formulae for the values of these fluxes. In particular,

$$\mathcal{F}_{ij} = -\frac{1}{4\pi}\log\left(\frac{\beta_i(\bar{\alpha}, \alpha^{-1})}{\beta_j(\bar{\alpha}, \alpha^{-1})}\right), \tag{6.25}$$

where the explicit formula (6.14) can be used.

7. The Kirchhoff–Routh path function

Using the representations for the hydrodynamic Green's and Robin functions in circular domains derived in §6, it is now possible to write down formulae for the Kirchhoff–Routh path function as given in equation (2.10) for any finite number N of vortices in a multiply connected circular region. However, our result is stronger than this once we exploit the second result (3.1) from Lin (1941*b*) showing how the Hamiltonian transforms under conformal mapping.

We summarize our results in a more explicit statement of theorem 2.3.

Theorem 7.1. *For the motion of vortices of strengths* $\{\Gamma_k | k=1, \ldots, N\}$ *in a general region* D_z *bounded by fixed boundaries, first construct the Kirchhoff–Routh path function* $H^{(\zeta)}(\alpha_1, \bar{\alpha}_1, \ldots, \alpha_N, \bar{\alpha}_N)$ *in a conformally equivalent circular region* D_ζ *of the form*

$$\begin{aligned} &H^{(\zeta)}(\alpha_1, \bar{\alpha}_1, \ldots, \alpha_N, \bar{\alpha}_N) \\ &= \sum_{k=1}^{N} \Gamma_k \psi_0(\alpha_k) + \sum_{\substack{k_1,k_2=1\\ k_1>k_2}}^{N} \Gamma_{k_1}\Gamma_{k_2} G(\alpha_{k_1}; \alpha_{k_2}) - \frac{1}{2}\sum_{k=1}^{N} \Gamma_k^2 \mathcal{R}(\alpha_k, \bar{\alpha}_k), \end{aligned} \tag{7.1}$$

where $G(\zeta; \alpha)$ *is given in equation (6.3),* $\mathcal{R}(\alpha, \bar{\alpha})$ *is given in equation (6.15) and* $\psi_0(\zeta)$ *is the contribution to the Hamiltonian from external agencies such as background flows or non-zero round-island circulations. Then, if* $z(\zeta)$ *is the conformal map from* D_ζ *to* D_z*, the Kirchhoff–Routh path function for the N-vortex motion is*

$$H^{(z)}(z_1, \bar{z}_1, \ldots, z_N, \bar{z}_N) = H^{(\zeta)}(\alpha_1, \bar{\alpha}_1, \ldots, \alpha_N, \bar{\alpha}_N) + \sum_{k=1}^{N} \frac{\Gamma_k^2}{4\pi}\log|z_\zeta(\alpha_k)|, \tag{7.2}$$

where $z_k = z(\alpha_k)$ *for* $k=1, \ldots, N$.

In cases where both $\psi_0(\zeta)$ and the conformal mapping $z(\zeta)$ are known explicitly, it follows that the Hamiltonian will also be given in analytical form by using equation (7.2). Even when either of the functions $\psi_0(\zeta)$ and $z(\zeta)$ is not known analytically (and must be computed numerically), these two functions are independent of the instantaneous point vortex positions, and can be computed at the start of any calculation (assuming the boundaries of the flow domain and the flow due to external agencies are not changing in time). In any case, the Hamiltonian given in equation (7.2) can still facilitate numerical calculation of even very complicated N-vortex flows.

8. The single vortex case

To illustrate the usefulness of the formulae derived in this paper, we include some examples for the motion of a single vortex. This also provides us the opportunity to examine how our general formulation reduces to the simply and doubly connected studies that have already appeared in the literature.

First, we write down the Hamiltonians for single-vortex flow in the three types of domain (cases 1–3) considered in §2.

Case 1. When just a single vortex is present, the double sum in equation (2.10) disappears and the Kirchhoff–Routh path function, in the absence of external sources of vorticity, for a circulation-Γ point vortex reduces to

$$H^{(\zeta)}(\alpha,\bar{\alpha}) = -\frac{\Gamma^2}{2}\mathcal{R}(\alpha,\bar{\alpha}). \tag{8.1}$$

By using equation (6.15), this becomes

$$H^{(\zeta)}(\alpha,\bar{\alpha}) = -\frac{\Gamma^2}{8\pi}\log\left|\frac{\omega'(\alpha,\alpha)\bar{\omega}'(\alpha^{-1},\alpha^{-1})}{\alpha^2\omega(\alpha,\bar{\alpha}^{-1})\bar{\omega}(\alpha^{-1},\bar{\alpha})}\right|. \tag{8.2}$$

Let $z(\zeta)$ be any map from the circular domain to a (conformally equivalent) multiply connected domain. Then, by equation (3.1), the Hamiltonian in the z-plane is given by

$$H^{(z)}(z_\alpha,\bar{z}_\alpha) = H^{(\zeta)}(\alpha,\bar{\alpha}) + \frac{\Gamma^2}{4\pi}\log|z_\zeta(\alpha)|, \tag{8.3}$$

where $z_\alpha = z(\alpha)$. Equivalently,

$$H^{(z)}(z_\alpha,\bar{z}_\alpha) = -\frac{\Gamma^2}{8\pi}\log\left|\frac{1}{\alpha^2}\,\frac{\omega'(\alpha,\alpha)\bar{\omega}'(\alpha^{-1},\alpha^{-1})}{\omega(\alpha,\bar{\alpha}^{-1})\bar{\omega}(\alpha^{-1},\bar{\alpha})}\,\frac{1}{z_\zeta(\alpha)^2}\right|. \tag{8.4}$$

If the conformal map $z(\zeta)$ is known explicitly, then equation (8.4) gives the Hamiltonian in explicit form.

Case 2. In the case where D is unbounded but has a single boundary that extends to infinity, C_0 is taken to map to the infinite boundary (so that the Green's function is zero on this boundary as required in the definition given in §2), and the Hamiltonian is again given by equation (8.4).

Case 3. In the case where D is unbounded in all directions, it is necessary to add a point vortex at infinity with circulation $-\Gamma$ in order to render zero the circulation around all islands. Let ζ_∞ be the point in D_ζ mapping to physical infinity. Then, the complex potential for the point vortex at infinity is

$$W_\infty(\zeta) = \frac{\mathrm{i}\Gamma}{4\pi}\log\left(\frac{\omega(\zeta,\zeta_\infty)\bar{\omega}(\zeta^{-1},\zeta_\infty^{-1})}{\omega(\zeta,\bar{\zeta}_\infty^{-1})\bar{\omega}(\zeta^{-1},\bar{\zeta}_\infty)}\right), \tag{8.5}$$

and the Hamiltonian equation (8.4) must be modified by the addition of

$$H_\infty(\alpha,\bar{\alpha}) = \Gamma\,\mathrm{Im}[W_\infty(\alpha)], \tag{8.6}$$

thereby yielding

$$H^{(z)}(z_\alpha,\bar{z}_\alpha) = -\frac{\Gamma^2}{8\pi}\log\left|\frac{1}{\alpha^2}\,\frac{\omega'(\alpha,\alpha)\bar{\omega}'(\alpha^{-1},\alpha^{-1})\omega^2(\alpha,\bar{\zeta}_\infty^{-1})\bar{\omega}^2(\alpha^{-1},\bar{\zeta}_\infty)}{\omega(\alpha,\bar{\alpha}^{-1})\bar{\omega}(\alpha^{-1},\bar{\alpha})\omega^2(\alpha,\zeta_\infty)\bar{\omega}^2(\alpha^{-1},\zeta_\infty^{-1})}\,\frac{1}{z_\zeta(\alpha)^2}\right|. \tag{8.7}$$

In the case of a single vortex, the Hamiltonian $H^{(z)}(z_\alpha,\bar{z}_\alpha)$ is a conserved quantity and the trajectories of the vortex are simply its level lines.

(*a*) *The simply connected case*

Consider a simply connected domain. In this case, the Schottky group is the trivial group and the associated Schottky–Klein prime function is just

$$\omega(\zeta,\gamma) = (\zeta-\gamma). \tag{8.8}$$

The hydrodynamic Green's function in a bounded domain reduces to

$$\begin{aligned} G(\zeta;\alpha) &= -\frac{1}{4\pi}\log\left|\frac{(\zeta-\alpha)(\zeta^{-1}-\alpha^{-1})}{(\zeta-\bar{\alpha}^{-1})(\zeta^{-1}-\bar{\alpha})}\right| = -\frac{1}{2\pi}\log\left|\frac{1}{\alpha}\,\frac{(\zeta-\alpha)}{(\zeta-\bar{\alpha}^{-1})}\right| \\ &= -\frac{1}{2\pi}\log|\zeta-\alpha| - \frac{1}{2\pi}\log\left|\frac{1}{\alpha}\,\frac{1}{(\zeta-\bar{\alpha}^{-1})}\right|. \end{aligned} \tag{8.9}$$

The Robin function is

$$\mathcal{R}(\alpha,\bar{\alpha}) = \frac{1}{2\pi}\log\left|\frac{1}{1-\alpha\bar{\alpha}}\right|. \tag{8.10}$$

The Hamiltonian for a single vortex in a domain mapping from the unit ζ-circle via a mapping $z(\zeta)$ (with the inverse mapping $\zeta=\zeta(z)$) is then

$$H(z_\alpha,\bar{z}_\alpha) = -\frac{\Gamma^2}{8\pi}\log\left|\frac{\zeta'(z_\alpha)\bar{\zeta}'(\bar{z}_\alpha)}{(1-\zeta(z_\alpha)\bar{\zeta}(\bar{z}_\alpha))^2}\right|. \tag{8.11}$$

It is now easy to verify directly that this Hamiltonian satisfies the elliptic Liouville equation

$$\nabla^2 H \equiv 4\frac{\partial^2 H}{\partial z_\alpha \partial \bar{z}_\alpha} = -\frac{\Gamma^2}{\pi}\mathrm{e}^{-8\pi H/\Gamma^2}. \tag{8.12}$$

Indeed, the Hamiltonian can be characterized as a solution of equation (8.12) in D_z which satisfies the boundary condition that it is everywhere infinite on the boundary of D_z. This result for a single vortex in a simply connected domain was pointed out by Flucher & Gustafsson (1997) and Richardson (1980).

On the other hand, the complex potential for a flow in which the round-island circulation is zero would be

$$W(\zeta;\alpha) = -\frac{\mathrm{i}}{2\pi}\log\left(\frac{(\zeta-\alpha)}{|\alpha|(\zeta-\bar{\alpha}^{-1})}\right) + \frac{\mathrm{i}}{2\pi}\log\zeta, \tag{8.13}$$

where an opposite circulation point vortex has been added at $\zeta=0$ in order to render the total circulation around the circular cylinder equal to zero. Under a conformal mapping $\zeta=z^{-1}$, which maps the interior of the unit ζ-circle to the exterior of the unit z-circle, we obtain the complex potential

$$w(z;\beta) \equiv W(1/z;1/\beta) = -\frac{1}{2\pi}\log\left(\frac{z}{|\beta|}\frac{(z-\beta)}{(z-\bar{\beta}^{-1})}\right), \tag{8.14}$$

where $\beta=\alpha^{-1}$ is the image in the z-plane of the point vortex at $\zeta=\alpha$. Equation (8.14) is the usual formula, which can be obtained by the Milne–Thomson circle theorem (Acheson 1990), for the complex potential of a single vortex at $z=\beta$ outside a circular cylinder in the case when the round-island circulation is taken to be zero.

(b) The doubly connected case

A doubly connected domain can be obtained by a conformal mapping from some annulus $q<|\zeta|<1$ in a parametric ζ-plane (the value of the parameter q is determined by the domain itself). In this case, the Schottky group is generated by the Möbius map

$$\theta_1(\zeta) = q^2\zeta, \tag{8.15}$$

and its inverse. Then,

$$\omega(\zeta,\gamma) = -\frac{\gamma}{C^2}P(\zeta/\gamma,q), \tag{8.16}$$

where

$$P(\zeta,q) \equiv (1-\zeta)\prod_{k=1}^{\infty}(1-q^{2k}\zeta)(1-q^{2k}\zeta^{-1}) \tag{8.17}$$

and

$$C \equiv \prod_{k=1}^{\infty}(1-q^{2k}). \tag{8.18}$$

Note that because $\bar{\theta}_1(\zeta)=\theta_1(\zeta)$, then $\bar{\omega}(\zeta,\gamma)=\omega(\zeta,\gamma)$.

In the case of a bounded doubly connected domain, the streamfunction becomes

$$G = -\frac{1}{4\pi}\log\left|\frac{P(\zeta\alpha^{-1},q)P(\alpha\zeta^{-1},q)}{P(\zeta\bar{\alpha},q)P(\zeta^{-1}\bar{\alpha}^{-1},q)}\right|. \tag{8.19}$$

Using the (easily established) property that $P(\zeta^{-1}, q) = -\zeta^{-1}P(\zeta, q)$, this reduces to

$$G = -\frac{1}{2\pi}\log\left|\frac{\alpha P(\zeta\alpha^{-1}, q)}{P(\zeta\bar{\alpha}, q)}\right|. \tag{8.20}$$

The function $P(\zeta, q)$ is related to the first Jacobi theta function Θ_1 (Whittaker & Watson 1927). Indeed, if we define

$$\tau = -\log\zeta, \quad \tau_\alpha = -\log\alpha, \tag{8.21}$$

then the annulus in the ζ-plane is mapped to a rectangle in the τ-plane. It can be shown (see Whittaker & Watson 1927) that

$$P(\zeta, q) = -\frac{\mathrm{i}C\,\mathrm{e}^{-\tau/2}}{q^{1/4}}\Theta_1(\mathrm{i}\tau/2, q). \tag{8.22}$$

Using equation (8.22), it follows that

$$\left.\begin{aligned} P(\zeta\alpha^{-1}, q) &= -\mathrm{i}Cq^{-1/4}\sqrt{\frac{\zeta}{\alpha}}\Theta_1(\mathrm{i}(\tau-\tau_\alpha)/2, q), \\ P(\zeta\bar{\alpha}, q) &= -\mathrm{i}Cq^{-1/4}\sqrt{\zeta\bar{\alpha}}\Theta_1(\mathrm{i}(\tau+\bar{\tau}_\alpha)/2, q), \end{aligned}\right\} \tag{8.23}$$

which, on substitution into equation (8.20), yields

$$G = -\frac{1}{2\pi}\log\left|\frac{\Theta_1(\mathrm{i}(\tau-\tau_\alpha)/2, q)}{\Theta_1(\mathrm{i}(\tau+\bar{\tau}_\alpha)/2, q)}\right|. \tag{8.24}$$

This is precisely the imaginary part of the complex potential given in eqn (2.11) of Johnson & McDonald (2004*a*).

(*c*) *Higher-connected examples*

To illustrate the usefulness of the formulae derived here, figures 2–5 show a series of examples in which the trajectories of a single vortex in various different bounded circular domains are computed. This is achieved by a straightforward calculation of the contours of $H^{(z)}(z_\alpha, \bar{z}_\alpha)$ as given by equation (8.4) in the case where the conformal map $z(\zeta)$ is taken to be the identity (i.e. $z(\zeta) = \zeta$). Figures 2 and 3, respectively, show two and four equal-sized circular cylinders equispaced on the real axis inside the unit disc. These figures show the existence of critical vortex trajectories splitting the flow into paths which encircle one or the other of the enclosed cylinders, none of the enclosed cylinders or both of the enclosed cylinders. Figure 4 shows four equal cylinders arranged at the four corners of a square. Figure 5 shows a case involving two different-sized cylinders.

The domains just considered have a reflectional symmetry about the real axis so that the Schottky–Klein prime functions satisfy the condition $\bar{\omega}(\zeta, \gamma) = \omega(\zeta, \gamma)$. As examples of cases in which this symmetry is not present, figure 6 shows the vortex trajectories around two non-symmetrically disposed cylinders of various diameters. Two general results on the motion of isolated point vortices established in corollary 10 of Flucher & Gustafsson (1997) are that there is always at least one stagnation point of the flow and that almost all trajectories

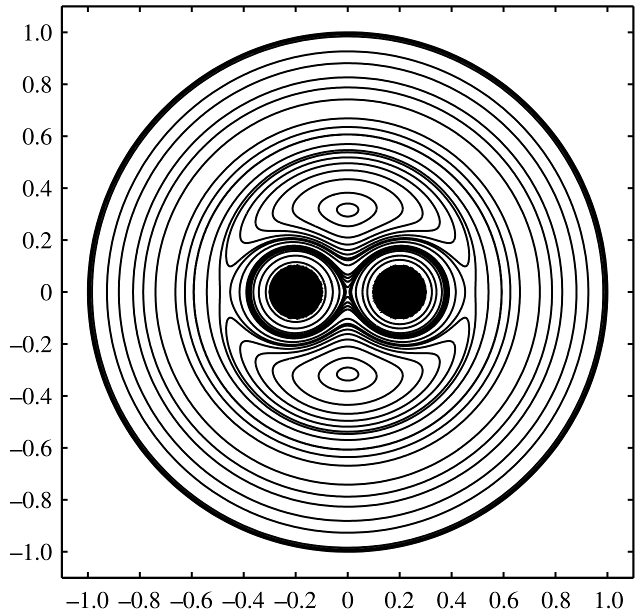

Figure 2. Typical vortex paths in the unit disc with two islands of radius 0.1 centred at ± 0.2. The domain D_ζ is triply connected.

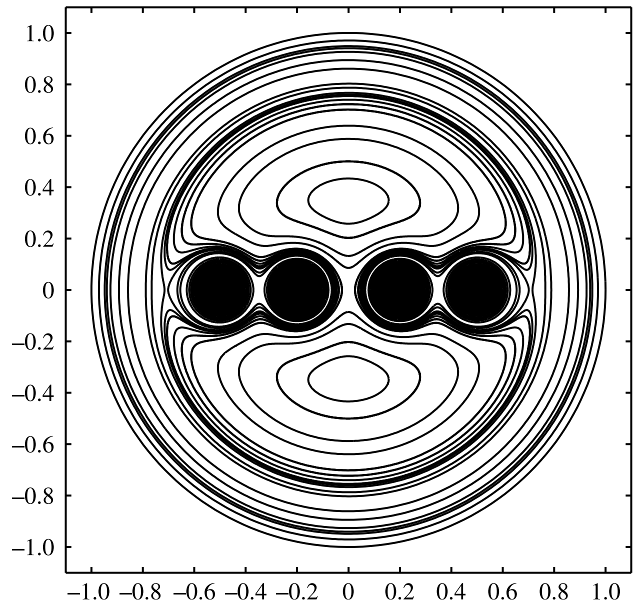

Figure 3. Typical vortex paths in the unit disc with four islands of radius 0.1 centred at ± 0.2, ± 0.5. The domain D_ζ is quintuply connected.

are periodic. All of the vortex paths shown in figures 2–6 are seen to be consistent with these general results.

Finally, we have not discussed in any detail the convergence properties of the infinite products defining the Schottky–Klein prime function. This is a detailed

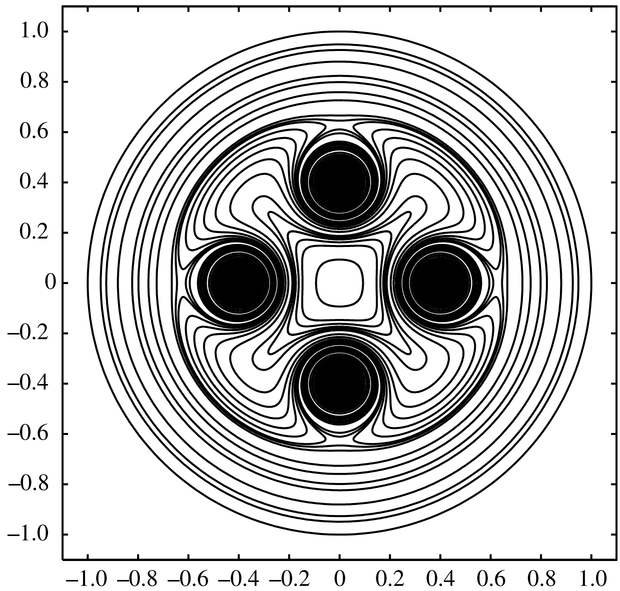

Figure 4. Typical vortex paths in the unit disc with four islands of radius 0.1 centred at ± 0.4, $\pm 0.4\mathrm{i}$. The domain D_ζ is quintuply connected.

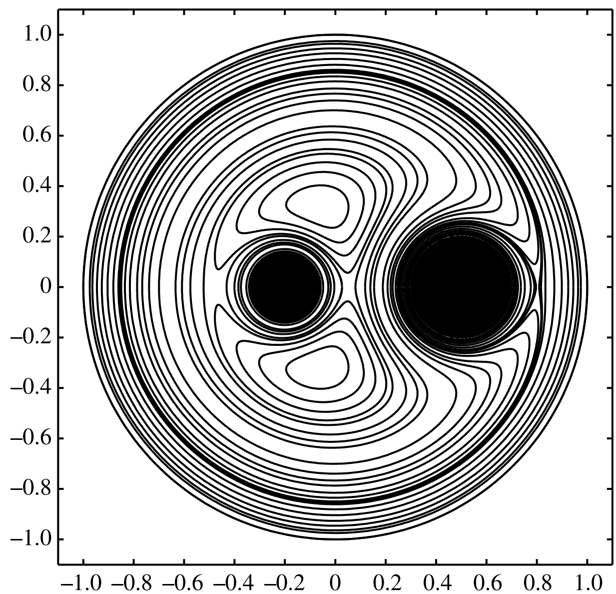

Figure 5. Typical vortex paths in the unit disc with two islands; one of radius 0.1 centred at -0.2 and another of radius 0.2 centred at 0.5. The domain D_ζ is triply connected.

mathematical issue. Roughly speaking, it is known that these products generally converge well, provided that the Schottky circles in the ζ-plane are well separated. In the examples above, as a numerical check on the choice of truncation at level three, the products are truncated at higher levels to verify that the choice of truncation is acceptable.

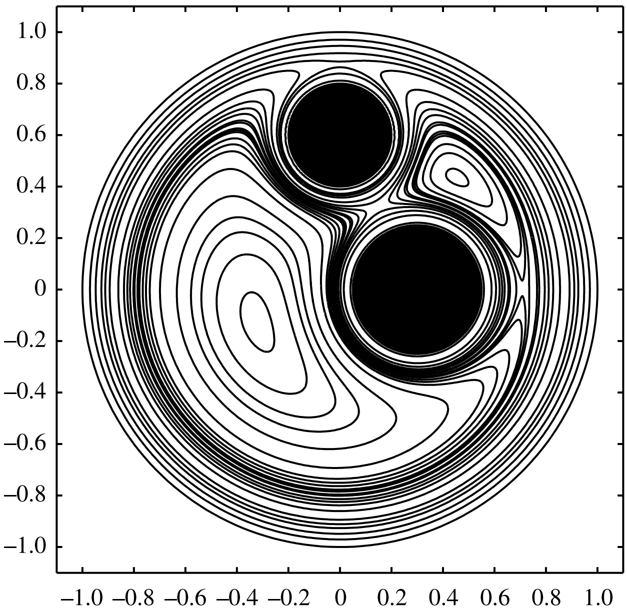

Figure 6. Typical vortex paths in the unit disc with two islands; one of radius 0.25 centred at 0.3 and another of radius 0.2 centred at 0.6i. The domain D_ζ is triply connected.

9. Discussion

The general formulae derived here are relevant to the motion of N-point vortices in *any* M-connected regions for which a conformal map $z(\zeta)$ to the region from some M-connected circular preimage region is known. If the conformal map is not known in analytical form, then a numerical determination of this map can still be used in the formulae above. The numerical determination of conformal maps is a standard procedure (Henrici 1986; DeLillo *et al.* 1999).

Referring back to the familiar 'method of images' mentioned in §1, it is worth pointing out that the Schottky–Klein prime functions that appear in the formulae for the Hamiltonians automatically place an appropriate distribution of 'image vortices' throughout the plane so that the streamline conditions on all the disjoint boundaries are simultaneously satisfied. The positioning of these image singularities is induced by the action of the Schottky group elements on the point vortex position α.

A limitation of the formulae derived here is that the round-island circulations are zero. This situation is relevant when a point vortex approaches an island cluster from far away so that the flow around the islands is initially quiescent. However, there are certain physical situations in which non-zero round-island circulations are appropriate. It would be of interest to generalize the formulae herein to this more general case (it turns out to be straightforward to do this in the special case of doubly connected regions).

The efficacy of our method has been demonstrated by a series of examples. It should be pointed out, however, that we have proceeded under the assumption that the infinite products defining the functions ω and $\bar{\omega}$ converge. In fact, these products do not converge for all choices of the parameters

$\{q_j, \delta_j | j=1, \ldots, M\}$. Broadly speaking, their convergence depends on the distribution of circles $\{C_j | j=1, \ldots, M\}$ in the preimage plane. If the circles are 'well-separated', then good convergence is assured. There is a large region of the parameter space $\{q_j, \delta_j | j=1, \ldots, M\}$ where the convergence is completely adequate for practical purposes. This region of parameter space is large enough to capture many physically interesting fluid domains. Examples of these can be found in a companion article by the authors (Crowdy & Marshall 2005), where the formulae of this paper have been combined with various choices of conformal maps to unbounded fluid domains of interest in applications.

On the subject of applications, we conclude by mentioning a few. In a recent paper, Johnson & McDonald (2004*a*) consider the motion of a vortex in the doubly connected region exterior to two circular cylinders whose boundaries act as impenetrable barriers for the flow. The motivation for this study was to provide a simple model to understand how an oceanic eddy/vortex interacts with topography (Simmons & Nof 2000). Such flow scenarios occur in a range of geophysical situations such as the interaction of Mediterranean salt lenses ('Meddies') with seamounts in the Canary basin (Dewar 2002) or the collision of North Brazil current rings with the islands of the Caribbean (Simmons & Nof 2002). In their study, Johnson & McDonald (2004*a*) consider the case in which the circulation around each island is zero, which is precisely the case considered here. An important result of Johnson & McDonald (2004*a*) was that, in many cases, the motion of the centroid of a finite-area vortex patch around topography is well-approximated by a point vortex model in the same domain. Thus, one application of our results will be to provide useful checks on numerical calculations of the motion of finite-area vortex patches around topography. Other studies of geophysical interest involve the motion of vortices near gaps in an impenetrable barrier (Nof 1995; Johnson & McDonald 2004*b*, 2005). The results here generalize the formulae of Johnson & McDonald (2004*a*) to any number of circular islands, and we expect our results to be of practical use in geophysical applications. Indeed, we have used the new formalism to study a number of island configurations of geophysical interest, including islands off an infinite coastline as well as island clusters in unbounded oceans (Crowdy & Marshall 2005).

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Appendix A. Proof of transformation property (5.9)

From the definition given in equations (5.1)–(5.3),

$$\omega(\zeta, \gamma) = (\zeta - \gamma) \prod_{\theta_j \in \Theta''} \{\zeta, \theta_j(\zeta), \gamma, \theta_j(\gamma)\}. \tag{A 1}$$

By using equation (A 1),

$$\omega(\zeta^{-1}, \gamma^{-1}) = (\zeta^{-1} - \gamma^{-1}) \prod_{\theta_j \in \Theta''} \{\zeta^{-1}, \theta_j(\zeta^{-1}), \gamma^{-1}, \theta_j(\gamma^{-1})\}. \tag{A 2}$$

Consider a general term of the form $\theta_j(\zeta^{-1})$. This is some composition of the generators of the Schottky group. Suppose, for example, that

$$\theta_j(\zeta^{-1}) = \theta_p(\theta_q(\theta_r(\zeta^{-1}))), \tag{A 3}$$

for some sequence of integers (p, q, r) labelling the level-one maps (such a sequence of integers is sometimes called a 'word'; Mumford *et al.* 2002). Recall from equation (4.6) that if θ_k is one of the basic level-one maps, then

$$\theta_k^{-1}(\zeta^{-1}) = \frac{1}{\bar{\theta}_k(\zeta)}. \tag{A 4}$$

Equivalently,

$$\theta_k(\zeta^{-1}) = \frac{1}{\bar{\theta}_k^{-1}(\zeta)}. \tag{A 5}$$

By using equation (A 5) repeatedly in equation (A 3),

$$\begin{aligned}\theta_j(\zeta^{-1}) = \theta_p(\theta_q(\theta_r(\zeta^{-1}))) &= \theta_p\left(\theta_q\left(\frac{1}{\bar{\theta}_r^{-1}(\zeta)}\right)\right) = \theta_p\left(\frac{1}{\bar{\theta}_q^{-1}(\bar{\theta}_r^{-1}(\zeta))}\right) \\ &= \frac{1}{\bar{\theta}_p^{-1}(\bar{\theta}_q^{-1}(\bar{\theta}_r^{-1}(\zeta)))} = \frac{1}{\overline{\theta_r\theta_q\theta_p}^{-1}(\zeta)}.\end{aligned} \tag{A 6}$$

We now introduce a general subscript 'r' notation; given the map θ_j (e.g. corresponding to the word $(p,\ q,\ r)$), then θ_{jr} will denote the map corresponding to the *reversed* word. In this example, the reversed word is $(r,\ q,\ p)$ so that

$$\theta_{jr} = \theta_r(\theta_q(\theta_p(\zeta))). \tag{A 7}$$

Then, equation (A 6) can be written

$$\theta_j(\zeta^{-1}) = \frac{1}{\overline{\theta_{jr}}^{-1}(\zeta)}. \tag{A 8}$$

It should be clear that the result in equation (A 8) will be true for *any* map θ_j. It follows that

$$\begin{aligned}\omega(\zeta^{-1}, \gamma^{-1}) &= (\zeta^{-1} - \gamma^{-1}) \prod_{\theta_j \in \Theta''} \left\{\frac{1}{\zeta}, \frac{1}{\overline{\theta_{jr}}^{-1}(\zeta)}, \frac{1}{\gamma}, \frac{1}{\overline{\theta_{jr}}^{-1}(\gamma)}\right\} \\ &= (\zeta^{-1} - \gamma^{-1}) \prod_{\theta_j \in \Theta''} \left\{\zeta, \overline{\theta_{jr}}^{-1}(\zeta), \gamma, \overline{\theta_{jr}}^{-1}(\gamma)\right\},\end{aligned} \tag{A 9}$$

where we have used the invariance of cross-ratios to Möbius transformation of all four arguments. Now, using the fact that inverses are excluded from the product, equation (A 9) can also be written

$$\omega(\zeta^{-1},\gamma^{-1}) = (\zeta^{-1}-\gamma^{-1}) \prod_{\theta_j\in\Theta''} \{\zeta, \overline{\theta_{j\mathrm{r}}}(\zeta), \gamma, \overline{\theta_{j\mathrm{r}}}(\gamma)\}, \tag{A 10}$$

where we have simply relabelled the maps in the product. Furthermore, if a mapping θ_j is included in the product, then it is easy to check that it can be arranged that the mapping θ_{jr} is also in the product. Thus, under a further relabelling of the maps, equation (A 10) becomes

$$\omega(\zeta^{-1},\gamma^{-1}) = (\zeta^{-1}-\gamma^{-1}) \prod_{\theta_j\in\Theta''} \{\zeta, \overline{\theta_j}(\zeta), \gamma, \overline{\theta_j}(\gamma)\}. \tag{A 11}$$

Thus,

$$\begin{aligned}\bar{\omega}(\zeta^{-1},\gamma^{-1}) &= \overline{\omega(\bar{\zeta}^{-1},\bar{\gamma}^{-1})}\\ &= (\zeta^{-1}-\gamma^{-1}) \prod_{\theta_j\in\Theta''} \{\zeta, \theta_j(\zeta), \gamma, \theta_j(\gamma)\}\\ &= (\zeta^{-1}-\gamma^{-1}) \frac{\omega(\zeta,\gamma)}{(\zeta-\gamma)}\\ &= -\frac{1}{\zeta\gamma}\omega(\zeta,\gamma).\end{aligned} \tag{A 12}$$

This completes the proof. ∎

Appendix B. Proof that $\{\beta_j(\alpha,\bar{\alpha}^{-1})\}$ are real and positive

We now verify that the formula on the right-hand side of equation (6.14) gives a real positive quantity.

First, note that the product in equation (6.14) defining $\beta_j(\alpha,\bar{\alpha}^{-1})$ is over maps in the set Θ_j, which excludes those maps in Θ with a power of θ_j or θ_j^{-1} on the right-hand end. Consider a typical term in the product, $t_k(\alpha,\bar{\alpha})$ say, associated with the map θ_k,

$$t_k(\alpha,\bar{\alpha}) = \left(\frac{\alpha-\theta_k(B_j)}{\alpha-\theta_k(A_j)}\right)\left(\frac{\bar{\alpha}^{-1}-\theta_k(A_j)}{\bar{\alpha}^{-1}-\theta_k(B_j)}\right). \tag{B 1}$$

Its complex conjugate is

$$\overline{t_k(\alpha,\bar{\alpha})} = \left(\frac{\bar{\alpha}-\bar{\theta}_k(\bar{B}_j)}{\bar{\alpha}-\bar{\theta}_k(\bar{A}_j)}\right)\left(\frac{\alpha^{-1}-\bar{\theta}_k(\bar{A}_j)}{\alpha^{-1}-\bar{\theta}_k(\bar{B}_j)}\right). \tag{B 2}$$

However, from equation (A 8), it is known that for all mappings in the group

$$\theta_k^{-1}(\zeta^{-1}) = \frac{1}{\bar{\theta}_{kr}(\zeta)}, \tag{B 3}$$

which implies, in particular, that

$$\bar{\theta}_k(\bar{A}_j) = \frac{1}{\theta_{kr}^{-1}(\bar{A}_j^{-1})}, \quad \bar{\theta}_k(\bar{B}_j) = \frac{1}{\theta_{kr}^{-1}(\bar{B}_j^{-1})}. \tag{B 4}$$

By using these in equation (B 2) and after some rearrangement, we get

$$\overline{t_k(\alpha,\bar{\alpha})} = \left(\frac{\alpha - \theta_{kr}^{-1}(\bar{A}_j^{-1})}{\alpha - \theta_{kr}^{-1}(\bar{B}_j^{-1})}\right)\left(\frac{\bar{\alpha}^{-1} - \theta_{kr}^{-1}(\bar{B}_j^{-1})}{\bar{\alpha}^{-1} - \theta_{kr}^{-1}(\bar{A}_j^{-1})}\right). \tag{B 5}$$

Now, observe that if A_j and B_j are the fixed points of θ_j (and hence also of θ_j^{-1}), then $\bar{A}_j$, $\bar{B}_j$ are necessarily the fixed points of $\bar{\theta}_j$ (and hence also of $\bar{\theta}_j^{-1}$). Substituting $\zeta = \bar{A}_j, \bar{B}_j$ in equation (B 3) yields

$$\theta_j^{-1}(\bar{A}_j^{-1}) = \bar{A}_j^{-1}, \quad \theta_j^{-1}(\bar{B}_j^{-1}) = \bar{B}_j^{-1}, \tag{B 6}$$

from which we deduce the relations

$$A_j = \frac{1}{\bar{B}_j}, \quad B_j = \frac{1}{\bar{A}_j}, \tag{B 7}$$

where we have enforced the ordering $|A_j| > |B_j|$. By using equation (B 7) in equation (B 5), we deduce

$$\overline{t_k(\alpha,\bar{\alpha})} = \left(\frac{\alpha - \theta_{kr}^{-1}(B_j)}{\alpha - \theta_{kr}^{-1}(A_j)}\right)\left(\frac{\bar{\alpha}^{-1} - \theta_{kr}^{-1}(A_j)}{\bar{\alpha}^{-1} - \theta_{kr}^{-1}(B_j)}\right). \tag{B 8}$$

However, the right-hand side of equation (B 8) is precisely the term appearing in the product associated with the map θ_{kr}^{-1}. It is straightforward to verify that if θ_k is in the set Θ_j, then so is the map θ_{kr}^{-1}. Thus, these two terms in the product are mutual complex conjugates and combine in pairs to give a positive real quantity.

θ_k and θ_{kr}^{-1} are distinct except if θ_k is the identity. The outstanding term is simply

$$\frac{(\alpha - B_j)(\bar{\alpha}^{-1} - A_j)}{(\alpha - A_j)(\bar{\alpha}^{-1} - B_j)}. \tag{B 9}$$

Given that $A_j = \overline{B_j}^{-1}$, this can be rewritten in the form

$$|A_j|^2 \left|\frac{\alpha - B_j}{\alpha - A_j}\right|^2, \tag{B 10}$$

which is clearly real, and positive.

In summary, the products defining the quantities $\{\beta_j(\alpha, \bar{\alpha}^{-1})\}$ are therefore real and positive. ■

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