Euro. Jnl of Applied Mathematics (2004), *vol.* 15, *pp.* 643–650.
doi:10.1017/S0956792505005814 Printed in the United Kingdom

Exact solutions for uniform vortex layers attached to corners and wedges

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(*Received 21 January 2004; revised 22 September 2004*)

A class of exact mathematical solutions describing distributed regions of uniform vorticity attached to two solid walls meeting at an angle 2α is derived. Exterior to the uniform vorticity region the flow is quiescent and irrotational. The mathematical method used is a generalization of ideas original presented in Crowdy [4] combined with elements of conformal mapping theory associated with a differential equation method due to Polubarinova-Kochina traditionally applied in finding the solution to various free boundary problems.

1 Introduction

Steady planar vortical flows involving regions of uniform vorticity are not only of great mathematical interest owing to the fact that contour dynamics techniques can be applied for the numerical solution [10, 11], but are also of great physical interest because of the Prandtl–Batchelor theorem [2]. This states that, in the limit of vanishing viscosity, the central effect of viscosity in a region of closed streamlines will be to diffuse vorticity across streamlines thereby leading to a uniform-vorticity region over long periods.

It is well known that one method of generating vorticity in a flow is due to viscous effects at solid boundaries. From the perspective of ideal flow theory, given its existence in the flow by whatever means, the problem of finding steady uniform-vorticity regions near a smooth or sharp-edged solid boundary is a difficult one and few results are known. Certainly, there currently appear to be no analytical solutions in the literature. Saffman & Tanveer [20] used numerical methods based on analytic function theory to compute a class of steady flows in which a uniform vortex is trapped in the wedge-type region between a finite-length flat plate and a finite-length forward-facing flap. Such a configuration has relevance to aerodynamic applications and involves a bounded recirculating region of uniform vorticity attached to a corner and which is otherwise surrounded by irrotational flow. In related work, Moore *et al.* [14] again used numerical methods to compute a class of Batchelor flows in which regions of uniform vorticity (vortex patches) were bounded by both walls and/or vortex sheets. Vanden-Broeck & Tuck [21] performed a similar numerical study of uniform vortex regions which were partly bounded by impenetrable slip walls and were partly free (adjacent to an irrotational fluid region). In terms of fully nonlinear calculations involving uniform vortex regions attached to walls, Pullin [17] has used contour dynamics methods to examine their behaviour.

In this paper, a class of exact solutions is derived describing layers of uniform vorticity, of finite thickness but infinite extent, attached to corners and wedges. Given the lack of analytical results for steady vortex patch regions attached to solid boundaries, it seems appropriate to report the existence of such exact solutions and give details of their mathematical construction. No specific applications of the solutions to this paradigmatic problem will be discussed but the ideas presented are rather general meaning that they might, in principle, be generalizable to more complicated geometries of practical interest.

Crowdy [4] finds a class of exact solutions of the Euler equations providing analytical models of a class of coherent structures known as multipolar vortices [15]. The solutions have distributed vorticity. The method involves consideration of a streamfunction having the form

$$\psi(x, y) = \begin{cases} -\frac{\omega}{4}\left(\bar{z}z - \int^{z} S(z')\,dz' - \int^{\bar{z}} \overline{S}(z')\,dz'\right) & z \in D \\ 0 & z \notin D \end{cases} \tag{1.1}$$

where D is the region of non-zero vorticity and $S(z)$ is the *Schwarz function* [9] of its boundary curve ∂D. For a given analytic curve, its associated Schwarz function $S(z)$ is the function, analytic in an annular neighbourhood of the curve, satisfying

$$S(z) = \bar{z} \tag{1.2}$$

everywhere on the curve. Mathematically, functions of the form (1.1) are known as *modified Schwarz potentials* [18].

When used in conjunction with elements of conformal mapping theory, streamfunctions having the form of modified Schwarz potentials can yield exact solutions describing vortical equilibria of the Euler equations. Indeed, the general ideas originally presented in Crowdy [4] have been extended to produce exact solutions involving rotating vortical equilibria involving both a single [6] and multiple [8] vortex patches, multipolar vortical equilibria existing on the surface of a sphere [7] and equilibria involving multiply-connected vortical regions [5].

All the studies just cited involve hydrid combinations of point vortices and uniform vortex patches (or “V-states” [10]). This paper presents the new result that streamfunctions of the same general form (1.1) can also yield exact solutions of the steady Euler equations involving solid boundaries. When combined with the appropriate elements of conformal mapping theory, it is possible to produce equilibrium solutions of the Euler equation in which both the region of distributed non-zero vorticity as well as the entire velocity field are given explicitly in terms of closed-form mathematical formulae.

2 Mathematical formulation

Consider a region D of uniform vorticity ω of infinite extent, bounded by two straight walls meeting at an angle 2α where $0 < \alpha \leqslant \pi$ and a vortex jump ∂D separating the region D from a region of irrotational flow. If the vortex region is in equilibrium, both the two bounding walls as well as the patch boundary ∂D must be streamlines. If $\alpha < \pi/2$, we shall refer to the flow as being in a corner, if $\alpha > \pi/2$ it will be flow around a wedge.

Depending on the nature of the irrotational flow outside the patch region, a multitude of possible equilibrium solutions are expected to exist. Here, we seek a special subclass in which the flow in the irrotational region away from the patch is quiescent. It will be shown that this subclass of solutions can be described in exact mathematical form.

Now, the general solution of

$$\nabla^2\psi = 4\psi_{z\bar{z}} = -\omega, \quad \text{in } D \tag{2.1}$$

is

$$\psi(x, y) = -\frac{\omega}{4}\left(\bar{z}z - \int^{z} F(z')\,dz' - \int^{\bar{z}} \overline{F}(z')\,dz'\right), \quad z \in D \tag{2.2}$$

for some function $F(z)$ which, if there are no singularities in the fluid, is analytic everywhere in D. The conjugate function $\overline{F}(z)$ is defined by

$$\overline{F}(z) = \overline{F(\bar{z})}. \tag{2.3}$$

The dynamic condition on the patch boundary that, at the vortex jump, the fluid pressures are continuous is known [19] to be equivalent to continuity of the fluid velocities. Since

$$u - iv = 2i\psi_z \tag{2.4}$$

this condition takes the form

$$\psi_z = 0, \quad \text{on } \partial D \tag{2.5}$$

so that the fluid velocity on the patch boundary as one approaches it from inside the patch is continous with the stagnant fluid velocity in the region of irrotational flow. There is an additional kinematic condition on the patch boundary that it must be a streamline if the flow is steady. However, if (2.5) holds, then this condition is also satisfied because

$$d\psi = \psi_z\,dz + \psi_{\bar{z}}\,d\bar{z} = 0, \quad \text{on } \partial D. \tag{2.6}$$

If condition (2.5) holds then, using (2.2),

$$\bar{z} = F(z), \quad \text{on } \partial D. \tag{2.7}$$

This means that, for the particular flow being considered here, $F(z)$ must be precisely the Schwarz function $S(z)$ of the curve ∂D. For a given *closed* analytic curve, its associated Schwarz function $S(z)$ is analytic in an annular neighbourhood of the curve. In this case of a curve of infinite extent (or a closed curve going through the point at infinity on the Riemann sphere), $S(z)$ is analytic in an infinite strip-like neighbourhood of the curve. Outside this neighbourhood, $S(z)$ will in general have a distribution of singularities dictated by the shape of the curve.

In Crowdy [4], the method then proceeds by picking special choices of domain D having boundary curves ∂D whose Schwarz functions $S(z)$ have just a finite distribution of simple pole singularities inside D. Physically, this corresponds to allowing a distribution of point vortices to be superposed on the patch of otherwise uniform vorticity. This construction produces mathematical models of multipolar vortices [15]. Mathematically, the requirement of a finite distribution of point vortices inside the patch restricts the class of admissible Schwarz functions $S(z)$ and hence determines the shape of the vortex patch boundary. It should also be noted that such special domains D are subject to the additional restriction that the point vortices must be steady under the effects of the local non-self-induced velocity. This is a requirement dictated by the Helmholtz laws of vortex motion [19].

Here we proceed differently and do not allow any vortical singularities in D. Rather, the requirement on the analytic continuation of the Schwarz function $S(z)$ out of the strip-like region containing ∂D takes a different form: its continuation into the fluid region D is required to be everywhere analytic in D and such that ψ is constant on the two straight walls. If so, the streamfunction (2.2) will then satisfy the kinematic conditions on the two walls that there is no flow through them – clearly a necessary condition for a consistent solution. Equivalently, both walls must be streamlines. It is this modified condition that will determine the Schwarz function $S(z)$ in this case, and hence will determine the shape of the vortex patch boundary ∂D.

It should be noted that there is a natural length-scale in the problem associated with the width of the vortex patch region infinitely far from the corner. This can be chosen arbitrarily and is here chosen to be $\sqrt{2}$ for reasons given in the next section.

3 Conformal mapping

As in Crowdy [4], an effective way to parametrize the boundaries of the special vortex regions D satisfying all the above conditions is by use of conformal maps. To construct the relevant conformal maps, we make the observation that the mathematical problem just described is identical to one arising in a formulation of the problem of coating wedges of arbitrary angle with viscous films. This problem has been solved, using conformal mappings from the upper-half of a parametric ζ-plane, by Craster [3] who generalized earlier work by Howison & King [13]. All these workers make use of a well-developed mathematical technique based on differential equations originally devised by Polubarinova-Kochina [16].

This observation affords us the luxury of simply writing down the required mathematical solution while referring readers interested in the derivation to Section 2 of Craster's paper [3]. There, the conformal mapping from the upper-half ζ-plane to the region D is found to be

$$z(\zeta) = \sqrt{2}\left(\frac{2}{\pi}Q_\nu(1-2\zeta) + iP_\nu(1-2\zeta)\right), \tag{3.1}$$

where $\nu = 2\alpha/\pi - 1$ and P_ν, Q_ν are the Legendre functions of the first and second kinds [1, 12]. The points $0, 1$ and ∞ on the real axis in the ζ-plane are taken to map respectively to the points $z = \infty$, $\infty e^{2i\alpha}$ and $z = 0$. The vortex jump, or ∂D, therefore corresponds to the image of the segment on the real axis between 0 and 1, i.e. $0 < \zeta < 1$. The factor $\sqrt{2}$ gives us the required far-field width of the vortex layer and is chosen simply because (3.1) is the form of the map given in Craster [3].

The Schwarz function $S(z(\zeta))$ is then

$$S(z(\zeta)) = \overline{z(\zeta)} = \bar{z}(\bar{\zeta}) = \bar{z}(\zeta) = \sqrt{2}\left(\frac{2}{\pi}Q_\nu(1-2\zeta) - iP_\nu(1-2\zeta)\right), \tag{3.2}$$

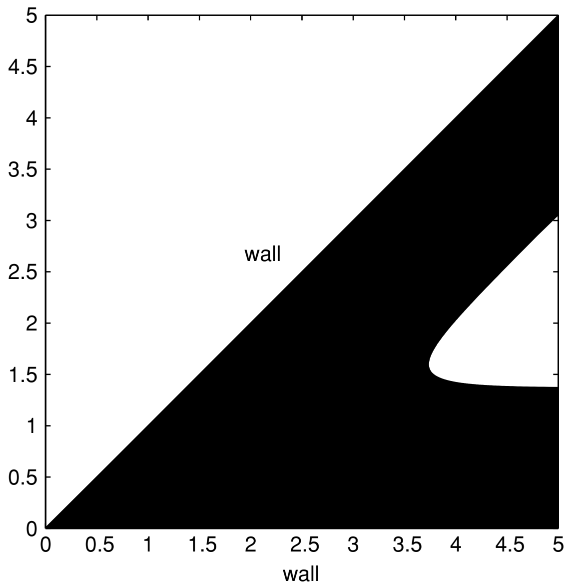

FIGURE 1. A uniform vortex layer (shaded) attached to a corner of angle $\pi/4$.

where we have used the fact that $\bar{\zeta} = \zeta$ on the segment $(0, 1)$. Using $u - iv = 2i\psi_z$, the complex velocity field is given explicitly as a function of ζ and $\bar{\zeta}$ by

$$u - iv = -\frac{i\omega}{\sqrt{2}}\left(\frac{2}{\pi}Q_v(1-2\bar{\zeta}) - iP_v(1-2\bar{\zeta}) - \frac{2}{\pi}Q_v(1-2\zeta) + iP_v(1-2\zeta)\right). \tag{3.3}$$

Thus, from (3.1) and (3.3), we have explicit formulae for both the equilibrium vortex patch boundary and the velocity field everywhere inside the patch.

It is of interest to examine the shape of the uniform vorticity region D. Thus we need to calculate (3.1) for $\zeta \in (0, 1)$. To do this, we first note that if $\zeta \in (0, 1)$ then $1 - 2\zeta \in (-1, 1)$. Next, we note that $P_v(\cos\phi)$ has the integral representation [12]

$$P_v(\cos\phi) = \frac{2}{\pi}\int_0^{\phi} \frac{\cos((v+1/2)\psi)}{(2\cos\psi - 2\cos\phi)^{1/2}}\, d\psi, \tag{3.4}$$

which can be used to calculate P_v for real values of its argument between -1 and 1. Finally, it is known [12] that for real arguments x,

$$\frac{2}{\pi}Q_v(x) = \frac{1}{2\sin(\pi v)}\left(P_v(x)\cos(\pi v) - P_v(-x)\right), \tag{3.5}$$

which, when used in conjunction with (3.4), can be used to evaluate Q_v for real values of its argument between -1 and 1.

The shaded regions of Figures 1–4 shows the vortex patch regions D for the values $\alpha = \pi/8, \pi/4, \pi/3$ and $2\pi/3$.

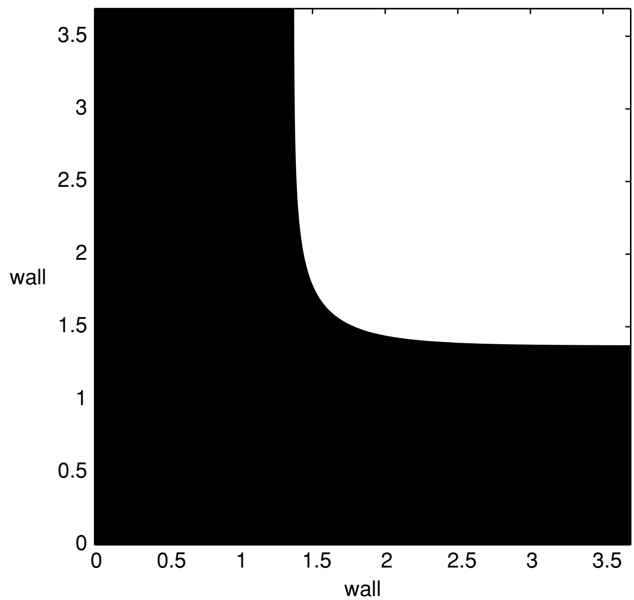

FIGURE 2. A uniform vortex layer (shaded) attached to a corner of angle $\pi/2$.

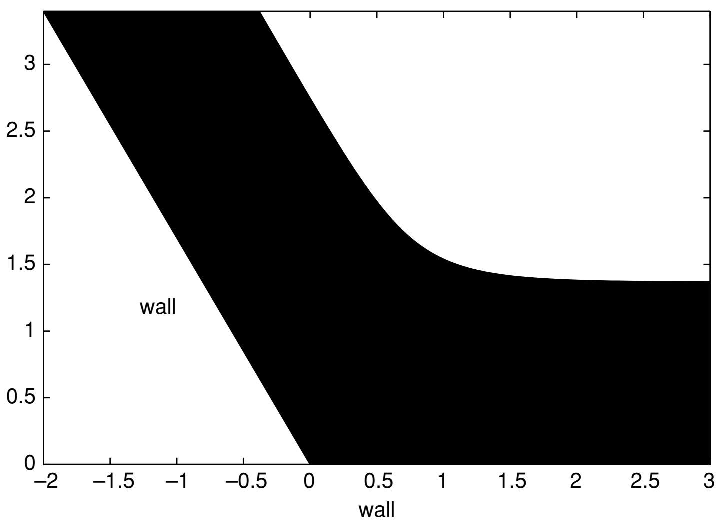

FIGURE 3. A uniform vortex layer (shaded) attached to a corner of angle $2\pi/3$.

4 Discussion

This paper has derived classes of exact solutions for uniform vortex regions attached to solid walls. The simplest examples of flows in corners and past wedges have been presented. In principle, the same method of combining consideration of streamfunctions which are modified Schwarz potentials with elements of conformal mapping theory associated with the differential equation method of Polubarinova-Kochina [16] can be

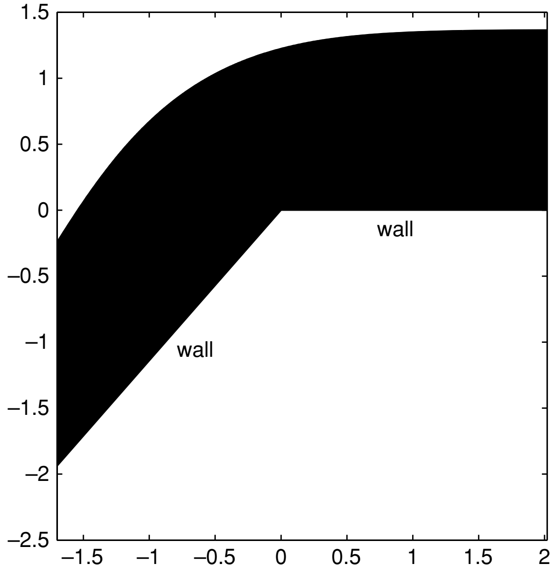

FIGURE 4. A uniform vortex layer (shaded) attached to a corner of angle $4\pi/3$.

applied in more complicated geometrical situations. While the latter method has found multifarious applications in a wide range of free boundary problems [13], the present paper appears to be the first time its applicability to the problem of finding equilibria of the Euler equation involving uniform vortex regions has been pointed out. This provides an addition to the compendium of physical problems to which the method of Polubarinova-Kochina [16] can be applied.

The stability of the equilibria remains to be investigated but is left for the future. It is also possible that more general mathematical solutions, for example, ones in which there is a non-trivial irrotational flow exterior to the uniform vorticity region, might be available analytically using perturbation methods about the leading-order non-trivial exact solutions found here.

Acknowledgement

This work is supported in part by a grant from EPSRC.

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