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Shapes of two-dimensional bubbles deformed by circulation

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Abstract. In this paper we study how the shape of a two-dimensional bubble is affected by circulation in an inviscid ambient flow. The shape of the bubble is determined by an equation which describes the balance of pressure forces and surface tension. Analysis of this equation shows that the bubble is circular for most values of circulation with the exception of an infinite series of discrete values where bifurcation occurs and the bubble's boundary develops ripples. It turns out that fully nonlinear deformations of the bubble as well as the associated circulatory flow around the bubble can be described explicitly in terms of rational functions.

AMS classification scheme numbers: 6B10, 76B45, 47H15, 30C20

1. Introduction

In this paper we analyse how circulation in an inviscid ambient flow field can affect the shape of a constant-pressure bubble with surface tension. The flows considered are two dimensional. This work is inspired by the recent paper of Crowdy [3] who found a class of explicit solutions to this problem with twofold symmetry. The question of whether additional bubble shapes exist was left open. In this paper, this question is answered in the affirmative.

The balance of the outside hydrodynamic pressure and the surface tension leads via Bernoulli's equation to a nonlinear equation for the analytic function which maps the exterior of the unit disc conformally to the flow region. A variant of this equation has recently been discussed by the first author [11]. When the same analysis is applied to the present situation it turns out that the bubble retains its circular shape for almost all values of the circulation. However, we find that for an infinite number of discrete values of the circulation, a bifurcation occurs. For any natural number m greater than or equal to 2 there exists a continuous one-parameter family of bubbles in equilibrium whose boundaries have m ripples.

The balance equation for pressure and surface tension has been considered by numerous authors for various physical flow configurations and represents a problem of classical interest in fluid dynamics dating back to Lord Rayleigh [7]. Surprisingly, explicit solutions have been found in the form of rational functions by several authors. McLeod [6] found an isolated exact solution for a bubble in a uniform flow field, Crapper [2] derived some for capillary water waves, Shankar [8] found some for the flow along a straight wall with a gap, and Crowdy [3] gave some solutions for a constant pressure bubble with circulation in the flow exterior to it. Longuet-Higgins [5] also provides a list of solutions to this general class of problems which

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are expressible in closed form. This paper adds further solutions to this list and generalizes the exact solutions of Crowdy [3]. In flow situations in which exact solutions do not exist, much numerical (e.g. [10]) and analytical (e.g. [9]) work has been performed in an attempt to understand this general class of problems.

2. Equations

Consider a single bubble B with constant pressure p_b inside and surface tension on its boundary. The outside flow covers a region G, which we consider as a subset of the complex plane $\mathbb{C}$. The flow is assumed to be irrotational and incompressible. It can then be described by a complex potential $W = \Phi + \mathrm{i}\Psi$ which is an analytic function in G. The components of W are the velocity potential Φ and the stream-function Ψ. The velocity vector $\boldsymbol{U} := (u, v)$ can be obtained either as $(u, v) = \operatorname{grad} \Phi$ or from the components of the derivative $W' = u - \mathrm{i}v$.

It is assumed that the flow has circulation Z around the bubble. This means that W behaves near infinity like

$$W(z) = -\frac{Z\mathrm{i}}{2\pi} \ln z + \mathrm{O}(1/z). \tag{1}$$

For a steady configuration, the boundary ∂G of the bubble is a streamline. The fluid density ρ is constant and, for convenience, we set $\rho = 2$. Then along the boundary the pressure p and the velocity $U := \sqrt{u^2 + v^2}$ satisfy Bernoulli's law

$$U^2 + p = C_B \tag{2}$$

with a constant C_B. The pressure difference across the boundary is balanced by surface tension

$$p_b - p = \sigma\kappa \tag{3}$$

with a coefficient $\sigma \geqslant 0$. The curvature κ of the boundary curve is positive (negative) when the curve is concave (convex) towards the bubble. For $\kappa > 0$ the force $\sigma\kappa$ acts towards the bubble and compensates for an overpressure inside.

There exists a uniquely determined conformal mapping f from the exterior $D^- := \{\zeta : |\zeta| > 1\}$ of the unit disc D to the flow region G subject to the hydrodynamic normalization

$$f(\infty) = \infty \qquad f'(\infty) > 0. \tag{4}$$

The second requirement of (4) which states that $f'(\zeta)$ tends to a positive real constant as $\zeta \to \infty$ fixes the rotational degree of freedom in the Riemann mapping theorem. The flow

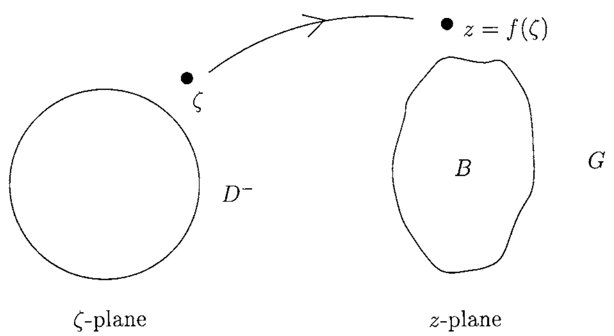

Figure 1. Conformal mapping domains.

field W in G is mapped to a flow field $w(\zeta) := W(f(\zeta))$ in D^- describing the flow around the disc. This must be a simple vortex with the same circulation Z, hence

$$w'(\zeta) = W'(f(\zeta))f'(\zeta) = -\frac{Z\mathrm{i}}{2\pi\zeta}. \tag{5}$$

Combining equations (2) and (3) we obtain the boundary condition for the derivative of the potential

$$|W'(z)|^2 = \sigma\kappa + C_B - p_b \tag{6}$$

for $z \in \partial G$.

Since the problem is invariant with respect to translations it is sufficient to find the derivative f' of the mapping function f. We use the parametrization $\eta(s) = x(s) + \mathrm{i}y(s) = f(\mathrm{e}^{\mathrm{i}s})$ to express W' and κ in (6) in terms of f'. This yields the condition

$$\frac{\Gamma}{|f'(\mathrm{e}^{\mathrm{i}s})|} + \gamma|f'(\mathrm{e}^{\mathrm{i}s})| - \sigma - \sigma\,\mathrm{Re}\left(\frac{\mathrm{e}^{\mathrm{i}s}f''(\mathrm{e}^{\mathrm{i}s})}{f'(\mathrm{e}^{\mathrm{i}s})}\right) = 0 \tag{7}$$

with the constants

$$\gamma = p_b - C_B \qquad \Gamma = Z^2/4\pi^2. \tag{8}$$

Equation (7) must be satisfied for all s—it represents the equilibrium condition which determines the free surface of the bubble. It is a necessary condition which the (as yet unknown) conformal mapping function must satisfy. Once f' is known, the flow field $W'(f(\zeta))$ can be calculated using (5).

By choosing suitable units, we can set $\sigma = 1$. Since only the derivative f' appears in (7), it can be written in the form of a functional equation

$$A(f') = 0 \tag{9}$$

for a nonlinear operator A which assigns to each function h, analytic in D^- and such that $h(\mathrm{e}^{\mathrm{i}s}) \neq 0$, the real function $\phi = A(h)$ defined by

$$\phi(s) := \frac{\Gamma}{|h(\mathrm{e}^{\mathrm{i}s})|} + \gamma|h(\mathrm{e}^{\mathrm{i}s})| - 1 - \mathrm{Re}\left(\frac{\mathrm{e}^{\mathrm{i}s}h'(\mathrm{e}^{\mathrm{i}s})}{h(\mathrm{e}^{\mathrm{i}s})}\right). \tag{10}$$

If f_0' is a solution of equation (7) then for any complex number $c \neq 0$, the function cf_0' solves the same equation for the parameters $|c|\Gamma$ and $\gamma/|c|$. Therefore, it is sufficient to determine solutions subject to the constraint $f'(\infty) = 1$.

3. Bifurcation analysis

The constant function $h_1 = 1$ (corresponding to a circular bubble shape) satisfies $A(h_1) = 0$ if and only if the parameters satisfy

$$\Gamma + \gamma = 1. \tag{11}$$

With the same arguments as in [11] one can show that the functional A is Fréchet differentiable. The derivative $DA(h)$ is an operator which assigns to an analytic function g in D^- the real function

$$DA(h)g(s) = -\mathrm{Re}\,\frac{\mathrm{e}^{\mathrm{i}s}g'(\mathrm{e}^{\mathrm{i}s})}{h(\mathrm{e}^{\mathrm{i}s})} + \mathrm{Re}(A_2(s)g(\mathrm{e}^{\mathrm{i}s})) \tag{12}$$

with

$$A_2(s) = \gamma \frac{|h(\mathrm{e}^{\mathrm{i}s})|}{h(\mathrm{e}^{\mathrm{i}s})} - \frac{\Gamma}{|h(\mathrm{e}^{\mathrm{i}s})| h(\mathrm{e}^{\mathrm{i}s})} + \frac{\mathrm{e}^{\mathrm{i}s} h'(\mathrm{e}^{\mathrm{i}s})}{h(\mathrm{e}^{\mathrm{i}s})^2}. \tag{13}$$

For the constant function $h_1 = 1$ the derivative is

$$DA(h_1)g(s) = \mathrm{Re}\,\big((\gamma - \Gamma)g(\mathrm{e}^{\mathrm{i}s}) - \mathrm{e}^{\mathrm{i}s} g'(\mathrm{e}^{\mathrm{i}s})\big). \tag{14}$$

With the coefficients of the Laurent series

$$g(\zeta) = \sum_{n=0}^{\infty} \frac{b_n}{\zeta^n} \tag{15}$$

the equation

$$DA(h_1)g(s) = \psi(s) = \sum_{n=-\infty}^{\infty} c_n \exp(\mathrm{i}ns) \tag{16}$$

can be rewritten as equations for the coefficients

$$(\gamma - \Gamma)\,\mathrm{Re}\, b_0 = c_0 \tag{17}$$

$$(\gamma - \Gamma + n)\bar{b}_n = 2c_n \qquad \text{for} \quad n = 1, 2, \ldots. \tag{18}$$

These equations can be solved for all n if and only if

$$\gamma - \Gamma + n \neq 0 \qquad \text{for all} \quad n = 0, 1, 2, \ldots. \tag{19}$$

From the solvability of equations (17) and (18) it follows that the operator DA is one-to-one. This implies a uniqueness result: for parameters γ, Γ which satisfy the condition (19) there are no bubble shapes (other than circular ones) possible in a neighbourhood of the unit circle.

If, on the other hand, for some integer $m \geqslant 2$ the condition

$$\gamma - \Gamma + m = 0 \tag{20}$$

holds, the operator $DA(h_1)$ is singular, and bifurcation from the circular shape can occur. We could investigate this by the method of Lyapounov–Schmidt (see [1]), but fortunately the bifurcating branches of solutions can be represented in explicit form by rational functions as we show in the next section.

The parameters where bifurcation occurs must satisfy equations (20) and (11). Therefore, bifurcation starts at the discrete set of parameter values given by

$$\gamma = -\frac{m-1}{2} \qquad \Gamma = \frac{m+1}{2}. \tag{21}$$

Since f' is a derivative, it has no $1/\zeta$-term. In view of the normalization $f'(\infty) = 1$, the constant term is fixed. Hence variations in g of the form given in (15) must be restricted by the conditions $b_0 = 0$ and $b_1 = 0$. Bifurcation occurs when the parameters satisfy (21) for integers $m \geqslant 2$.

4. Explicit solutions

For any function g analytic in the exterior D^- of the unit disc, the reflected function g^* defined by

$$g^*(\zeta) := \overline{g(1/\bar{\zeta})} \tag{22}$$

is analytic in the unit disc D. On the circle $|\zeta| = 1$ the equation

$$g^*(\zeta) = \overline{g(\zeta)} \tag{23}$$

holds. In view of this property reflection is a convenient means to calculate the moduli and real parts of analytic functions. The curvature $\kappa(s)$ of the curve $\eta(s) := f(\mathrm{e}^{\mathrm{i}s})$ can be represented using the reflected function f'^* of f'

$$\kappa(s) = \frac{1}{f'}\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(\frac{\zeta^2 f'}{f'^*}\right)^{1/2}. \tag{24}$$

The sign of the square root must be chosen in accordance with our convention of the sign of the curvature. Using this formula (24), we obtain instead of (7) the equation

$$\frac{\Gamma}{f'^*} + \gamma f' + \frac{\mathrm{d}}{\mathrm{d}\zeta}\left(\frac{\zeta^2 f'}{f'^*}\right)^{1/2} = 0 \tag{25}$$

which must be satisfied for $|\zeta| = 1$.

Consider for any integer $m \geqslant 2$ the rational function

$$h_m(\zeta) = \left(\frac{\zeta^m - (m+1)a^m}{\zeta^m + (m-1)a^m}\right)^2 \tag{26}$$

depending on the complex parameter a. Insertion of h_m for f' into (25) gives the relation

$$\Gamma\left(\frac{1 + (m-1)\bar{a}^m\zeta^m}{1 - (m+1)\bar{a}^m\zeta^m}\right)^2 + \gamma\left(\frac{\zeta^m - (m+1)a^m}{\zeta^m + (m-1)a^m}\right)^2 - \frac{\mathrm{d}}{\mathrm{d}\zeta}\left[\frac{\zeta(\zeta^m - (m+1)a^m)(1 + (m-1)\bar{a}^m\zeta^m)}{(\zeta^m + (m-1)a^m)(1 - (m+1)\bar{a}^m\zeta^m)}\right] = 0. \tag{27}$$

The sign of the curvature term is chosen so that it gives the correct result $\kappa = +1$ for the unit circle ($a = 0$). All terms in the relation (27) can be integrated in closed form by rational functions. The integrated equation is

$$\Gamma\left(\frac{m-1}{m+1}\right)^2\left(\zeta + \frac{4m\zeta}{(m-1)^2(1 - (m+1)\bar{a}^m\zeta^m)}\right) + \gamma\left(\zeta + \frac{4ma^m\zeta}{(m-1)(\zeta^m + (m-1)a^m)}\right) - \frac{\zeta(\zeta^m - (m+1)a^m)(1 + (m-1)\bar{a}^m\zeta^m)}{(\zeta^m + (m-1)a^m)(1 - (m+1)\bar{a}^m\zeta^m)} = C \tag{28}$$

with a constant C. Insertion of $\zeta = 0$ on the left-hand side shows that C must be 0. If equation (28) is written with a common denominator, then only the three powers ζ, ζ^{m+1}, ζ^{2m+1} appear in the numerator. Straightforward calculation shows that (28) is satisfied for the values of the parameters

$$\gamma = -\left(\frac{m-1}{2}\right)\frac{1 - (m-1)^2|a|^{2m}}{1 + (m^2-1)|a|^{2m}} = -\frac{m-1}{2}\left(1 - 2m(m-1)|a|^{2m} + \cdots\right) \tag{29}$$

$$\Gamma = \left(\frac{m+1}{2}\right)\frac{1 - (m+1)^2|a|^{2m}}{1 + (m^2-1)|a|^{2m}} = \frac{m+1}{2}\left(1 - 2m(m+1)|a|^{2m} + \cdots\right) \tag{30}$$

where the series expansions are for $|a| \ll 1$.

This means that $f'_m := h_m$ is a solution of equations (25) and (7). The corresponding conformal mapping

$$f_m(\zeta) = \zeta + \frac{4ma^m\zeta}{(m-1)(\zeta^m + (m-1)a^m)} \tag{31}$$

is the primitive of the function $h_m(\zeta)$.

The ansatz (26) is motivated by the following argument. In order to obtain m-fold symmetry one must search for h_m as a function of ζ^m. Instead of (26) one could try the more general ansatz

$$h_m(\zeta) = \left(\frac{\zeta^m - B^m}{\zeta^m - C^m}\right)^2 \tag{32}$$

with complex constants B and C. The condition (27) implies that h_m must have a rational function as primitive. However, a function of form (32) is the derivative of a rational function if and only if

$$(m-1)B^m + (m+1)C^m = 0 \tag{33}$$

holds. Therefore, all functions h_m of form (32), which solve equation (25), must have the special form (26).

The area F covered by the bubble can be expressed in terms of f'_m and the reflection f^*_m of the function f_m

$$F = \tfrac{1}{2}\,\mathrm{Im}\int \bar{f}_m f'_m\,\mathrm{d}\zeta = \tfrac{1}{2}\,\mathrm{Im}\int f^*_m f'_m\,\mathrm{d}\zeta. \tag{34}$$

The path of integration is the unit circle. Evaluation of the last integral in (34) by means of the residue theorem yields the result

$$F = \pi\left(1 - \frac{16m^2|a|^{2m}}{m-1}\,\frac{1+(m-1)|a|^{2m}}{(1-(m-1)^2|a|^{2m})^2}\right) \approx \pi\left(1 - \frac{8m^2|a|^{2m}}{m-1}\right)^2. \tag{35}$$

The perimeter L of the bubble can be written as

$$L = \int |f'_m(\mathrm{e}^{\mathrm{i}s})|\,\mathrm{d}s = \int (f'_m f'^*_m)^{1/2}\,\mathrm{d}s \tag{36}$$

$$= -\mathrm{i}\int \frac{(\zeta^m - (m+1)a^m)(1-(m+1)\bar{a}^m\zeta^m)}{\zeta(\zeta^m + (m-1)a^m)(1+(m-1)\bar{a}^m\zeta^m)}\,\mathrm{d}\zeta \tag{37}$$

with the unit circle as the path of integration. Evaluation of the integral by means of the residue theorem yields the result

$$L = 2\pi\frac{1+(3m-1)(m+1)|a|^{2m}}{1-(m-1)^2|a|^{2m}} = 2\pi(1 + 4m^2|a|^{2m} + 8(m-1)^2|a|^{4m} + \cdots). \tag{38}$$

The mapping f_m is one-to-one only for sufficiently small values of the parameter $|a| < \alpha_m$. The bound depends on m. We list the first few values: $\alpha_2 = 0.3333, \alpha_3 = 0.4696, \alpha_4 = 0.5426, \alpha_5 = 0.5934, \alpha_6 = 0.6318, \alpha_7 = 0.6625$. Figures 2–4 show typical (superposed) bubble shapes for $m = 3$, 4 and 7, respectively. The bubbles are all normalized to total area π. The sequence of shapes may therefore be considered as the shape of a single incompressible bubble under the influence of varying parameters Γ and γ. Note that for the special case $m = 2$ formula (26) with (29) and (30) retrieves the results of Crowdy [3].

The bubbles shown in the figures are all calculated with real positive values of the parameter a. For complex values of a the shapes are the same as for $|a|$ but rotated. The second

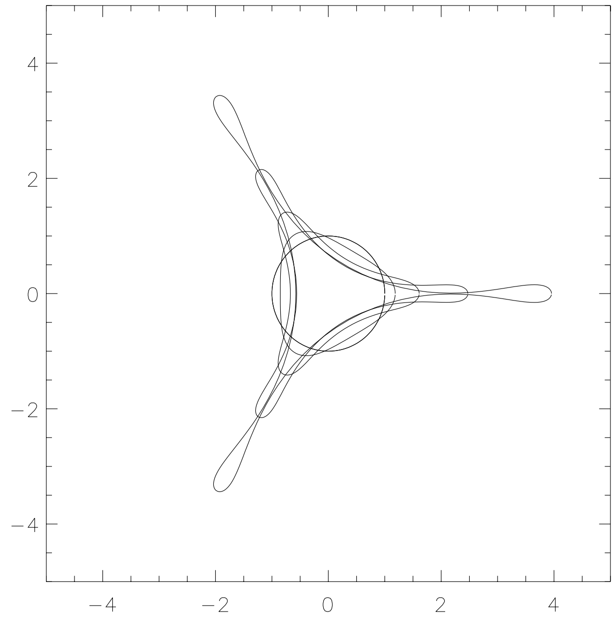

Figure 2. Bubble shapes for $m = 3$ and parameter values $a = 0$, 0.3, 0.4, 0.45, 0.469.

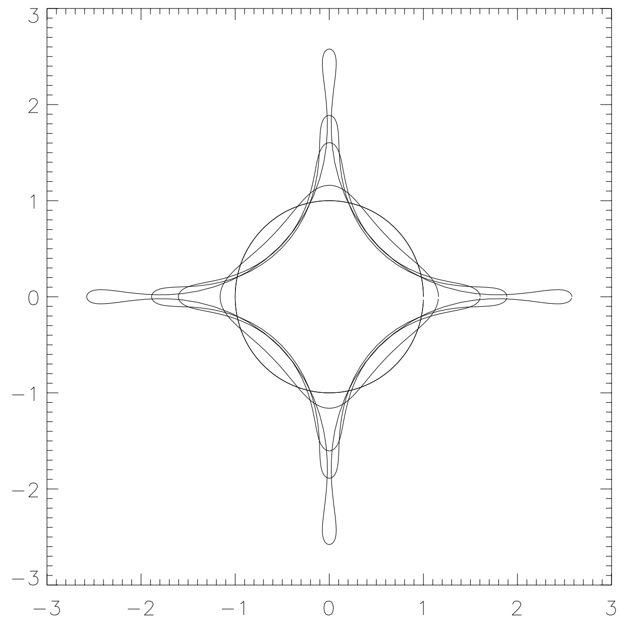

Figure 3. Bubble shapes for $m = 4$ and parameter values $a = 0$, 0.4, 0.5, 0.52, 0.54.

normalization condition (5) is satisfied. One could also rotate the bubbles by an angle θ by multiplication of h_m by a factor of $e^{i\theta}$. However, then (4) would be violated.

In figure 5, we plot some typical streamlines of the steady circulatory flow around a near-critical $m = 7$ bubble.

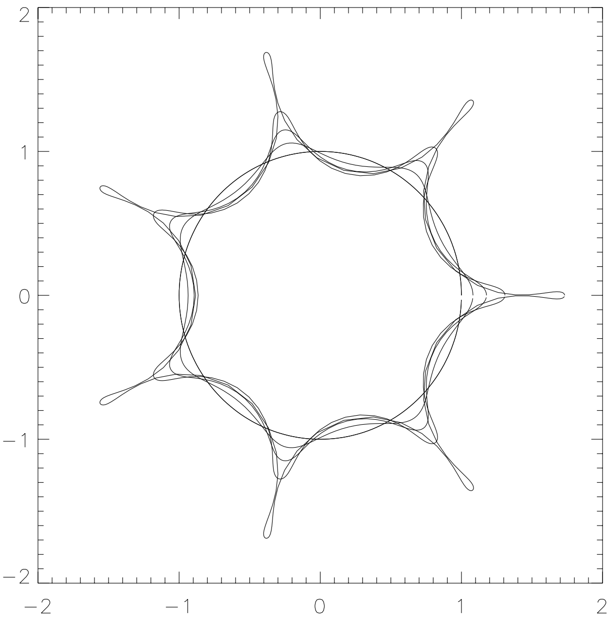

Figure 4. Bubble shapes for $m = 7$ and parameter values $a = 0, 0.55, 0.6, 0.63, 0.662$.

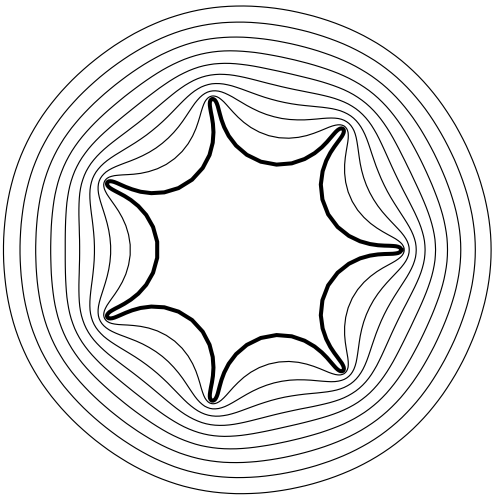

Figure 5. Typical streamlines for the case $m = 7$, $a = 0.65$.

One can use the formula (35) for the area to normalize all bubbles to area π and to change the parameters accordingly. The parameters $\tilde{\Gamma}$ and $\tilde{\gamma}$ for the normalized solutions satisfy the approximate relation

$$\tilde{\Gamma} = \frac{m+1}{2} + \frac{1+4m-m^2}{m^2-1}\delta + \frac{8m(2+m+m^2)}{(m-1)^2(m+1)^3}\delta^2 + \cdots \tag{39}$$

with $\delta := \tilde{\gamma} + (m-1)/2$. Figure 6 shows the values of the normalized parameters $\tilde{\Gamma}$ and $\tilde{\gamma}$ corresponding to the solutions mentioned before and the quadratic approximation of formula

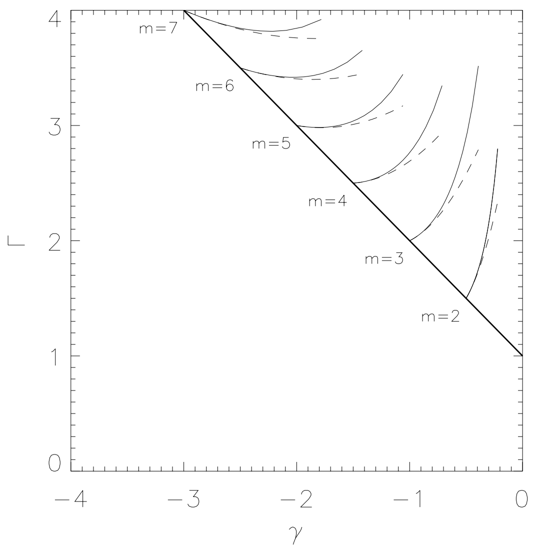

Figure 6. Pairs of normalized parameters $\tilde{\gamma}$, $\tilde{\Gamma}$ for which bubble shapes with area $= \pi$ exist. The broken curves give the approximation of (39).

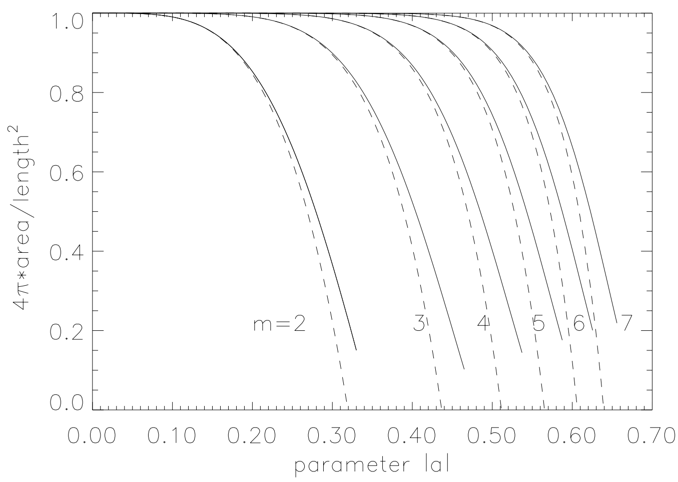

Figure 7. The isoperimetric ratio $4\pi F/L^2$ as a function of the parameter $|a|$ for exponents $m = 2, 3, \ldots, 7$. The broken curves give the approximation of (40).

(39). The straight line from top left to bottom right is given by equation (11), $\Gamma + \gamma = 1$, corresponding to circular bubbles. For the parameter values determined by (21) branches of solutions bifurcate with parameter given by (29) and (30) normalized using (35) to a bubble area of π.

In figure 7, for several values of m, we plot the isoperimetric ratio which is given by

$$\frac{4\pi F}{L^2} = 1 - 8\frac{m+1}{m-1}|a|^{2m} \cdots \tag{40}$$

as a function of $|a|$.

5. Conclusion

The formulae derived in this paper enable one to compute the steady deformation of capillary bubbles in a circulatory ambient flow when the area of the bubble is fixed. Physically, this is convenient because one question of physical interest is how large the perimeter-to-area ratio becomes as the circulation outside a given bubble (containing, for example, a fixed amount of insoluble gas) changes. Large perimeter–area ratios might help to explain enhanced heat and mass transfer rates (between the bubble and host fluid) in the presence of circulation. For a more complete discussion of the possible relevance of this mathematical model to various applications, we refer the reader to Crowdy [3].

It is also noted that the critical bubble shapes at which the conformal mapping function loses univalency are typified by an m-fold 'pinching' of the bubbles leading to the formation of m smaller satellite bubbles. Crapper observed a similar pinching phenomenon in his early study of capillary water waves [2] (for details of how the exact solution of Crapper [2] is related to the new exact solutions found here we refer the reader to Crowdy [3, 4]). This suggests that the effect of circulation outside a bubble is to deform it in such a way that it might lead to a splitting or 'atomizing' of the bubble into several smaller ones. In a liquid dispersion, this 'atomizing' of an enclosed fluid (i.e. the bubble) in a liquid host is believed to be at least partly responsible for enhanced heat and mass transport rates between the enclosed fluid and its liquid host. See [3] for additional references.

While this paper was under review, one of the referees pointed out that the stationary flow around the bubble can be understood as the composition of a circulatory flow with a Crapper-type capillary wave running along the fluid surface in the opposite direction. The flow around the unit disc with circulation Z has along the circle the velocity $V = Z/2\pi$ and constant pressure p_e. Equation (11) yields simply $p_b - p_e = \sigma$, i.e. the equilibrium condition (3) with $\kappa = 1$. The bifurcation condition (20) then gives an equation for the velocity

$$V^2 = (m+1)\frac{\sigma}{\rho} = \left(\frac{m+1}{m}\right)\frac{2\pi\sigma}{\lambda\rho} \tag{41}$$

with the wavelength $\lambda = 2\pi/m$. This can be compared with the speed $c^2 = 2\pi\sigma/\rho\lambda$ of Crapper's waves [2] for small amplitudes. Both velocities agree in the limit $m \to \infty$. For large m the wavelength becomes small compared with the radius of curvature of the unit circle. The short waves travelling around the circle feel little influence from the curvature and propagate along an almost straight surface in the same way as Crapper's plane waves.

When we insert the physical constants for water ($\sigma = 72.9$ dyn cm^{-1}, $\rho = 1$ cm^{-3}) we obtain for the first wave ($m = 2$) and a bubble of radius 1 cm a velocity of $V_2 = 14.8$ cm s^{-1}. This means that the first wave is excited when the flow rotates 2.3 times in each second. The $m = 7$ waves shown in figures 4 and 5 are excited at a velocity $V_7 = 24$ cm s^{-1}, when the fluid rotates 3.8 times per second.

The linear stability of the newly derived equilibrium solutions is of great importance in determining the possible physical relevance of the solutions. Moreover, it might provide insight into the nature of any transitions between the various equilibria presented in this paper. For general values of the parameters a and m, although the base state solutions are known exactly, the linear stability calculation must be performed numerically. A detailed investigation of the linear stability properties and its physical ramifications is left for the future.

Notwithstanding these remarks on the possible physical application of our results, the emphasis of this paper has been principally mathematical. To summarize, it has been demonstrated that a highly nonlinear free boundary problem of classical interest involving the balance of hydrodynamic pressure forces and surface tension forces admits a family of exact solutions parametrized by an integer m (reflecting a rotational symmetry) and a continuous parameter a (reflecting the degree of deformation from a circular bubble shape).

Finally, we remark that a bubble with capillarity in a circulating flow presents a nice example of a physically motivated nonlinear equation where bifurcation occurs. Mathematically, it is a particularly instructive example because the bifurcating branches of fully nonlinear solutions can be described explicitly.

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