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A Theory of Exact Solutions for Plane Viscous Blobs

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Summary. We present a new general theory of exact solutions for a simply connected, plane, slow viscous fluid blob with surface tension. The formulation reveals the existence of an infinite number of conserved quantities associated with the exact solutions. This new theoretical approach simplifies the calculation of concrete solutions.

Key words. Stokes flow, viscous drop, conservation laws, exact solutions

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1. Introduction

This paper presents a reformulation of the problem of the slow viscous quasi-steady flow of a two-dimensional, simply connected fluid blob with surface tension. Many exact solutions for special cases of this problem have already appeared in the literature [1] [2] [3] [4] [9] and rely on a complexification of the problem first exploited by Richardson [5]. The closely related problem of the Stokes flow around a single bubble in a strain field has also received much attention recently [8] [10] [11]. It is noted that the general theory presented in this paper for the case of a viscous blob is readily extended, with only minor changes in detail, to the case of Stokes flow around a single bubble.

The approach adopted in this paper, while employing the same formulation in terms of complex analytic functions, is essentially different from previous methods in that the problem for the boundary evolution of the blob is recast in terms of the time evolution of a very general set of purely geometrical line integral quantities defined around the boundary

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of the blob. This approach seems to simplify greatly much of the unwieldy analysis that has characterized previous treatments. The reformulation also reveals important mathematical properties of the equations—in particular, the evolution equations for the specially defined line integral quantities all have a very special *upper-triangular* structure. With suitable initial conditions, this special structure of the evolution equations can be shown to lead to the existence of an infinite number of conserved quantities associated with a very general class of exact solutions. The existence of such conserved quantities has not been generally recognized using previous methods. Another general result that is easily demonstrated using the new approach is a “theorem of invariants” which automatically provides a further finite set of invariants for a subset of solutions in this general class of “exact solutions.” It is noted that the phrase “exact solutions” is used to refer to solutions of this free boundary value problem that can be described exactly in terms of a *finite* set of first-order ordinary differential equations (as opposed to an infinite set). While this paper was in preparation, the authors learned of recent related work of Cummings, Howison, and King [7] who employ some of the same ideas to a more restricted class of exact solutions (described by polynomial maps) for the simpler problem of a blob **without** surface tension.

Finally, since many examples of the slow viscous flow of simply connected fluid blobs have already been explicitly calculated using alternative solution methods, we do not attempt to calculate further examples in this paper. The purpose of this paper is to present a novel theoretical approach. We do, however, give details of a special class of exact solutions with a particularly appealing mathematical structure that comes to light as a result of the reformulation in this paper. This example is presented as a case study and represents a generalization of solutions found by Richardson [1]. Again, while this paper was in preparation, the authors learned that Richardson [6] has also recently produced this generalization of Richardson [1] but using the more traditional techniques.

The paper is organized as follows: In Section 2 the equations and boundary conditions are rewritten in complex form. In Section 3 a conformal mapping representation is introduced and the complexified problem is now rewritten in terms of an appropriately defined conformal mapping function. Section 4 introduces the new conservation law approach to finding exact solutions to the (complexified) problem. This section contains the important details of the new method. Section 5 presents what is herein called a “theorem of invariants”—a general result which can be deduced immediately from the new mathematical formulation. Finally, in Section 6 a case study is presented to show the utility of the reformulation in deducing the appropriate evolution equations for the time-dependent parameters appearing in the exact solutions.

2. Mathematical Formulation

We consider the unsteady evolution of a general simply connected plane blob of fluid of viscosity μ under the assumptions of no inertial effects and no gravitational effects or effects from other body forces. The equations of motion of the fluid are

$$\mu\nabla^2\mathbf{u} = \nabla p, \tag{1}$$

$$\nabla.\mathbf{u} = 0, \tag{2}$$

where $\mathbf{u}(x, y)$ is the fluid velocity, $p(x, y)$ is the pressure, and μ the fluid viscosity. We choose to nondimensionalize the problem using a as a typical length-scale (e.g., an effective radius where πa^2 is the initial area of the blob). If σ is the surface tension parameter, we nondimensionalize velocities by $\frac{\sigma}{\mu}$, the pressure by $\frac{\sigma}{a}$, length by a, and time by $\frac{a\mu}{\sigma}$. Introducing a streamfunction $\psi(x, y)$ such that

$$\mathbf{u} = (\psi_y, -\psi_x), \tag{3}$$

it is well-known that the two-dimensional Stokes flow can be reformulated in terms of this streamfunction, which satisfies a biharmonic equation in the fluid region, i.e.

$$\nabla^4 \psi = 0. \tag{4}$$

On the blob boundary we must ensure continuity of shear stress and satisfy the requirement that the jump in the normal stress across the interface equals the product of the surface tension σ and the curvature κ. These two conditions can be written as

$$-pn_j + 2e_{jk}n_k = -\kappa n_j, \tag{5}$$

where e_{jk} are given by

$$e_{jk} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\right). \tag{6}$$

Additionally, there is a kinematic boundary condition that the normal velocity V_n of a point on the boundary equals the normal fluid velocity at that point, that is,

$$\mathbf{u}.\mathbf{n} = V_n. \tag{7}$$

It will also be seen that further conditions will need to be imposed at any singularities within the fluid in order to specify a solution completely.

To complexify the problem, all fields are written as functions of $z_1 = x + iy$ and $\bar{z}_1 = x - iy$. According to the Goursat representation for biharmonic functions, we can then write

$$\psi(z_1, \bar{z}_1) = Im[\bar{z}_1 f_1(z_1) + g_1(z_1)], \tag{8}$$

where $f_1(z_1)$ and $g_1(z_1)$ are two functions which are analytic in the fluid region. Note that since the blob boundary evolves with time, each of f_1 and g_1 also depend on time t, though this dependence is suppressed in (8) for purposes of brevity. All physically relevant quantities can now be written in terms of these two functions $f_1(z_1)$ and $g_1(z_1)$. In particular,

$$\frac{p}{\mu} - i\omega = 4f_1'(z_1), \tag{9}$$

$$u_1 + iv_1 = -f_1(z_1) + z_1\bar{f}_1'(\bar{z}_1) + \bar{g}_1'(\bar{z}_1), \tag{10}$$

$$e_{11} + ie_{12} = z_1\bar{f}_1''(\bar{z}_1) + \bar{g}_1''(\bar{z}_1), \tag{11}$$

where $\bar{f}_1$ denotes the conjugate function, $\bar{f}_1(z_1) = \overline{f_1(\bar{z}_1)}$, and u_1, v_1 represent the components of velocity in the x and y directions, respectively.

The stress condition must be rewritten in a more convenient form. To do this, we define a complex normal as

$$N \equiv n_1 + in_2 = -i(x_s + iy_s) = -iz_{1_s} = -i\exp(i\theta), \tag{12}$$

where s is the arclength around the blob traversed in the anticlockwise direction and θ is the angle between the tangent and the real positive axis. The stress condition can then be rewritten as

$$-pN + 2(e_{11} + ie_{12})\bar{N} = -\kappa N, \tag{13}$$

where κ is the curvature. Substituting for the various quantities in this equation, a straightforward calculation reveals that it can be written as

$$\frac{\partial S(z_1, \bar{z}_1)}{\partial z_1} z_{1_s} + \frac{\partial S(z_1, \bar{z}_1)}{\partial \bar{z}_1} \bar{z}_{1_s} = -i\frac{z_{1_{ss}}}{2}, \tag{14}$$

where

$$S(z_1, \bar{z}_1) \equiv f_1(z_1) + z_1 \bar{f}_1'(\bar{z}_1) + \bar{g}_1'(\bar{z}_1). \tag{15}$$

(14) can be integrated immediately to give

$$f_1(z_1) + z_1 \bar{f}_1'(\bar{z}_1) + \bar{g}_1'(\bar{z}_1) = -i\frac{z_{1_s}}{2} + B(t), \tag{16}$$

where $B(t)$ is a complex constant of integration.

There is a certain amount of arbitrariness in the functions $f_1(z_1), g_1'(z_1)$, which provide a given stress distribution on the blob boundary. Physically, the choices of $f_1(z_1)$ and $g_1'(z_1)$ that leave the pressure and stresses in (9) and (11) invariant correspond to velocity fields that differ from each other by time-dependent, uniform translations or rotations. However, such changes of velocity field clearly cannot affect the shape of the blob boundary in any fundamental way—they simply translate or rotate it. In light of this, a judicious transformation of $f_1(z_1)$ and $g_1'(z_1)$ will be made in order to simplify the resulting evolution equations as much as possible, as seen below.

Consider the following (time-dependent) change of origin in physical space and rotation of the physical plane expressed via

$$z_1 = z_0(t) + e^{i\phi(t)} z, \tag{17}$$

where $z_0(t)$ is a complex function of time, and $\phi(t)$ is a real function of time. Given this transformation of z_1, the transformations of $f_1(z_1), g_1'(z_1)$ that leave the stress and pressure expressions in (9) and (11) invariant can be written

$$f_1(z_1) = e^{i\phi}[f(z) + iC(t)z] + \gamma(t), \tag{18}$$

$$g_1'(z_1) = e^{-i\phi} g'(z) - \bar{z}_0 f_1'(z_1) - \bar{\gamma} + \bar{B}, \tag{19}$$

where $C(t)$ is real, but $\gamma(t)$ is generally complex. The boundary condition (16) then becomes

$$f(z) + z\bar{f}'(\bar{z}) + \bar{g}'(\bar{z}) = -i\frac{z_s}{2}. \tag{20}$$

Under this same transformation the velocity field becomes

$$\begin{aligned} u_1 + i v_1 &= e^{i\phi}\left[-f(z) + z\bar{f}'(\bar{z}) + \bar{g}'(\bar{z}) - 2iCz\right] - 2\gamma + B \\ &= e^{i\phi}\left[u + iv - 2iCz\right] - 2\gamma + B, \end{aligned} \tag{21}$$

where $u + iv$ denotes the velocity field in the z-reference frame. It is clear from (21) that the arbitrariness expressed by the transformation above corresponds to a velocity field that is determined only up to a rigid body motion, i.e., an arbitrary translation and rotation. The suitability of the transformations (17)–(19), and the choices of the remaining degrees of freedom in the choice of $\phi(t)$, $z_0(t)$, and $\gamma(t)$ will become clear once the kinematic condition (7) is recast in terms of a conformal mapping representation as in the following section.

3. Conformal Mapping Representation

Consider the conformal map $z_1(\zeta, t)$ from the interior of the unit circle in the ζ plane into the simply connected region occupied by the fluid so that $\zeta = 0$ is mapped to a point $z_0(t)$ inside the fluid blob. The existence of such a map is guaranteed by Riemann's Theorem. We choose $z_0(0)$ to be any convenient point inside the blob initially. The choice of $\dot{z}_0(t)$ will be made to simplify the problem appropriately, as will be seen shortly. It is clear that, for sufficiently small time t, $z_0(t)$ will remain inside the blob when $\dot{z}_0(t)$ is finite. A priori, that is all that is needed to derive the dynamical equations and the exact solutions—examination of the exact solutions themselves will then determine the time of validity of a particular solution. The remaining rotational degree of freedom of the Riemann mapping theorem will be used later by fixing a rotational freedom in the ζ plane in a convenient way.

The kinematic boundary condition on the blob can be written as the following boundary condition on the unit circle, $\zeta = e^{i\nu}$:

$$Im\left[\frac{(z_{1_t} - (u_1 + iv_1))}{z_{1_\nu}}\right] = 0. \tag{22}$$

If we now use the substitution (17), where z is now viewed as a function of t and ζ (or ν on the circular boundary), then it is clear that (22) is equivalent to

$$Im\left[\frac{(z_t + i(\dot{\phi} + 2C)z - e^{-i\phi}(u + iv + 2\gamma - B) + e^{-i\phi}(\dot{z}_0 + 2\gamma - B))}{z_\nu}\right] = 0. \tag{23}$$

We now choose

$$\dot{\phi}(t) = -2C(t), \tag{24}$$

$$\dot{z}_0(t) = B(t) - 2\gamma(t), \tag{25}$$

so that on using (21), (23) simplifies to

$$Im\left[\frac{z_t - \{-f(z) + z\bar{f}'(\bar{z}) + \bar{g}'(\bar{z})\}}{z_\nu}\right] = 0. \tag{26}$$

Note that since the function $z(\zeta, t)$ is simply a translation and rotation of $z_1(\zeta, t)$ then $z(\zeta, t)$ is also a conformal map. Since $\zeta = 0$ corresponds to $z_1 = z_0$, it follows from (17) that

$$z(0, t) = 0. \tag{27}$$

We further make the arbitrary but convenient specification that

$$\gamma(t) = f_1(z_0(t)); \tag{28}$$

then it is clear from (18) that

$$f(0) = 0. \tag{29}$$

Note also that the specific choice of real function $C(t)$ is unimportant in the simplification to (20) and (26), with auxiliary conditions (27) and (29), provided that $\phi(t)$ evolves according to $\dot{\phi} = -2C$. It is found that the above conditions are enough to uniquely determine the velocity field, with the evolution equations given by (20) and (26).

Previous authors [1] [9] have suggested various physical arguments that might be used to specify uniquely the velocity field rather than the purely mathematical condition used above (namely, the choice $\gamma(t) = f_1(z_0(t))$). The most plausible suggestion is the requirement of conservation of global momentum. Although we are considering the zero Reynolds number asymptotic limit of the Navier-Stokes equation where, *locally*, inertial effects (momentum transfer) have been neglected in comparison with the viscous stresses, it is argued [9] that this does not obviate the need to respect *global* conservation of momentum. Assuming that global momentum conservation is the appropriate physical principle to invoke, unless the solutions are suitably symmetric, in general the mathematical condition leading to (29) above does *not* provide conservation of global momentum. However, in the case when there are no flow singularities in the blob, this is of no consequence as there are then no special points in the fluid and an appropriate rigid body motion can be added a posteriori to the solution (so that global momentum is conserved) without affecting any other aspect of the flow. Thus, in that case, there is really no need to appeal to any physical principle to specify uniquely the velocity field, and the convenient mathematical condition above serves perfectly well. The case where there does exist a distribution of singularities in the flow is discussed in later sections.

Using (20) and the fact that on $|\zeta| = 1$,

$$z_s = \frac{i\zeta z_\zeta}{|z_\zeta|}, \tag{30}$$

the kinematic boundary condition (26) becomes the following condition on $|\zeta| = 1$:

$$Re\left[\frac{z_t + 2F(\zeta, t)}{\zeta z_\zeta(\zeta, t)}\right] = \frac{1}{2|z_\zeta|}, \tag{31}$$

where we define

$$F(\zeta, t) \equiv f(z(\zeta, t), t). \tag{32}$$

We also define

$$G(\zeta, t) \equiv g'(z(\zeta, t)). \tag{33}$$

Formally, in the following analysis, we assume that $F(\zeta, t)$ is analytic in $|\zeta| \le 1$, but we allow $G(\zeta, t)$ possibly to have a pole of order r_0 at $\zeta = 0$ and poles of order r_j at $\zeta = \bar{\zeta}_j^{-1}$ inside the unit circle, with

$$0 \le r_0 \le M - M_0, \qquad 0 \le r_j \le \gamma_j, \qquad j = 1 \ldots N, \tag{34}$$

where ζ_j, $j = 1 \ldots N$, are the poles of order γ_j of the conformal map $z(\zeta, t)$ outside the unit circle (see (39)), and we define

$$M_0 = \sum_{j=1}^{N} \gamma_j. \tag{35}$$

Physically, these singularities represent general multipoles (e.g., a source/sink, dipole) at z_0 and at z-locations corresponding to $\zeta = \bar{\zeta}_j^{-1}$.

We now convert the boundary condition (31) into a differential equation for z valid everywhere in $|\zeta| \le 1$. Because of the restriction (29) (which implies $F(0, t) = 0$) and (27), it is easily seen that $\zeta = 0$ is a removable singularity of the expression within the square parentheses on the left-hand side of (31). The left-hand side of (31) is clearly the real part of an analytic function in $|\zeta| \le 1$. Using the Poisson integral formula for $|\zeta| < 1$,

$$z_t + 2F = \zeta I(\zeta, t) z_\zeta, \tag{36}$$

where

$$I(\zeta, t) = \frac{1}{2\pi i} \oint_{|\zeta'|=1} \frac{d\zeta'}{\zeta'} \left[\frac{\zeta' + \zeta}{\zeta' - \zeta} \right] \frac{1}{2|z_\zeta|} + i D(t) \tag{37}$$

and $D(t)$ is a real function of time. The remaining rotational degree of freedom of the Riemann mapping theorem is used by insisting $D(t) = 0$. That such a freedom exists can be observed readily by replacing ζ by $\zeta e^{i\theta(t)}$ in (36), with $\dot{\theta} = D(t)$. Thus, without any loss of generality,

$$I(\zeta, t) = \frac{1}{2\pi i} \oint_{|\zeta'|=1} \frac{d\zeta'}{\zeta'} \left[\frac{\zeta' + \zeta}{\zeta' - \zeta} \right] . \frac{1}{2|z_\zeta|}. \tag{38}$$

Since $z(\zeta, t)$ must be analytic in $|\zeta| \le 1$, it is possible to express it in the form

$$z(\zeta, t) = \frac{h(\zeta, t)}{\prod_{j=1}^{N}, (\zeta - \zeta_j(t))^{\gamma_j}}, \tag{39}$$

where N, γ_j are arbitrary positive integers and the corresponding poles ζ_j are all outside the unit circle (i.e., $|\zeta_j| > 1$), while $h(\zeta, t)$ is analytic for $|\zeta| \le 1$.

The principal result of this paper is to demonstrate the important fact that, if $h(\zeta, 0)$ is an arbitrary polynomial of sufficiently high order, then *it remains a polynomial of the same order for all times that the solution exists* (provided the poles $\zeta_j(t)$ evolve in an appropriate manner). Thus, it will be shown that the conformal mappings of certain initial blobs as they evolve under Stokes flow with surface tension **remain** describable in terms of a **finite set** of time-evolving parameters for as long as the solution exists. It is in this sense that we refer to such solutions as *exact*. Moreover, the conservation law approach used here is seen to simplify the computation of the evolution of the coefficients of $h(\zeta, t)$ (in comparison to other known methods), while simultaneously producing a set of conserved quantities—the existence of which was not explicit in previous methods.

4. Conservation Laws and Exact Solutions

To demonstrate the existence of exact solutions and the conserved quantities associated with them, the problem is now reformulated in terms of a set of very general line integral quantities given by

$$J_K(t) = \oint_C K(\zeta, t)\bar{z}(\bar{\zeta}, t) z_\zeta(\zeta, t)\, d\zeta, \tag{40}$$

where $K(\zeta, t)$ is a general function of ζ and t which will be taken to be analytic on and within the unit circle and C denotes the boundary of the unit circle $|\zeta| = 1$ traversed anticlockwise. Later, special choices of the function $K(\zeta, t)$ will be made in order to establish various results. Notice that these line integral quantities can be defined at any instant of time given **only** the conformal map representing the fluid region at that time. In this sense, such quantities are *purely geometrical* and do not depend on the flow within the blob. The time evolution of such quantities **does**, however, depend on the flow within the blob. First we state and prove a theorem about how $J_K(t)$ evolves in time, assuming that the blob evolves under the equations of Stokes flow with surface tension:

Theorem 4.1. *For $J_K(t)$ defined as in (40), where $z(\zeta, t)$ is the conformal mapping function as defined earlier,*

$$\begin{aligned} \dot{J}_K(t) \;=\; & \oint_C K(\zeta, t) 2G(\zeta, t) z_\zeta(\zeta, t)\, d\zeta \\ & + \oint_C \left[K_t(\zeta, t) - \zeta I(\zeta, t) K_\zeta(\zeta, t)\right] \bar{z}(\bar{\zeta}, t) z_\zeta(\zeta, t)\, d\zeta. \end{aligned} \tag{41}$$

Proof. Differentiating $J_K(t)$ with respect to time gives

$$\frac{d}{dt}\oint_C K(\zeta, t)\bar{z}(\bar{\zeta}, t) z_\zeta(\zeta, t)\, d\zeta = \oint_C K(\zeta, t)\left[\bar{z}_t z_\zeta + \bar{z} z_{\zeta t}\right] + K_t(\zeta, t)\bar{z} z_\zeta\, d\zeta. \tag{42}$$

Using (36) (and its complex conjugate) to substitute for z_t, $\bar{z}_t$ gives

$$\begin{aligned} \dot{J}_K(t) \;=\; & \oint_C K(\zeta, t)\bigg[-2\bar{F}(\bar{\zeta}, t) z_\zeta + \frac{1}{\zeta}\bar{I}(\bar{\zeta}, t)\bar{z}_\zeta(\bar{\zeta}, t) z_\zeta(\zeta, t) \\ & + \bar{z}[-2F + \zeta I(\zeta, t) z_\zeta]_\zeta\bigg] + K_t \bar{z} z_\zeta\, d\zeta. \end{aligned} \tag{43}$$

Rearranging terms and integrating one of the terms by parts, this becomes

$$\begin{aligned} \dot{J}_K(t) \;=\; & \oint_C K(\zeta, t)\left[-2\bar{F} z_\zeta - 2\bar{z} F_\zeta + \frac{1}{\zeta}[I(\zeta, t) + \bar{I}(\bar{\zeta}, t)] z_\zeta \bar{z}_\zeta\right] d\zeta \\ & + \oint_C \left[K_t - \zeta I(\zeta, t) K_\zeta\right] \bar{z} z_\zeta\, d\zeta. \end{aligned} \tag{44}$$

Using the stress condition (20), which using (30) can be written as

$$\bar{F}(\bar{\zeta}, t) z_\zeta + \bar{z} F_\zeta(\zeta, t) + G(\zeta, t) z_\zeta = \frac{1}{2\zeta} z_\zeta^{1/2} \bar{z}_\zeta^{1/2}, \tag{45}$$

and the fact that on C

$$I(\zeta,t)+\bar{I}(\bar{\zeta},t)=\frac{1}{z_\zeta^{1/2}\bar{z}_\zeta^{1/2}}, \tag{46}$$

we then obtain the required result. □

In order to demonstrate the existence of exact solutions of the form (39), with $h(\zeta,t)$ a polynomial, we will make special choices of the function $K(\zeta,t)$.

Definition. Define integrals $J_{k_0}^0(t)$ for each $k_0=0,1,2,\ldots$ as

$$J_{k_0}^0(t)=\oint_C K_0(\zeta,t;k_0)\bar{z}(\bar{\zeta},t)z_\zeta(\zeta,t)\,d\zeta, \tag{47}$$

where

$$K_0(\zeta,t;k_0)=\zeta^{k_0}\prod_{p=1}^{N}(\zeta-\bar{\zeta}_p^{-1})^{\gamma_p}. \tag{48}$$

We now state a theorem that connects the properties of the function $h(\zeta,t)$ to the properties of $J_{k_0}^0(t)$. It is noted that the following theorem has nothing to do with any dynamics of the physical problem at hand.

Theorem 4.2. *Assume M is an integer such that $M \ge M_0$. Then,*

$$J_{k_0}^0(t)=0 \qquad \textit{for all} \quad k_0 \ge M-M_0, \tag{49}$$

if and only if $h(\zeta,t)$ is a polynomial of degree at most M.

Proof. First, assume that $J_{k_0}^0(t)=0$ for $k_0 \ge M-M_0$. Then, from (39) and the definition of $J_{k_0}^0$ in (47), it follows that

$$\oint_C \zeta^j H(\zeta,t)\,d\zeta=0 \qquad \text{for} \quad j=k_0-M+M_0 \ge 0, \tag{50}$$

where

$$H(\zeta,t)=\zeta^M\bar{h}(1/\zeta,t)\,z_\zeta. \tag{51}$$

Since $H(\zeta,t)$ is known to be analytic on $|\zeta|=1$, it must have a Laurent series convergent for $|\zeta|=1$ (and locally in an enclosing annulus). Writing this as

$$H(\zeta,t)=\sum_{j=-\infty}^{\infty}H_n(t)\,\zeta^n, \tag{52}$$

it is clear that (50) implies that $H_{-j-1}=0$ for $j\ge 0$, i.e., all negative coefficients of the Laurent expansion for $H(\zeta,t)$ are zero. Thus, $H(\zeta,t)$ is analytic in $|\zeta|\le 1$. Since it is known that z_ζ is analytic and nonzero there, then it follows that $\zeta^M\bar{h}(\frac{1}{\zeta},t)$ is also analytic in $|\zeta|\le 1$. We conclude that $h(\zeta,t)$ must be polynomial of degree at most M.

Conversely, assume that $h(\zeta, t)$ is a polynomial of degree M or less. It follows that $\zeta^M \bar{h}(1/\zeta, t) z_\zeta$ is analytic for $|\zeta| \leq 1$. By Cauchy's theorem, we deduce $J^0_{k_0}(t) = 0$ for $k_0 \geq M - M_0$, and the proof of Theorem 4.2 is complete. □

Using Theorem 4.1, the following theorem concerning $\dot{J}^0_{k_0}$ is useful:

Theorem 4.3. *Define $\{d_j \mid j \geq 0\}$ as the Taylor series coefficients of the following analytic function in $|\zeta| \leq 1$:*

$$-k_0 I(\zeta, t) + \sum_{p=1}^{N} \gamma_p \frac{\zeta I(\zeta, t) - \bar{\zeta}_p^{-1} I(\bar{\zeta}_p^{-1}, t)}{\zeta - \bar{\zeta}_p^{-1}} = \sum_{j=0}^{\infty} d_j \zeta^j. \tag{53}$$

Also, assume that each $\zeta_j(t)$ evolves according to

$$\frac{d}{dt}\zeta_j^{-1} = -\zeta_j^{-1}(t)\ I(\zeta_j^{-1}(t), t). \tag{54}$$

Then, for each integer $k_0 \geq 0$,

$$\dot{J}^0_{k_0} = \sum_{j=0}^{\infty} d_j\ J^0_{(k_0+j)} + \oint_{|\zeta|=1} d\zeta\ \ K_0\, 2\, G\, z_\zeta. \tag{55}$$

Further, if $k_0 \geq r_0$,

$$\dot{J}^0_{k_0} = \sum_{j=0}^{\infty} d_j\ J^0_{(k_0+j)}. \tag{56}$$

Proof. On substituting $K_0(\zeta, t; k_0)$ for K in Theorem 4.1, it follows that

$$\begin{aligned} \dot{J}^0_{k_0} = &\oint_{|\zeta|=1} d\zeta\ K_0 \bar{z} z_\zeta \sum_{p=1}^{N} \frac{\gamma_p}{(\zeta - \bar{\zeta}_p^{-1})} \left[-\frac{d}{dt}\bar{\zeta}_p^{-1} - \bar{\zeta}_p^{-1} I(\bar{\zeta}_p^{-1}, t) \right] \\ &+ \oint_{|\zeta|=1} d\zeta\ K_0 \bar{z} z_\zeta \sum_{p=1}^{N} \frac{\gamma_p}{\zeta - \bar{\zeta}_p^{-1}} \left[-\zeta I(\zeta, t) + \bar{\zeta}_p^{-1} I(\bar{\zeta}_p^{-1}, t) \right] \\ &+ \oint_{|\zeta|=1} d\zeta\ K_0\, z_\zeta\ (2\,G - k_0 I \bar{z}). \end{aligned} \tag{57}$$

On taking the complex conjugate of (54) and using the property that the complex conjugate of $I(\zeta_p^{-1}, t)$ is $I(\bar{\zeta}_p^{-1}, t)$ (which follows from (38)), the first integral in (57) vanishes. Using the series representation (53) (which is uniformly convergent for $|\zeta| \leq 1$) in (57), the result (55) immediately follows. It is readily seen that if $k_0 \geq r_0$, then $K_0\, z_\zeta\, G$ must be analytic for $|\zeta| \leq 1$. So, the result (56) follows. □

It is important to point out the *upper-triangular* nature of the evolution equations (56). Note that the evolution of each $J^0_{k_0}(t)$ (for given k_0) depends **only** on the values of $J^0_j(t)$

for $j \geq k_0$. This observation opens up the possibility that if the initial configuration of the blob is such that all the $J_j^0(0)$ vanish for j greater than (or equal to) some sufficiently large integer, then these line integrals can **remain** zero for all times that the solution exists. We now show that this is, in fact, the case.

Remark 1. From the definition of $I(\zeta, t)$ in (38), it is clear that on $|\zeta| = 1$, $Re\ I$ is given by the right-hand side of (31), which is always positive. Since $Re\ I$ is a harmonic function for $|\zeta| \leq 1$, it follows from the maximum principle that $Re\ I(\zeta, t) > 0$ in that domain for as long as the integral (38) exists. From (54), this immediately implies that $Re\ [\dot{\zeta}_j/\zeta_j] > 0$, which shows that all pole singularities of the conformal mapping function (39) move away from $|\zeta| = 1$. Earlier, Tanveer and Vasconcelos [8] presented a more general argument to show that any initial singularity of $z(\zeta, t)$ in $|\zeta| > 1$ moves outward with time.

Remark 2. If surface tension effects are ignored in the analysis, then it is clear that $I(\zeta, t) \equiv 0$ and, in that case, all the d_k coefficients are zero. Therefore from (54)–(56), it follows that the singularities $\zeta_j(t)$ and all but a finite number of the line integral quantities are time invariant even when $h(\zeta, t)$ is not restricted to a polynomial. Such results for zero surface tension when $z(\zeta, t)$ is a general analytic function have been derived systematically by Cummings et al. [7] in a manner similar to Theorem 4.1, although these results follow directly from earlier work of Tanveer and Vasconcelos [8] (X_k in the notation of Section 4 of [8]), who found such invariants in an *ad hoc* manner for the closely related problem of a single bubble in an arbitrary strain field.

Theorem 4.4. *If $J_{k_0}^0(0) = 0$ for $k_0 \geq M - M_0$, then $J_{k_0}^0(t) = 0$ for $t > 0$.*

Proof. This crucial theorem contains the essential dynamics of the problem. Note that since $M - M_0 \geq r_0$ by (34), then (56) gives the appropriate evolution equation for $J_{k_0}^0$ when $k_0 \geq M - M_0$. By inspection of (56), $J_{k_0}^0(t) = 0$ for $k_0 \geq M - M_0$ is clearly a solution of the initial value problem. However, this does not address the question of uniqueness. In order to show uniqueness for $|\zeta| \leq 1$, it is convenient to express

$$I(\zeta, t) = \sum_{n=0}^{\infty} I_n\, \zeta^n, \tag{58}$$

$$\sum_{p=1}^{N} \gamma_p \frac{\zeta I(\zeta, t) - \bar{\zeta}_p^{-1} I(\bar{\zeta}_p^{-1}, t)}{\zeta - \bar{\zeta}_p^{-1}} = \sum_{n=0}^{\infty} T_n\, \zeta^n = T(\zeta). \tag{59}$$

It is clear from (53) that

$$d_n = -k_0 I_n + T_n, \qquad n \geq 0. \tag{60}$$

Note that I_n and T_n are not dependent on k_0, unlike d_n as defined in (53). (56) can be then be rewritten as

$$\dot{J}_{k_0}^0 = -\sum_{j=0}^{\infty} k_0 I_j\, J_{(k_0+j)}^0 + \sum_{j=0}^{\infty} T_j\, J_{(k_0+j)}^0. \tag{61}$$

It is convenient to extend the definition of I_j and T_j for $j < 0$ by setting them to zero. Then, for $k_0 \geq M - M_0$,

$$\dot{J}^0_{k_0} = -\sum_{j=-\infty}^{\infty} k_0 I_j J^0_{(k_0+j)} + \sum_{j=-\infty}^{\infty} T_j J^0_{(k_0+j)}. \tag{62}$$

We define new variables

$$U_k(t) = J^0_k(t) \qquad \text{for} \quad k \geq M - M_0. \tag{63}$$

Then, for $k \geq M - M_0$,

$$\dot{U}_k = -\sum_{j=-\infty}^{\infty} kI_j U_{(k+j)} + \sum_{j=-\infty}^{\infty} T_j U_{(k+j)}. \tag{64}$$

We extend U_k to $k < M - M_0$ by requiring $U_k(0) = 0$ and demanding that it satisfies (64), even for $k < M - M_0$. If we now define

$$U(\zeta, t) = \sum_{k=-\infty}^{\infty} U_k(t)\zeta^k, \tag{65}$$

$$\hat{I}(\zeta, t) = \sum_{n=-\infty}^{\infty} I_n \zeta^{-n} = I(\zeta^{-1}, t), \tag{66}$$

$$\hat{T}(\zeta, t) = \sum_{n=-\infty}^{\infty} T_n \zeta^{-n} = T(\zeta^{-1}, t). \tag{67}$$

By multiplying (64) by ζ^k and summing over k from $-\infty$ to ∞, it is clear that $U(\zeta, t)$ satisfies the following partial differential equation:

$$U_t + \zeta(\hat{I}\, U)_\zeta - \hat{T}\, U = 0. \tag{68}$$

We know that as long as $z_\zeta \neq 0$ in $|\zeta| \leq 1$, $I(\zeta, t)$ defined by (38) is analytic for $|\zeta| \leq 1$. This implies $\hat{I}(\zeta, t) = I(\zeta^{-1}, t)$ is analytic for $|\zeta| \geq 1$. Further, by inspection, it is clear that $\hat{T}(\zeta, t)$ is analytic in this domain as well. The initial conditions on $J^0_{k_0}$ for $k_0 \geq M - M_0$ imply $U_{k_0} = 0$ for all k_0, and hence $U(\zeta, 0) = 0$. From the well-known theory of first-order partial differential equations, whose coefficients are known a priori to be analytic over some domain, it follows from (68) that the unique solution is $U(\zeta, t) = 0$. This implies all $U_k(t)$ (and hence all $J^0_k(t)$) for $k \geq M - M_0$ are zero. Thus, Theorem 4.4 is proved. □

Remark 3. If $J^0_{k_0}(0) = 0$ for $k \geq M - M_0$, as is true when $h(\zeta, t)$ is a polynomial of degree M, then the summation index j in (55) ranges only from 0 to $M - M_0 - k_0 - 1$.

Theorem 4.5. *If $h(\zeta, 0)$ is a polynomial of degree at most M, then so is $h(\zeta, t)$.*

Proof. If $h(\zeta, 0)$ is a polynomial of degree at most M, it follows from Theorem 4.2 that $J^0_{k_0}(0) = 0$ for $k_0 \geq M - M_0$. From Theorem 4.4, it follows that $J^0_{k_0}(t) = 0$ for $t > 0$. Theorem 4.2 implies $h(\zeta, t)$ is a polynomial of degree at most M. The proof is then complete. □

From this point onwards we will only be concerned with initial conditions for which $h(\zeta, 0)$ is a polynomial of order M, where $M \geq M_0$. From Theorem 4.5, if follows that as long as the solution exists, $h(\zeta, t)$ will remain a polynomial of degree M, and this will be assumed henceforth. While the evolution of the poles of the conformal mapping outside the unit circle is known, it remains to determine the evolution of the (finite set of) coefficients of powers of ζ in the polynomial $h(\zeta, t)$. It will now be shown that the evolution of the coefficients of $h(\zeta, t)$ can be deduced by considering further sets of line integral quantities whose evolution equations also have an upper-triangular structure.

Definition. For each integer j between 1 and N, and integer $k_j = 0, 1, 2, \ldots$, we define $J^j_{k_j}(t)$ as

$$J^j_{k_j}(t) = \oint_C K_j(\zeta, t; k_j)\bar{z}(\bar{\zeta}, t) z_\zeta(\zeta, t) d\zeta, \tag{69}$$

where

$$K_j(\zeta, t; k_j) = \zeta^{M-M_0}(\zeta - \bar{\zeta}_j^{-1})^{k_j} \prod_{\substack{p=1 \\ p\neq j}}^{N} (\zeta - \bar{\zeta}_p^{-1})^{\gamma_p}. \tag{70}$$

We now introduce a theorem about the evolution of $J^j_{k_j}(t)$.

Theorem 4.6. *Assume that* $\{\hat{d}^j_n \mid n \geq 0\}$ *are defined as the Taylor series coefficients of the following analytic function around* $\zeta = \bar{\zeta}_j^{-1}$:

$$\begin{aligned} -(M - M_0)\, I(\zeta, t) + k_j \left[\frac{-\frac{d}{dt}\bar{\zeta}_j^{-1} - \zeta I(\zeta, t)}{\zeta - \bar{\zeta}_j^{-1}} \right] \\ + \sum_{\substack{p=1 \\ p\neq j}}^{N} \gamma_p \frac{-\zeta I(\zeta, t) + \bar{\zeta}_p^{-1} I(\bar{\zeta}_p^{-1}, t)}{\zeta - \bar{\zeta}_p^{-1}} \\ = \sum_{n=0}^{\infty} \hat{d}^j_n\, (\zeta - \bar{\zeta}_j^{-1})^n. \end{aligned} \tag{71}$$

Also, assume that each $\zeta_j(t)$ *evolves according to (54). Then, for each integer* $k_j \geq 0$,

$$\dot{J}^j_{k_j} = \sum_{n=0}^{\infty} \hat{d}^j_n\; J^j_{(k_j+n)} + \oint_{|\zeta|=1} K_j(\zeta, t; k_j)\; 2\; G\; z_\zeta\; d\zeta. \tag{72}$$

Further, if $k_j \geq r_j$,

$$\dot{J}^j_{k_j} = \sum_{n=0}^{\infty} \hat{d}^j_n\; J^j_{(k_j+n)}. \tag{73}$$

Proof. We use Theorem 4.1 and the expression for K_j in (70) to conclude that

$$\begin{aligned} \dot{J}^j_{k_j}(t) &= k_j \oint_C K_j(\zeta, t; k_j) \left[\frac{-\frac{d}{dt}(\bar{\zeta}_j^{-1}) - \zeta I(\zeta, t)}{\zeta - \bar{\zeta}_j^{-1}} \right] \bar{z}(\bar{\zeta}, t) z_\zeta(\zeta, t)\, d\zeta \\ &\quad + \sum_{\substack{p=1 \\ p\neq j}}^{N} \gamma_p \oint_C K_j(\zeta, t; k_j) \left[\frac{-\frac{d}{dt}\bar{\zeta}_p^{-1} - \zeta I(\zeta, t)}{\zeta - \bar{\zeta}_p^{-1}} \right] \bar{z}(\bar{\zeta}, t) z_\zeta\, d\zeta \\ &\quad - (M - M_0) \oint_C K_j(\zeta, t; k_j) I(\zeta, t) \bar{z}(\bar{\zeta}, t) z_\zeta\, d\zeta \\ &\quad + \oint_C K_j(\zeta, t; k_j) 2G(\zeta, t) z_\zeta(\zeta, t)\, d\zeta. \end{aligned} \tag{74}$$

Using (54), the integrands in (74) are seen to be analytic for $|\zeta| \le 1$, except possibly at $\zeta = \bar{\zeta}_j^{-1}$. We deform the contour and rewrite (74) as

$$\begin{aligned} \dot{J}^j_{k_j}(t) &= k_j \oint_{|\zeta - \bar{\zeta}_j^{-1}| = \epsilon} K_j(\zeta, t; k_j) \left[\frac{-\frac{d}{dt}(\bar{\zeta}_j^{-1}) - \zeta I(\zeta, t)}{\zeta - \bar{\zeta}_j^{-1}} \right] \bar{z}(\bar{\zeta}, t) z_\zeta(\zeta, t)\, d\zeta \\ &\quad + \sum_{\substack{p=1 \\ p\neq j}}^{N} \gamma_p \oint_{|\zeta - \bar{\zeta}_j^{-1}| = \epsilon} K_j(\zeta, t; k_j) \left[\frac{-\frac{d}{dt}(\bar{\zeta}_p^{-1}) - \zeta I(\zeta, t)}{\zeta - \bar{\zeta}_p^{-1}} \right] \bar{z}(\bar{\zeta}, t) z_\zeta(\zeta, t)\, d\zeta \\ &\quad - (M - M_0) \oint_{|\zeta - \bar{\zeta}_j^{-1}| = \epsilon} K_j(\zeta, t; k_j) I(\zeta, t) \bar{z}(\bar{\zeta}, t) z_\zeta(\zeta, t)\, d\zeta \\ &\quad + \oint_{|\zeta| = 1} K_j(\zeta, t; k_j) 2G(\zeta, t) z_\zeta(\zeta, t)\, d\zeta, \end{aligned} \tag{75}$$

where ϵ is chosen small enough to ensure that the series in (71) is convergent for $|\zeta - \bar{\zeta}_j^{-1}| \le \epsilon$. Using (71), and carrying out term by term integration (valid since the convergence is uniform), the result (72) immediately follows. Further, if $k_j \ge r_j$, it is clear that the integrand $K_j(\zeta, t; k_j) 2G(\zeta, t) z_\zeta(\zeta, t)$ is analytic everywhere in $|\zeta| \le 1$, and hence (73) follows. The proof of the Theorem 4.6 is then complete. ☐

We can now state an important lemma concerning the line integral quantities $J^j_{k_j}(t)$ for $k_j \ge \gamma_j$.

Lemma 4.1. $J^j_{k_j}(t) = 0$ *for* $k_j \ge \gamma_j$.

Proof. On substituting (39) into (69) and using $\bar{\zeta} = 1/\zeta$ on $|\zeta| = 1$, as well as using the definition of K_j in (70), it is observed easily that the integrand in (69) is analytic in $|\zeta| \le 1$ for $k_j \ge \gamma_j$, and therefore the Lemma follows by Cauchy's theorem. ☐

Remark 4. Note that the result in the Lemma is consistent with (73).

Remark 5. Because of the Lemma above, the summation index n in (72) ranges from 0 to $\gamma_j - k_j - 1$.

We now discuss some ramifications of all the theorems above. An immediate observation is that an infinite set of integral invariants associated with solutions for which $h(\zeta, 0)$ is a polynomial of degree M have been identified. Associated with such solutions, only a *finite* set of integral quantities will be (in general) nonzero and time evolving in a nontrivial fashion, namely,

$$\{J^j_{k_j} \mid k_j = 0, 1 \dots \gamma_j - 1\}; \qquad j = 1 \dots N, \tag{76}$$

$$\{J^0_{k_0} \mid k_0 = 0, 1 \dots M - M_0 - 1\}. \tag{77}$$

These are determined by solving the differential equations (55) and (72). (Note simplifications due to Remark 4 and Remark 5 above). Thus there are in general $\sum_{p=1}^N \gamma_p + M - M_0 = M$ nonzero time-evolving line integral quantities. Writing the polynomial $h(\zeta, t)$ as follows,

$$h(\zeta, t) = \sum_{n=0}^{M} h_n(t)\zeta^n, \tag{78}$$

condition (27) then implies that $h_0(t) \equiv 0$, leaving only M as yet undetermined functions $h_1(t) \dots h_M(t)$. We now state a conjecture that is so far supported only by numerical evidence.

Conjecture. For given $\zeta_1(t)$, $\zeta_2(t)$, $\dots \zeta_N(t)$ outside the unit ζ circle, the set of M quantities in (76)–(77), as defined in (47) and (69), implicitly determine $h_1(t)$ through $h_M(t)$.

Remark 6. It is clear from the definition of $J^0_{k_0}(t)$ and $J^j_{k_j}(t)$ in (47), (69) and the relations (39), (78) that these are quadratically dependent on $h_1(t)$ through $h_M(t)$; hence, a globally unique relation between the set of J's and h's is unlikely. However, a Newton iterative procedure gives a unique solution locally when subjected to the constraint that $h_j(0)$ are as specified.

5. A Theorem of Invariants

For a certain subset of the solutions (39), it is possible to deduce immediately a further finite set of invariants which greatly facilitates the calculation of such solutions. We now state and prove a theorem involving solutions in which the mapping function $z(\zeta, t)$ has *simple* poles outside the unit circle.

Theorem 5.1. (Theorem of Invariants) *If the initial conformal map for a viscous blob has the form*

$$z(\zeta, 0) = \frac{h(\zeta, 0)}{\prod_{j=1}^N (\zeta - \zeta_j(0))^{\gamma_j}}, \tag{79}$$

where $h(\zeta, 0)$ is a polynomial of degree $M \geq M_0$, then for any j for which $\gamma_j = 1$ and $r_j = 0$ (so that $G(\zeta, t)$ has no singularity at $\zeta = \bar{\zeta}_j^{-1}$), there exists an invariant of the motion given by

$$B_j = \frac{J_0^j(t)\bar{\zeta}_j^{M-M_0}}{\prod_{\substack{p=1 \\ p \neq j}}^{N} (\bar{\zeta}_j^{-1} - \bar{\zeta}_p^{-1})^{\gamma_p}}. \tag{80}$$

Proof. It is clear from the results of previous sections that the evolution of the blob is given by

$$z(\zeta, t) = \frac{h(\zeta, t)}{\prod_{j=1}^{N}(\zeta - \zeta_j(t))^{\gamma_j}}, \tag{81}$$

where $h(\zeta, t)$ remains a polynomial of degree M, and the poles ζ_j, $j = 1 \ldots N$, evolve according to (54). Suppose there exists an index j such that $\gamma_j = 1$ with $r_j = 0$, i.e., $G(\zeta, t)$ is free of any singularity at $\zeta = \bar{\zeta}_j^{-1}$. Consider $k_j = 0$ in (72); it is clear from (72) that since $r_j = 0$,

$$\dot{J}_0^j = -\hat{d}_0^j J_0^j. \tag{82}$$

Using (54) and (71), it follows that

$$\hat{d}_0^j = \sum_{\substack{p=1 \\ p \neq j}}^{N} \gamma_p \left[\frac{-\frac{d}{dt}(\bar{\zeta}_p^{-1}) + \frac{d}{dt}(\bar{\zeta}_j^{-1})}{\bar{\zeta}_j^{-1} - \bar{\zeta}_p^{-1}} \right] - (M - M_0) I(\bar{\zeta}_j^{-1}, t). \tag{83}$$

From (82) and (83),

$$\frac{d}{dt}\log(J_0^j(t)) = \sum_{\substack{p=1 \\ p \neq j}}^{N} \gamma_p \frac{d}{dt}\log(\bar{\zeta}_j^{-1} - \bar{\zeta}_p^{-1}) - (M - M_0)\frac{d}{dt}\log(\bar{\zeta}_j). \tag{84}$$

Integrating with respect to time yields.

$$B_j = \frac{J_0^j(t)\bar{\zeta}_j^{M-M_0}}{\prod_{\substack{p=1 \\ p \neq j}}^{N} (\bar{\zeta}_j^{-1} - \bar{\zeta}_p^{-1})^{\gamma_p}}, \tag{85}$$

where the complex constants B_j are determined from initial conditions. Hence the theorem is proved. □

6. Case Study

Since the aim of this paper is to present a reformulation of the theory of exact solutions for the problem of Stokes flow of a simply connected viscous blob, and since previous studies in the literature have already computed specific examples illustrating the behaviour of viscous fluid blobs, we do not intend to compute further examples here. We do, however, include details of a case study with a particularly appealing mathematical structure that becomes clear as a result of the preceding analysis. We consider the special class of solutions having n *simple* poles (and no other poles) outside the unit circle, i.e., $\{\gamma_j = 1 \mid j = 1 \ldots n\}$ giving

$$z(\zeta, t) = \frac{h(\zeta, t)}{\prod_{j=1}^{n}(\zeta - \zeta_j)}, \tag{86}$$

where $h(\zeta, 0)$ is taken as a polynomial of degree n. [Note that we could equally well find a solution with $h(\zeta, t)$ as any polynomial of degree at least n by taking a suitable initial condition.] We assume the flow is driven purely by surface tension so that there are no flow singularities in the blob and $r_j = 0$ for all $j = 1 \ldots n$. The results of this paper allow the evolution equations for the parameters in this map to be written down in a particularly concise and mathematically appealing way. From Theorem 4.3, we deduce that provided the poles $\{\zeta_j \mid j = 1 \ldots n\}$ evolve according to the equations,

$$\frac{d}{dt}\zeta_j{}^{-1} = -\zeta_j{}^{-1} I(\zeta_j{}^{-1}, t), \qquad j = 1 \ldots n, \tag{87}$$

then a solution of the form (86) can be found. It only remains to determine the n coefficients of $h(\zeta, t)$, i.e., $\{h_k(t) \mid k = 1 \ldots n\}$ (since we know $h_0(t) \equiv 0$). However, Theorem 5.1 tells us that there are n invariants (or first integrals) of the motion associated with this solution given by

$$B_j = \frac{J_0^j(t)}{\prod_{\substack{p=1 \\ p \neq j}}^{n}(\bar{\zeta}_j^{-1} - \bar{\zeta}_p^{-1})}, \qquad j = 1 \ldots n. \tag{88}$$

The invariants $\{B_j \mid j = 1 \ldots n\}$ are determined by initial conditions. These n equations then provide n nonlinear *algebraic* equations for the coefficients $\{h_k(t) \mid k = 1 \ldots n\}$ once the pole positions are known. Thus, the $2n$ equations, (87) and (88), provide a complete and concise set of equations for this problem.

Finally, we remark that the special case of this example where $n = 2$ includes the problem of the coalescence of two viscous cylinders of unequal radius analysed by Richardson [1], using a direct approach of combining the kinematic boundary condition and the stress condition and adjusting the time evolution of the parameters in the map $z(\zeta, t)$ to give the required analyticity properties of $G(\zeta, t)$ in the unit circle. Such a solution is obtained by making appropriate choices of initial conditions. After extensive algebraic manipulation, Richardson [1] also deduces the existence of two invariant quantities that can be shown to be equivalent to (88) in the case $n = 2$. He also deduces two evolution equations for the poles of the mapping, which can be shown to be equivalent to the more concise equations (87). The above case study represents a generalization of these results to general n. After this work was completed, the present authors became aware that recently Richardson [6] has also identified the generalization of Richardson [1] presented in the case study above by studying maps of the form

$$z(\zeta, t) = \sum_{j=1}^{n} \frac{\beta_j \zeta}{1 - \gamma_j \zeta}. \tag{89}$$

However, the method he used is different from that presented here and is, in essence, a simplified version of the method used in Richardson [1]. Richardson [6] goes on to study numerically a class of solutions with initial conditions corresponding to *n* touching circular cylinders.

7. Discussion

With no flow singularities present in the blob, the results of this paper essentially provide exact solutions, describable in terms of a finite set of parameters, for the physical problem of the time evolution of certain initial boundary shapes for viscous blobs driven by surface tension. The solutions are given by conformal maps of the form (39) with $h(\zeta, t)$ a polynomial (of sufficiently high order).

Mathematically, the analysis also allows for a distribution of multipole singularities to exist within the blob, and again exact solutions for the evolution can also be found in this case. It has been found that while it is possible to specify externally the nature and strength of such singularities (i.e., specify the strength of the residue contributions from the last integral in (55) and (72)), it is not in general possible to specify externally the singularity positions after the initial time (the singularities necessarily evolve according to (54)). Thus, except in very special cases (for example, a single singularity at the origin or at infinity [4] [7] [8] [10] [11]), these mathematical solutions are physically untenable in that they are solutions to a problem where the singularities must move in very special ways determined implicitly by the solution itself. While this is something of a drawback in the use of these solutions to solve particular initial value problems with a given distribution of known singularities at specified points in the flow, such solutions may be adequate *qualitative* models of this physical scenario.

In summary, a new theoretical approach to the problem of the slow quasi-steady viscous flow of a two-dimensional simply connected blob of fluid with surface tension has been presented, which improves upon and unifies those used by previous authors. The central result is that it is possible to find an infinite set of conserved quantities associated with a very general class of rational conformal maps describing the boundary evolution of the blob and a finite set of line integral quantities which implicitly determines the evolution of such maps.

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