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A Theory of Exact Solutions for Annular Viscous Blobs

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Summary. A new theory of exact solutions is presented for the problem of the slow viscous Stokes flow of a plane, doubly connected annular viscous blob driven by surface tension. The formulation reveals the existence of an infinite number of conserved quantities associated with the flow for a certain general class of initial conditions. These conserved quantities are associated with a class of exact solutions. This work is believed to provide the first exact solutions for the evolution of a doubly connected fluid region evolving under Stokes flow with surface tension.

Key words. Stokes flow, viscous drop, conservation laws, exact solutions

MSC numbers. 30XX, 35XX, 76XX

PAC numbers. 47.15.Gf; 47.55.Dz; 68.10-m

1. Introduction

This paper presents a theory of exact solutions for the quasi-steady evolution of a plane annular viscous blob of fluid driven by surface tension. Although many exact solutions have been identified for the Stokes flow of a simply connected fluid region both with and without surface tension (e.g., [7]–[17], [20]–[22]), this paper represents the first successful attempt to extend the solution techniques to a doubly connected topology. The method relies upon a complexification of the problem originally exploited by Richardson [17] in the context of a single bubble in a strain field.

The last few years have seen a revival of interest in this problem. All of the methods used by previous authors up to now are essentially the same and rely on the use of

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conformal mapping techniques. Most of these methods ([11]–[17], [20]–[22]) consist of conjecturing a form (ansatz) for the conformal map (from a standard region in a parametric plane to the region in physical space representing the fluid) in terms of a finite set of time-evolving parameters and showing, essentially by inspection, that the time evolution of these parameters can be adjusted such that the appropriate analyticity properties of the solution hold inside the fluid region. Recently, the present authors devised an alternative approach using a reformulation of the problem in terms of a general set of purely geometrical line integrals [7] [8]. This theory, which includes surface tension, was presented in the context of the evolution of a single *simply connected* fluid blob with allowance made for a finite set of multipole singularities in the fluid. With minor modifications, the same theory also applies to the case of an infinite expanse of fluid with a single bubble or a semi-infinite expanse of fluid with an infinite free surface. The reformulation revealed some important mathematical properties of the equations for Stokes flow with surface tension—principally, that the system of evolution equations for the specially defined set of line integral quantities had an *upper-triangular* structure, which helped to explain both the existence of exact solutions and the existence of an infinite set of conserved quantities associated with a very general class of initial conditions. The reformulation also led to the identification of another interesting result—referred to in [7] and [8] as a "theorem of invariants." This theorem provides the existence of a further finite set of invariants of the motion (or "first integrals") associated with a certain subset of the exact solutions. Earlier, Cummings et al. [11] used a line integral approach with **zero** surface tension to derive a special class of exact solutions (of polynomial form) for the evolution of a simply connected blob. An infinite set of conserved quantities was identified, although this specific result itself is implicit in a general study [9] of the closely related mathematical problem of a single bubble evolving in an infinite straining flow.

In the present paper, the new theoretical approach developed in [7] and [8] is extended in a natural way to deal with the problem of an annular blob, which constitutes a doubly connected fluid region requiring suitable adjustments of the solution method. The present method combines the theoretical reformulation presented in [7] and [8] with elements of loxodromic function theory to produce a wide class of exact solutions.

Although, in the present paper, we concentrate on presenting the mathematical theory, the results of this paper represent a significant step forward and are likely to have great utility in problems of real physical interest. The original motivation for the recent revival of interest in the Stokes flow of a two-dimensional fluid region was *sintering*, a term loosely referring to the consolidation of an assemblage of particles in which surface tension provides the principal mechanism for mass transport. Sintering is a complex topic with a huge literature, and the study of sintering is difficult owing to geometrical complexities. It is therefore natural to isolate parts of the overall problem for individual study, and one of the basic paradigms in the study of sintering is the coalescence of two viscous cylinders (particles). Exact solutions for the coalescence (under Stokes approximation) of two viscous cylinders (particles) driven purely by surface tension have been identified recently by Richardson [12], with further developments to the case of the coalescence of multiple touching cylinders made even more recently [17]. Earlier, Hopper [13] studied the evolution of two touching cylinders of equal size. These studies cover only the case of a *simply connected* fluid domain; Richardson's study [17] of the coalescence of multiple cylinders deals with an arbitrary number of cylinders in a linear concatenation (so that no cylinder touches more than two other cylinders, the overall fluid region being simply connected). The relevance of such solutions (even as a model) in a study of the more useful scenario of, say, a general collection (say, a pile) of cylinders/particles, where the fluid region has a connectivity greater than one, is not clear.

This paper presents a theory of exact solutions for the case of a doubly connected fluid region for a certain general class of initial conditions, and facilitates the study of the physical scenario where two coalescing cylinders have, say, a small air bubble between them. This would seem to be a natural paradigm for the study of the coalescence of a *general* assemblage of cylinders/particles (rather than a linear concatenation) where, of course, there would inevitably be small air bubbles between the cylinders if they were arbitrarily piled together. As a simple example calculation to verify the validity of the theory presented in this paper, the simple paradigm of the coalescence of two (unequal) touching cylinders is extended in a natural way to include the case of the evolution of two (unequal) touching cylindrical blobs but now with a small air bubble between them. More involved calculations using the general theory developed here will be presented in future work.

Finally, we note that certain similarities between the free boundary problems in simply connected domains for Hele-Shaw flow (with zero surface tension) and Stokes flow with nonzero surface tension have been discussed by Howison and Richardson [20], although there are essential differences as well [10]. The problem of Hele-Shaw flow (again with zero surface tension) in a doubly connected region has been studied both by Tanveer [19] in the context of a steady nonsymmetric bubble and more recently by Richardson [18] for unsteady flows. In both cases, the theory of elliptic/loxodromic functions was also found to play an important role.

2. Mathematical Formulation

Consider the slow viscous flow of an arbitrary annular blob of fluid. The equations of motion in the fluid is given by

$$-\nabla p + \nabla^2 \mathbf{u} = 0, \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0. \tag{2}$$

Length and time scales have been nondimensionalized with respect to a and $\frac{a\mu}{\sigma}$, respectively, where a is an effective radius (with πa^2 a measure of the initial area of the blob), σ is the surface tension parameter, and μ is the viscosity. Velocities have been rescaled by $\frac{\sigma}{\mu}$ and pressures by $\frac{\sigma}{a}$. The blob has two boundaries, the boundary conditions on each consisting of a stress condition which, for the two boundaries, can be written as

$$-pn_j + 2e_{jk}n_k = -\kappa n_j, \tag{3}$$

where n_1 and n_2 are the x and y components of the unit normal vector pointing outwards from the bubble boundary and κ is the curvature. e_{jk} are the components of the nondimensionalized stress tensor given by

$$e_{jk} = \frac{1}{2}\left[\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\right]. \tag{4}$$

In addition, there is a kinematic boundary condition that the normal velocity of a point on the blob boundary is the same as the normal component of the fluid velocity at that point.

Introducing a streamfunction $\psi(x, y)$ such that

$$\mathbf{u} = (\psi_y, -\psi_x), \tag{5}$$

then it is well-known that two-dimensional Stokes flow in the plane can be reformulated in terms of this streamfunction, which satisfies the **biharmonic** equation in the fluid region, i.e.,

$$\nabla^4\psi = 0. \tag{6}$$

From the Goursat representation of a general biharmonic function, it is known that we can introduce two functions $f(z)$ and $g(z)$, analytic in the fluid region, such that

$$\psi = Im\left[\bar{z}f(z) + g(z)\right], \tag{7}$$

where $z = x + iy$. Some elementary manipulation reveals that it is now possible to write all quantities of physical interest in terms of these two functions. In particular,

$$p - i\omega = 4f'(z), \tag{8}$$

$$u + iv = -f(z) + z\bar{f}'(\bar{z}) + \bar{g}'(\bar{z}), \tag{9}$$

$$e_{11} + ie_{12} = z\bar{f}''(\bar{z}) + \bar{g}''(\bar{z}), \tag{10}$$

where the conjugate function $\bar{f}(z)$ is defined by the expression $\bar{f}(z) = \overline{f(\bar{z})}$ and $u,\ v$ denote the fluid velocities in the x and y directions, respectively. The vorticity ω is given by the relation

$$\nabla^2\psi = -\omega. \tag{11}$$

To obtain a more convenient expression for the stress conditions on the blob boundaries, we now define the complex normal to a boundary to be

$$N = n_1 + in_2 = -i(x_s + iy_s) = -iz_s = -i\exp(i\theta), \tag{12}$$

where s is the arclength assumed to increase in the *anticlockwise* direction on the outer boundary of the blob, and in the *clockwise* direction on the inner boundary. θ is defined as the angle made at each point on the boundary by the tangent at that point to the positive real axis. In terms of this notation, we make the following observations. First, the stress condition on each boundary can be written in complex form as

$$-pN + 2(e_{11} + ie_{12})\bar{N} = -\kappa N. \tag{13}$$

In deriving (13), the fluid flow on either side of the annular fluid region is neglected—an asymptotically valid assumption when the viscosity ratio between fluids is small. Further, the same constant pressure on either side of the annular blob is assumed. There is some loss of generality in this assumption; for instance, it is not true for a steady annular blob with no motion that lies between two concentric circles. However, there is no additional loss of generality in assuming that the constant pressure on either side is zero, as assumed in deriving (13).

Using the expressions for p, N, and $e_{11} + ie_{12}$ in (8)–(10), (13) becomes

$$-2iz_s \frac{\partial S(z,\bar{z})}{\partial z} - 2i\bar{z}_s \frac{\partial S(z,\bar{z})}{\partial \bar{z}} = -i\theta_s e^{i\theta} \tag{14}$$

on each boundary where

$$S(z,\bar{z}) \equiv f(z) + z\bar{f}'(\bar{z}) + \bar{g}'(\bar{z}). \tag{15}$$

These two conditions can be integrated immediately with respect to s to give

$$f(z) + z\bar{f}'(\bar{z}) + \bar{g}'(\bar{z}) = -i\frac{z_s}{2} + A_O(t), \tag{16}$$

$$f(z) + z\bar{f}'(\bar{z}) + \bar{g}'(\bar{z}) = -i\frac{z_s}{2} + A_I(t), \tag{17}$$

on the outer and inner boundaries of the blob, respectively, where $A_O(t)$ and $A_I(t)$ are constants of integration which are, in general, functions of time. It follows, using (9), that on the outer blob boundary,

$$u + iv = -i\frac{z_s}{2} + A_O(t) - 2f(z), \tag{18}$$

while on the inner blob boundary,

$$u + iv = -i\frac{z_s}{2} + A_I(t) - 2f(z). \tag{19}$$

Although the main problem of physical interest in this paper is that where the evolution of the annular blob is driven purely by surface tension (corresponding to the function $g'(z)$ having no singularities in the fluid region), *mathematically* the formulation developed in this paper can be extended to find solutions corresponding to $g'(z)$ having an arbitrary distribution of poles in the fluid region. Physically, it is well-known that an nth order multipole singularity at a point z_{sing} in the fluid corresponds to $g'(z)$ having an nth-order pole at z_{sing}. The present solution method can be extended to include an arbitrary but finite number of such singularities in the flow domain. However, so far, we are only able to get a restricted set of solutions for which it seems to be possible to specify only the strengths of the multipole singularities and their *initial* positions. For the exact solutions obtained here, the subsequent *positions* of the singularities evolve in a way determined by the solution itself and cannot be externally specified. The *physical* relevance of such mathematical solutions appears to be limited because, in general, singularity positions and strengths should be specifiable at each instant of time.

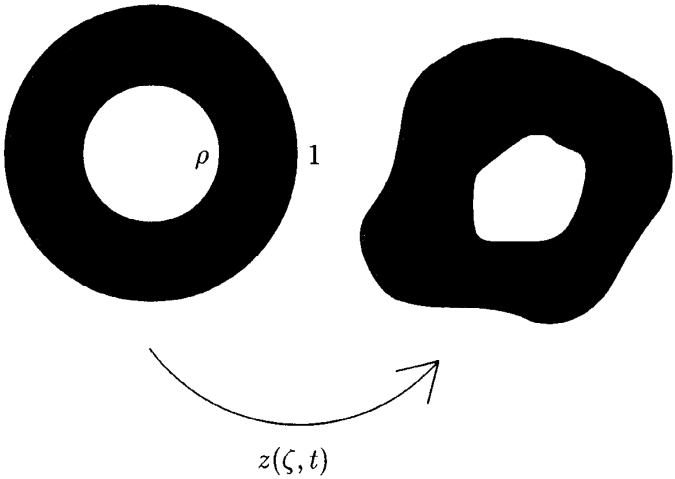

Fig. 1. Conformal mapping domains.

3. Conformal Mapping

The analysis proceeds by defining a conformal map from an annulus C in a complex ζ-plane where C is $\rho < |\zeta| < 1$ to the fluid region in physical space, the circle $|\zeta| = 1$ mapping to the outer boundary of the fluid annulus, the circle $|\zeta| = \rho$ mapping to the inner boundary, see Figure 1. This conformal map will be called $z(\zeta, t)$. It is known, by Riemann's theorem, that any *given* doubly connected fluid domain can be mapped to such an annulus C for *some* ρ. For a general time-evolving domain in physical space, the *conformal modulus* of the region (see [4] for a definition) must therefore be assumed, a priori, to change in time. Thus we suppose that $\rho(t)$ is a function of time to be determined as part of the solution. The remaining degree of freedom of the Riemann Mapping Theorem will be fixed in a convenient way later in the analysis. We will seek solutions for which $z(\zeta, t)$ is analytic in C and, for blobs with smooth boundaries with no corners or cusps, has the property that $|z_\zeta| \neq 0$ everywhere inside C and on the boundary ∂C. Further, in order for the solutions to be relevant physically, $z(\zeta, 0)$ will be restricted to functions that are univalent in C. A posteriori examination of the exact solutions obtained clarifies if and when $z(\zeta, t)$ fails to be univalent beyond a certain time that marks a change in topology. The solutions fail to be relevant physically beyond such a time (if it exists).

The kinematic boundary condition on both blob boundaries can be written

$$Im\left[\frac{\frac{dz}{dt} - (u + iv)}{z_s}\right] = 0. \tag{20}$$

Using the facts that

$$z_s = \frac{i\zeta z_\zeta}{|z_\zeta|} \qquad \text{on} \qquad |\zeta| = 1, \tag{21}$$

$$z_s = -\frac{i\zeta z_\zeta}{\rho|z_\zeta|} \qquad \text{on} \qquad |\zeta| = \rho(t), \tag{22}$$

and then substituting these expressions into the kinematic conditions and using (18) and (19) yields

$$Re\left[\frac{z_t + 2F(\zeta,t)}{\zeta z_\zeta}\right] = \frac{1}{2|z_\zeta|} + Re\left[\frac{A_O}{\zeta z_\zeta}\right]. \tag{23}$$

on the outer boundary, where we define

$$F(\zeta,t) \equiv f(z(\zeta,t),t), \tag{24}$$

and on the inner boundary,

$$Re\left[\frac{z_t + 2F(\zeta,t)}{\zeta z_\zeta}\right] = -\frac{1}{2|z_\zeta|} - \frac{\dot{\rho}}{\rho} + Re\left[\frac{A_I}{\zeta z_\zeta}\right]. \tag{25}$$

We also define the function

$$G(\zeta,t) \equiv g'(z(\zeta,t),t). \tag{26}$$

Our solution method and results are so far restricted to the case where $A_O = A_I$. It is emphasized that this choice involves a definite *loss of generality* in the class of solutions being considered. However, it is clear that *without* any further loss of generality, the value of A_O (and hence A_I) can be taken to be zero. This can be seen by redefining $f(z)$ and $g'(z)$ as follows:

$$f(z) \mapsto f(z) + \frac{A_O}{2}, \tag{27}$$

$$g'(z) \mapsto g'(z) + \frac{\bar{A}_O}{2}, \tag{28}$$

this is a transformation that does not alter the velocity field, but which effectively removes the constants of integration in (16) and (17) once the choice $A_O = A_I$ has been made. There seems to be no straightforward way to interpret the *physical* implications of the special mathematical choice $A_I = A_O$.

It remains to specify the rotational degree of freedom in the problem, but once that is done, the evolution of the annular blob is uniquely determined, as shall be seen later. We remark that since it is only the geometrical evolution of the blob boundaries that is of interest, it is of no importance if the solution is such that the global momentum of the blob is not conserved. Any overall translation or rotation of the blob can be subtracted a posteriori, without altering the validity of the solution for the blob shape.

It is immediately clear that the function in square brackets on the left-hand sides of (23) and (25) is an analytic function inside C. Thus, using the well-known integral formula for a harmonic function in terms of the values of its real part on the boundary of C (also known as Villat's formula—see, for example, [5]), we deduce that for ζ within C,

$$z_t(\zeta,t) + 2F(\zeta,t) = \zeta I(\zeta,t) z_\zeta(\zeta,t), \tag{29}$$

where $I(\zeta,t)$ is given by

$$I(\zeta,t) = I^{+}(\zeta,t) - I^{-}(\zeta,t) + C_1(t) + iC_2(t), \tag{30}$$

where

$$I^{+}(\zeta,t) = \frac{1}{4\pi i}\oint_{|\zeta'|=1}\frac{d\zeta'}{\zeta'}\left(1-2\frac{\zeta}{\zeta'}\frac{P'(\frac{\zeta}{\zeta'})}{P(\frac{\zeta}{\zeta'})}\right)\left[\frac{1}{z_\zeta^{1/2}(\zeta',t)\bar{z}_\zeta^{1/2}(\frac{1}{\zeta'},t)}\right] \tag{31}$$

and

$$I^{-}(\zeta,t) = \frac{1}{4\pi i}\oint_{|\zeta'|=\rho}\frac{d\zeta'}{\zeta'}\left(1-2\frac{\zeta}{\zeta'}\frac{P'(\frac{\zeta}{\zeta'})}{P(\frac{\zeta}{\zeta'})}\right)\left[-\frac{1}{\rho z_\zeta^{1/2}(\zeta',t)\bar{z}_\zeta^{1/2}(\frac{\rho^2}{\zeta'},t)} - \frac{2\dot{\rho}}{\rho}\right], \tag{32}$$

and where the function $P(\zeta)$ is defined via the infinite product expansion,

$$P(\zeta) = (1-\zeta)\prod_{n=1}^{\infty}(1-\rho^{2n}\zeta)(1-\rho^{2n}/\zeta). \tag{33}$$

It is to be noted that $P(\zeta)$ has simple zeroes at $\zeta = \rho^{2m}$ for any integer m, positive or negative or zero. $C_1(t)$ is a real function of time given by

$$C_1(t) = -\frac{1}{4\pi i}\oint_{|\zeta'|=\rho}\frac{d\zeta'}{\zeta'}\left[-\frac{1}{\rho z_\zeta^{1/2}(\zeta',t)\bar{z}_\zeta^{1/2}(\frac{\rho^2}{\zeta'})} - \frac{2\dot{\rho}}{\rho}\right], \tag{34}$$

and $C_2(t)$ is an arbitrary real function of time. The remaining rotational degree of freedom of the Riemann Mapping Theorem is now used up by choosing $C_2(t) = 0$.

Theorem 3.1. *With the choice $A_0 = A_I = 0$ in the boundary conditions (16) and (17), the evolution of the conformal modulus of the corresponding class of solutions is given by the real equation*

$$\dot{\rho} = -\frac{\rho}{4\pi i}\left(\oint_{|\zeta|=1}\frac{d\zeta}{\zeta}\frac{1}{z_\zeta^{1/2}\bar{z}_\zeta^{1/2}} + \oint_{|\zeta|=\rho}\frac{d\zeta}{\zeta}\frac{1}{\rho z_\zeta^{1/2}\bar{z}_\zeta^{1/2}}\right). \tag{35}$$

Proof. This result follows from the fact that the "average" of the real part of $I(\zeta,t)$ around the two bounding circles of the annulus C must be the same. This is necessary in order that $I(\zeta,t)$ is a *single-valued* analytic function everywhere in C. Note that the right-hand side of (23) and (25), with $A_0 = 0 = A_I$, determines *Re* I on the two boundaries. □

Remark 1. For the class of solutions under consideration, the sign of $\dot{\rho}$ is always negative, implying that $\rho(t) \to 0$ as time evolves.

As mentioned earlier, for a flow driven purely by surface tension, $g'(z)$ is regular everywhere in the fluid. $f(z)$ is also taken to be analytic everywhere in the fluid. By conformality of $z(\zeta,t)$ in C, this implies that both $G(\zeta,t)$ and $F(\zeta,t)$ are analytic everywhere in C.

4. Conserved Quantities

The analysis will proceed by reformulating the problem for the time evolution of the boundary into a problem for the time evolution of a number of purely geometrical line integral quantities defined thus:

$$J_K(t) = \oint_{\partial C} K(\zeta, t)\bar{z}(\bar{\zeta}, t)z_\zeta(\zeta, t)\, d\zeta, \tag{36}$$

where ∂C denotes the boundary of the annulus C with $|\zeta| = 1$ traversed anticlockwise and the boundary $|\zeta| = \rho$ traversed clockwise. Note that the contour C (and hence ∂C) also evolves in time. The function $K(\zeta, t)$ appearing in the integrand is any function of ζ and t which will be taken to be analytic in C. Note that, given the conformal map $z(\zeta, t)$ at any instant of time t, the line integrals in (36) are completely defined once $K(\zeta, t)$ is specified. In this sense the line integrals are purely geometrical and are independent of the flow inside the blob. Of course, the *time evolution* of the line integrals will be determined by the dynamics of the blob. Special choices will be made for the function $K(\zeta, t)$ in order to establish the required results. The following theorem on the time evolution of $J_K(t)$ will prove particularly useful in what follows:

Theorem 4.1. *The time evolution of the integral quantity $J_K(t)$ defined above under the equations of motion for the Stokes flow of an annular blob of fluid is given by*

$$\begin{aligned} \dot{J}_K(t) &= \oint_{\partial C} 2K(\zeta, t)G(\zeta, t)z_\zeta(\zeta, t)\, d\zeta \\ &\quad + \oint_{\partial C} \left(K_t(\zeta, t) - \zeta I(\zeta, t)K_\zeta\right) \bar{z}(\bar{\zeta}, t)z_\zeta(\zeta, t)\, d\zeta. \end{aligned} \tag{37}$$

Proof. Note that this theorem describes the essential dynamics of the blob under Stokes flow with surface tension. First, note that on $|\zeta| = 1$,

$$\frac{d}{dt}z = -2F + \zeta I z_\zeta, \tag{38}$$

while on $|\zeta| = \rho$,

$$\frac{d}{dt}z = -2F + \zeta I z_\zeta + \frac{\dot{\rho}}{\rho}\zeta z_\zeta, \tag{39}$$

where $\frac{d}{dt}z$ is defined to be the time derivative of z, keeping $\nu = Arg\ \zeta$ fixed. Note, this is **not** the same as $z_t(\zeta, t)$ on the inner boundary $\zeta = \rho e^{i\nu}$. Throughout this proof, all conjugate functions are understood to be functions of the conjugate variable $\bar{\zeta}$. Also, $z_\zeta(\zeta, t)$ is understood to mean the partial derivative of z with respect to the first variable. Consider the time derivative of $J_K(t)$,

$$\frac{d}{dt}J_K(t) = \oint_{\partial C} \frac{dK}{dt}\bar{z}z_\zeta d\zeta + K\frac{d\bar{z}}{dt}z_\zeta d\zeta + K\bar{z}\frac{d}{dt}\left(z_\zeta d\zeta\right). \tag{40}$$

Using (38) and (39), this becomes (writing out the integrals on each boundary separately)

$$\frac{d}{dt}J_K(t) = \oint_{|\zeta|=1} K_t \bar{z} z_\zeta + K\left(-2\bar{F} + \frac{1}{\zeta}\bar{I}\bar{z}_\zeta\right) z_\zeta + K\bar{z}\left(-2F_\zeta + \frac{\partial}{\partial\zeta}\left(\zeta I z_\zeta\right)\right) d\zeta$$
$$- \oint_{|\zeta|=\rho}\left(K_t + \frac{\dot{\rho}}{\rho}\zeta K_\zeta\right)\bar{z} z_\zeta + K z_\zeta\left(-2\bar{F} + \frac{\rho^2}{\zeta}\bar{I}\bar{z}_\zeta + \frac{\rho\dot{\rho}}{\zeta}\bar{z}_\zeta\right)$$
$$+ K\bar{z}\left(-2F_\zeta + \frac{\partial}{\partial\zeta}\left(\zeta I z_\zeta\right) + \frac{\dot{\rho}}{\rho}\frac{\partial}{\partial\zeta}\left(\zeta z_\zeta\right)\right) d\zeta. \tag{41}$$

Now we use the stress conditions on the boundary circles of the annulus, which takes the form on $|\zeta| = 1$,

$$-2\bar{F} z_\zeta - 2F_\zeta \bar{z} = 2G z_\zeta - \frac{z_\zeta^{1/2}\bar{z}_\zeta^{1/2}}{\zeta}, \tag{42}$$

while on $|\zeta| = \rho$,

$$-2\bar{F} z_\zeta - 2F_\zeta \bar{z} = 2G z_\zeta + \frac{\rho z_\zeta^{1/2}\bar{z}_\zeta^{1/2}}{\zeta}. \tag{43}$$

Consider first the integral around $|\zeta| = 1$ in (41). Using integration by parts and the stress condition (42), this takes the form

$$\oint_{|\zeta|=1}\left(K_t - \zeta I K_\zeta\right)\bar{z} z_\zeta + K\left(2G z_\zeta - \frac{z_\zeta^{1/2}\bar{z}_\zeta^{1/2}}{\zeta} + \left(I + \bar{I}\right)\frac{z_\zeta \bar{z}_\zeta}{\zeta}\right). \tag{44}$$

Now, using the fact that on $|\zeta| = 1$,

$$I + \bar{I} = 2\,Re\,I = \frac{1}{z_\zeta^{1/2}\bar{z}_\zeta^{1/2}}, \tag{45}$$

which follows from (23) (with $A_O = 0$). This reduces to

$$\oint_{|\zeta|=1}\left(K_t - \zeta I K_\zeta\right)\bar{z} z_\zeta + 2KG z_\zeta\, d\zeta. \tag{46}$$

Now consider the integral around $|\zeta| = \rho$ in (41). Using integration by parts and the stress condition (43), we get

$$\oint_{|\zeta|=\rho}\left(K_t + \frac{\dot{\rho}}{\rho}\zeta K_\zeta - \zeta I K_\zeta\right)\bar{z} z_\zeta + K\left(2G z_\zeta + \frac{\rho z_\zeta^{1/2}\bar{z}_\zeta^{1/2}}{\zeta} + \frac{\rho^2}{\zeta}\left(I + \bar{I}\right) z_\zeta \bar{z}_\zeta\right)$$
$$- \frac{\dot{\rho}}{\rho}\zeta K_\zeta \bar{z} z_\zeta + \frac{2\dot{\rho}\rho}{\zeta} K z_\zeta \bar{z}_\zeta\, d\zeta. \tag{47}$$

Using the fact that on $|\zeta| = \rho$,

$$I + \bar{I} = 2\,Re\,I = -\frac{1}{\rho z_\zeta^{1/2}\bar{z}_\zeta^{1/2}} - \frac{2\dot{\rho}}{\rho}, \tag{48}$$

which follows from (25) (with $A_I = 0$), (47) becomes

$$\oint_{|\zeta|=\rho} \left(K_t - \zeta I K_\zeta\right) \bar{z} z_\zeta d\zeta + 2KGz_\zeta \, d\zeta. \tag{49}$$

Subtracting (49) from (46) gives the required result. □

In the case of a single simply connected blob of fluid with surface tension, exact solutions in the form of an arbitrary rational function, conformal in the unit circle in the ζ plane, have been identified (e.g., [7] [8] [11]–[15]). By analogy with these solutions, we now seek solutions which are again meromorphic in the ζ plane (except at zero and infinity) but with an infinite number of poles outside the annulus C.

We introduce our solution by first defining a set of N parameters $\{\zeta_j \mid j = 1 \ldots N\}$, each of which satisfies the condition $1 < |\zeta_j| < \rho^{-1}$, at least initially. $\zeta_j(t)$ will evolve in time according to equations to be determined. The solution will cease to be valid if and when the above condition is violated. Note that each $\bar{\zeta}_j^{-1}$ is within C. We define the analytic function $h(\zeta, t)$ through the relation

$$z(\zeta, t) = \frac{h(\zeta, t)}{\prod_{j=1}^{N} \left[P(\frac{\zeta}{\zeta_j})\right]^{\gamma_j}}, \tag{50}$$

where $\{\gamma_j \mid j = 1 \ldots N\}$ are arbitrary nonnegative integers, and where $P(\zeta)$ is the function defined in (33). Two properties of $P(\zeta)$ that will prove useful in what follows can be written

$$P(\zeta^{-1}) = -\frac{1}{\zeta} P(\zeta), \tag{51}$$

$$P(\rho^2 \zeta) = -\frac{1}{\zeta} P(\zeta). \tag{52}$$

These two properties can be shown by straightforward manipulation of the infinite product definition in (33). Since the zeroes in the denominator on the right-hand side of (50) are clearly outside C and $z(\zeta, t)$ is analytic within C, it follows that an analytic h in C implies an analytic z in C and vice versa. We also define

$$M_0 = \sum_{j=1}^{N} \gamma_j, \tag{53}$$

where we assume $M_0 \geq 2$. The reason for this last restriction will become clear later.

It is now appropriate to consider integral quantities of the form

$$J^0_{k_0}(t) = \oint_{\partial C} K_0(\zeta, t; k_0) \bar{z}(\bar{\zeta}, t) z_\zeta(\zeta, t) \, d\zeta, \tag{54}$$

where

$$K_0(\zeta, t; k_0) \equiv \zeta^{k_0} \prod_{j=1}^{N} \left[P(\zeta \bar{\zeta}_j)\right]^{\gamma_j}, \qquad k_0 = \cdots - 2, -1, 0, 1, 2 \ldots. \tag{55}$$

Note that $K_0(\zeta, t; k_0)$ is analytic in C (indeed, it is analytic in the entire ζ-plane except for essential singularities at 0 and ∞). We now state an important theorem concerning conserved quantities.

Theorem 4.2. *(Dynamics) If $J^0_{k_0}(0) = 0$ for all k_0, then*

$$J^0_{k_0}(t) = 0 \qquad \text{for all integers } k_0, \tag{56}$$

provided

$$\frac{d}{dt}\bar{\zeta}_j^{-1} = -\bar{\zeta}_j^{-1} I\left(\bar{\zeta}_j^{-1}, t\right) \qquad \text{for all } j = 1 \ldots N. \tag{57}$$

Proof. This result lies at the heart of the existence of the exact solutions and conserved quantities, as will now be demonstrated. First, we observe that

$$\frac{\partial}{\partial t} K_0(\zeta, t; k_0) = K_0(\zeta, t; k_0)\left[\sum_{j=1}^{N} \gamma_j \zeta \frac{d\bar{\zeta}_j}{dt} \frac{P'(\zeta\bar{\zeta}_j)}{P(\zeta\bar{\zeta}_j)} + \gamma_j \dot{\rho} \frac{P_\rho(\zeta\bar{\zeta}_j)}{P(\zeta\bar{\zeta}_j)}\right], \tag{58}$$

where $P'(\zeta)$ denotes the usual differentiation with respect to the argument, and P_ρ denotes differentiation with respect to the parameter ρ. Note that the notation suppresses the implicit dependence of $P(\zeta)$ on the parameter $\rho(t)$. Also,

$$\frac{\partial}{\partial \zeta} K_0(\zeta, t; k_0) = K_0(\zeta, t; k_0)\left[\frac{k_0}{\zeta} + \sum_{j=1}^{N} \gamma_j \bar{\zeta}_j \frac{P'(\zeta\bar{\zeta}_j)}{P(\zeta\bar{\zeta}_j)}\right]. \tag{59}$$

Applying the results of Theorem 4.1, we deduce

$$\dot{J}^0_{k_0}(t) = \oint_{\partial C} K_0 2G z_\zeta \, d\zeta + \oint_{\partial C} K_0 \left[-k_0 I + \sum_{j=1}^{N}\left(\gamma_j \zeta \frac{P'(\zeta\bar{\zeta}_j)}{P(\zeta\bar{\zeta}_j)}\left(\frac{d}{dt}\bar{\zeta}_j - \bar{\zeta}_j I(\zeta, t)\right) + \gamma_j \dot{\rho} \frac{P_\rho(\zeta\bar{\zeta}_j)}{P(\zeta\bar{\zeta}_j)}\right)\right] \bar{z} z_\zeta \, d\zeta. \tag{60}$$

We now define the following function:

$$T(\zeta, t) \equiv \sum_{j=1}^{N} \gamma_j \zeta \frac{P'(\zeta\bar{\zeta}_j)}{P(\zeta\bar{\zeta}_j)}\left(\frac{d\bar{\zeta}_j}{dt} - \bar{\zeta}_j I(\zeta, t)\right) + \gamma_j \dot{\rho} \frac{P_\rho(\zeta\bar{\zeta}_j)}{P(\zeta\bar{\zeta}_j)}. \tag{61}$$

Note that

$$P_\rho(\zeta\bar{\zeta}_j) = -2P(\zeta\bar{\zeta}_j) \sum_{n=1}^{\infty} n\, \rho^{2n-1} \left(\frac{1}{\zeta\bar{\zeta}_j - \rho^{2n}} + \frac{1}{\frac{1}{\zeta\bar{\zeta}_j} - \rho^{2n}}\right). \tag{62}$$

Notice P_ρ/P is singular at $\zeta = \bar{\zeta}_j^{-1}\rho^{2n}$ for a *nonzero* integer n and therefore is analytic in C. The first term on the right-hand side of (61) is also free of singularities in C provided $\zeta_j(t)$ is evolved according to $\frac{d\bar{\zeta}_j}{dt} = \bar{\zeta}_j I\left(\bar{\zeta}_j^{-1}, t\right)$, which can be written equivalently as

$$\frac{d\bar{\zeta}_j^{-1}}{dt} = -\bar{\zeta}_j^{-1} I\left(\bar{\zeta}_j^{-1}, t\right), \qquad j = 1 \ldots N. \tag{63}$$

Since $T(\zeta, t)$ is then analytic in C, it therefore has a Laurent series which we denote

$$T(\zeta, t) = \sum_{j=-\infty}^{\infty} T_j \zeta^j. \tag{64}$$

This expansion is convergent everywhere in C. We also define the following Laurent series:

$$I(\zeta, t) = \sum_{j=-\infty}^{\infty} I_j \zeta^j, \tag{65}$$

which is also convergent in C. Using these expansions, it is straightforward to see that equation (60) can be written

$$\dot{J}_{k_0}^0(t) = -\sum_{j=-\infty}^{\infty} k_0 I_j J_{k_0+j}^0 + \sum_{j=-\infty}^{\infty} T_j J_{k_0+j}^0. \tag{66}$$

Note that the first integral on the right-hand side of (60) gives no contribution because $G(\zeta, t)$ is assumed to be analytic in C. We now define the function $J(\zeta, t)$ via

$$J(\zeta, t) \equiv \sum_{j=-\infty}^{\infty} J_j^0(t) \zeta^j \tag{67}$$

and also the following functions:

$$\hat{I}(\zeta, t) = I(\zeta^{-1}, t), \tag{68}$$

$$\hat{T}(\zeta, t) = T(\zeta^{-1}, t). \tag{69}$$

Multiplying (66) by ζ_0^k and summing over all integers k_0 yields the following partial differential equation for $J(\zeta, t)$:

$$\frac{\partial J}{\partial t} + \zeta \frac{\partial}{\partial \zeta}(\hat{I} J) - \hat{T} J = 0. \tag{70}$$

Note that in the annulus $1 < |\zeta| < \rho^{-1}$, the coefficient functions of the first-order partial differential equation (70) are known a priori to be analytic. Thus, if $J(\zeta, 0) = 0$ in this domain, then by the well-known theory of first-order partial differential equations whose coefficients are known a priori to be analytic over some domain, we deduce that the *unique* solution is $J(\zeta, t) = 0$ for all times that the solution exists. Hence, the theorem is proved. □

Now we state some important theorems which relate the properties of the function $h(\zeta, t)$ to the properties of the countably infinite set of quantities $\{J_{k_0}^0(t) \mid k_0 \text{ any integer}\}$. Note that the next two theorems have *nothing to do with the dynamics* of the physical problem being studied.

Theorem 4.3. *Consider the quantities $J^0_{k_0}(t)$ defined in (54)–(55). Then,*

$$J^0_{k_0}(t) = 0 \quad \forall\, k_0 \tag{71}$$

if and only if the function $\bar{h}(\zeta, t)$ as defined in (50) is analytic everywhere in the ζ plane except at 0 and ∞ and satisfies the functional equation

$$R(t)\zeta^{-M_0}\bar{h}(\zeta, t) = \bar{h}(\rho^2\zeta, t), \tag{72}$$

for all $\zeta \neq 0$ where $R(t) = \prod_{j=1}^{N}[-\bar{\zeta}_j(t)]^{\gamma_j}$.

Proof. Define the Laurent series of z_ζ as follows:

$$z_\zeta(\zeta, t) = \sum_{-\infty}^{\infty} Z_n \zeta^n. \tag{73}$$

Since z_ζ is analytic in C, this series converges everywhere inside C and on the boundary ∂C. We also denote the Laurent expansion of $\bar{h}(\zeta, t)$ by

$$\bar{h}(\zeta, t) = \sum_{-\infty}^{\infty} H_n \zeta^n, \tag{74}$$

which is also known to be convergent everywhere inside C and on the boundary ∂C. First, note the following facts that result from the relation (50) for $z(\zeta, t)$ and the properties (51)–(52) of $P(\zeta)$:

$$\bar{z}(\zeta^{-1}, t) = R(t)\frac{\zeta^{M_0}\bar{h}(\zeta^{-1}, t)}{\prod_{j=1}^{N}\left[P(\zeta\bar{\zeta}_j)\right]^{\gamma_j}}, \tag{75}$$

$$\bar{z}(\rho^2\zeta^{-1}, t) = \frac{\bar{h}(\rho^2\zeta^{-1}, t)}{\prod_{j=1}^{N}\left[P(\zeta\bar{\zeta}_j)\right]^{\gamma_j}}. \tag{76}$$

Now suppose that $J^0_{k_0}(t) = 0$ for all k_0. This implies that $\forall k_0$

$$R(t)\oint_{|\zeta|=1} \zeta^{k_0+M_0}\bar{h}(\zeta^{-1}, t) z_\zeta(\zeta, t)\, d\zeta = \oint_{|\zeta|=\rho} \zeta^{k_0}\bar{h}(\rho^2\zeta^{-1}, t) z_\zeta(\zeta, t)\, d\zeta, \tag{77}$$

where each of the contour integrals in (77) is taken in the anticlockwise sense. Since $|\zeta^{-1}| = 1$ on $|\zeta| = 1$ and $|\rho^2\zeta^{-1}| = \rho$ on $|\zeta| = \rho$, we can write

$$\bar{h}(\zeta^{-1}, t) = \sum_{-\infty}^{\infty} H_n \zeta^{-n} \qquad \text{on } |\zeta| = 1, \tag{78}$$

$$\bar{h}(\rho^2\zeta^{-1}, t) = \sum_{-\infty}^{\infty} H_n \rho^{2n} \zeta^{-n} \qquad \text{on } |\zeta| = \rho. \tag{79}$$

Similarly, expanding $z_\zeta(\zeta, t)$ as a Laurent series (valid on both boundaries of C), (77) becomes

$$R(t)\oint_{|\zeta|=1}\sum_{m=-\infty}^{\infty}\zeta^{k_0+m+M_0}\sum_{n=-\infty}^{\infty}H_n Z_{n+m}\,d\zeta$$
$$=\oint_{|\zeta|=\rho}\sum_{m=-\infty}^{\infty}\zeta^{k_0+m}\sum_{n=-\infty}^{\infty}H_n\rho^{2n}Z_{n+m}\,d\zeta, \tag{80}$$

for all k_0. Computing the integrals gives

$$R(t)\sum_{n=-\infty}^{\infty}H_n Z_{n-k_0-1-M_0}=\sum_{n=-\infty}^{\infty}H_n\rho^{2n}Z_{n-k_0-1}\quad\forall k_0. \tag{81}$$

Now multiply equation (81) by ζ^{k_0+1} and sum over all k_0. Using k instead of k_0, we then get

$$R(t)\sum_{k=-\infty}^{\infty}\zeta^{k+1}\sum_{n=-\infty}^{\infty}H_n Z_{n-1-k-M_0}=\sum_{k=-\infty}^{\infty}\zeta^{k+1}\sum_{n=-\infty}^{\infty}H_n\rho^{2n}Z_{n-k-1}. \tag{82}$$

Define the function $H(\zeta, t)$ by the Laurent series

$$H(\zeta, t)\equiv\sum_{n=-\infty}^{\infty}H_n\rho^{2n}\zeta^n, \tag{83}$$

where the coefficients $\{H_n\}$ are the same as in (74). It is known that on $|\zeta|=\frac{1}{\rho}$ this series converges and equals $\bar{h}(\rho^2\zeta, t)$ there. Note also that the left-hand side of (82) is equal to $R(t)\zeta^{-M_0}\bar{h}(\zeta, t)z_\zeta(\zeta^{-1}, t)$ on $|\zeta|=1$, where it is known to be analytic with a convergent Laurent expansion. This equality must hold anywhere the series converges. From consideration of the right-hand side of (82), it is clear that it is equal to $H(\zeta, t)z_\zeta(\zeta^{-1}, t)$ on $|\zeta|=\rho^{-1}$, where it is known to be analytic with a convergent Laurent expansion. Thus the series in (82) is convergent on $|\zeta|=\rho^{-1}$, as well. From the principle of analytic continuation, this implies that

$$R(t)\zeta^{-M_0}\bar{h}(\zeta, t)z_\zeta(\zeta^{-1}, t)=H(\zeta, t)z_\zeta(\zeta^{-1}, t), \tag{84}$$

with each side of the equality (84) having the same convergent Laurent series. From the relation of H with $\bar{h}$, it follows that

$$R(t)\zeta^{-M_0}\bar{h}(\zeta, t)=\bar{h}(\rho^2\zeta, t). \tag{85}$$

Since we know that $\bar{h}(\zeta, t)$ is analytic (at least) in the entire annulus $\rho\le|\zeta|\le\rho^{-1}$, (85) furnishes the analytic continuation of $\bar{h}(\zeta, t)$ into the entire plane, excluding the points at 0 and ∞. This is the required result.

Conversely, if $\bar{h}(\zeta, t)$ satisfies condition (72) for all ζ, and is analytic everywhere except at 0 and ∞, then it is clear that $J_{k_0}^0(t)$ reduces to the integral around ∂C of the following function of ζ, i.e.,

$$R(t)\zeta^{k_0+M_0}\bar{h}(\zeta^{-1}, t)z_\zeta(\zeta, t), \tag{86}$$

which is known to be analytic in C for all k_0, and thus the result follows by Cauchy's theorem. □

Theorem 4.4. *The function* $\bar{h}(\zeta, t)$ *satisfies the functional equation*

$$R(t)\zeta^{-M_0}\bar{h}(\zeta, t) = \bar{h}(\rho^2\zeta, t), \tag{87}$$

for all $\zeta \neq 0$*, and is analytic everywhere except possibly at* 0 *and* ∞ *if and only if*

$$\bar{h}(\zeta, t) = \bar{S}(t)\prod_{m=1}^{M_0} P\left(\frac{\zeta}{\bar{\eta}_m}\right), \tag{88}$$

for some $\bar{S}(t)$ *and some* $\{\bar{\eta}_m(t) \mid m = 1 \ldots M_0\}$ *satisfying the condition*

$$\prod_{m=1}^{M_0}[-\bar{\eta}_m(t)] = R(t). \tag{89}$$

Proof. Suppose that $\bar{h}(\zeta, t)$ satisfies (87) and is analytic everywhere except possibly at 0 and ∞. Consider the function $M(\zeta, t)$ defined by

$$M(\zeta, t) = \prod_{m=1}^{M_0} P\left(\frac{\zeta}{\beta_m}\right), \tag{90}$$

where $\{\beta_m(t) \mid m = 1 \ldots M_0\}$ are taken to satisfy

$$\prod_{m=1}^{M_0}[-\beta_m(t)] = R(t). \tag{91}$$

Note that $M(\zeta, t)$ satisfies the following functional equation, which results from the properties of $P(\zeta)$ as given in (51)–(52).

$$M(\rho^2\zeta) = R(t)\zeta^{-M_0}M(\zeta, t). \tag{92}$$

Now define the function $N(\zeta, t)$ by

$$N(\zeta, t) \equiv \frac{\bar{h}(\zeta, t)}{M(\zeta, t)}. \tag{93}$$

Then it is clear from the definitions of $N(\zeta, t)$ and $M(\zeta, t)$ and the known analyticity of $\bar{h}(\zeta, t)$ everywhere except at 0 and ∞ that $N(\zeta, t)$ is a meromorphic function everywhere (excluding 0 and ∞) with poles at the points $\{\beta_m(t)\}$ and all equivalent points $\{\rho^{2n}\beta_m(t)\}$, where n is an arbitrary integer. It is also easily seen that it satisfies the functional equation

$$N(\rho^2\zeta, t) = N(\zeta, t) \tag{94}$$

for all $\zeta \neq 0$. Thus, $N(\zeta, t)$ is a *loxodromic* function—i.e., a meromorphic function everywhere in the finite ζ-plane ($\zeta \neq 0$) which also satisfies the functional equation (94) for $\zeta \neq 0$. By the well-known representation theorem for loxodromic functions (see [3] [18]), we conclude that $N(\zeta, t)$ necessarily has a representation of the form

$$N(\zeta, t) = \bar{S}(t)\frac{\prod_{m=1}^{M_0} P(\frac{\zeta}{\bar{\eta}_m})}{\prod_{m=1}^{M_0} P(\frac{\zeta}{\beta_m})}, \tag{95}$$

for some $\bar{S}(t)$ and some functions $\{\bar{\eta}_m(t) \mid m = 1 \ldots M_0\}$ satisfying the condition

$$\prod_{m=1}^{M_0}[-\bar{\eta}_m(t)] = \prod_{m=1}^{M_0}[-\beta_m(t)] = R(t). \tag{96}$$

Thus, by comparison with the definition of $N(\zeta, t)$, we conclude that $\bar{h}(\zeta, t)$, can be written

$$\bar{h}(\zeta, t) = \bar{S}(t) \prod_{m=1}^{M_0} P\left(\frac{\zeta}{\bar{\eta}_m}\right), \tag{97}$$

for some $\bar{S}(t)$ with the $\{\bar{\eta}_m(t)\}$ satisfying (96). This is the required result.

The converse result is established trivially by using the properties (51)–(52) of the function $P(\zeta, t)$. □

We now state the important theorem of this paper:

Theorem 4.5. *If initially $h(\zeta, 0)$ has the form*

$$\bar{h}(\zeta, 0) = \bar{S}(0) \prod_{m=1}^{M_0} P\left(\frac{\zeta}{\bar{\eta}_m(0)}\right), \tag{98}$$

where $\prod_{m=1}^{M_0}[-\bar{\eta}_m(0)] = \prod_{j=1}^{N}[-\bar{\zeta}_j(0)]^{\gamma_j}$ and $\zeta_j(0)$ (and equivalent points) are the positions of the poles of $z(\zeta, 0)$ and provided that

$$\frac{d\bar{\zeta}_j^{-1}}{dt} = -\bar{\zeta}_j^{-1} I\left(\bar{\zeta}_j^{-1}, t\right) \qquad \text{for all } j = 1 \ldots N, \tag{99}$$

then

$$\bar{h}(\zeta, t) = \bar{S}(t) \prod_{m=1}^{M_0} P\left(\frac{\zeta}{\bar{\eta}_m(t)}\right), \tag{100}$$

where

$$\prod_{m=1}^{M_0}[-\bar{\eta}_m(t)] = \prod_{j=1}^{N}[-\bar{\zeta}_j(t)]^{\gamma_j} \tag{101}$$

for all future times that the solution exists.

Proof. From the initial form for $h(\zeta, 0)$, it follows from Theorems 4.3 and 4.4 that

$$J_{k_0}^0(0) = 0 \ \ \forall k_0. \tag{102}$$

By Theorem 4.2 it is known that, provided the poles evolve according to (99), $J_{k_0}^0(t) = 0$ $\forall\ k_0$ is the **unique** solution for all time that the solution exists. By Theorem 4.3, we then deduce that $\bar{h}(\zeta, t)$ satisfies (87) which then implies, by Theorem 4.4, that $\bar{h}(\zeta, t)$ necessarily has the form (100) for all times. Hence, Theorem 4.5 is proved. □

Theorem 4.5 provides the crucial result, and we will now limit the discussion to initial conditions of the form (98), thus implying that $z(\zeta, t)$ has the following form for all time that the solution exists, i.e.,

$$z(\zeta, t) = S(t) \frac{\prod_{m=1}^{M_0} P(\frac{\zeta}{\eta_m})}{\prod_{j=1}^{N} \left[P(\frac{\zeta}{\zeta_j}) \right]^{\gamma_j}}, \tag{103}$$

where $\{\zeta_j(t)\}$, $\{\eta_j(t)\}$ satisfy (101). Note that we immediately see that such maps are *loxodromic functions*—a *loxodromic function* is defined as a *meromorphic* function everywhere in the finite ζ-plane ($\zeta \neq 0$) that also satisfies the functional equation $z(\rho^2\zeta, t) = z(\zeta, t)$ for all $\zeta \neq 0$. Richardson [18] recently made use of the theory of loxodromic functions in a very different mathematical context to find solutions to the problem of the Hele-Shaw flow of a doubly connected fluid region, when surface tension is completely neglected. A convenient summary of the general properties of loxodromic functions is given in an appendix to Richardson's paper. Richardson's appendix is modelled on the presentation of the material given in Valiron [3]. We note also that, by means of the following exponential transformation,

$$\zeta = e^{i\pi u}, \tag{104}$$

it is seen that loxodromic functions are intimately related to elliptic functions, i.e., defining

$$\tilde{z}(u, t) \equiv z(\zeta, t), \tag{105}$$

it is clear that, for loxodromic, single-valued $z(\zeta, t)$, $\tilde{z}(u, t)$ is a doubly periodic, meromorphic (i.e., elliptic) function of u with a purely real and a purely imaginary period,

$$\tilde{z}(u + 2, t) = \tilde{z}(u, t), \tag{106}$$

$$\tilde{z}(u - i\frac{2}{\pi} \log \rho, t) = \tilde{z}(u, t). \tag{107}$$

Given this association, it is clear that all the standard theory of elliptic functions (e.g., [1] [3] [5] [6]) also pertains directly to loxodromic functions.

Remark 2. Since we have deduced $z(\zeta, t)$ is a loxodromic function, it is known from the general theory (e.g., [3] [18]) that such a function is uniquely defined once its poles and zeroes in the *fundamental annulus* $\rho^2 < |\zeta| \leq 1$ are known, as well as its value at one other point. Note that the fundamental annulus is *not* the same as C.

Remark 3. To be physically acceptable, $z(\zeta, t)$ must be univalent in C. It is therefore necessary to pick initial values of $\{\eta_m(0) \mid m = 1 \ldots M_0\}$ and $\{\zeta_j(0) \mid j = 1 \ldots N\}$ such that $z(\zeta, 0)$ is a univalent map for $\rho \leq |\zeta| \leq 1$. If the map subsequently evolves such as to violate this condition, then the solution will be deemed invalid thereafter. A necessary though not sufficient condition for this is to ascertain that there are no zeroes of the derivative $z_\zeta(\zeta, t)$ in this region. Since $z(\zeta, t)$ is a loxodromic function with fundamental annulus $\rho^2 < |\zeta| \leq 1$, (i.e., an elliptic function in the variable u), it follows from the well-known theory of elliptic functions of order M_0 that any value of z will be taken M_0 times in this fundamental annulus (or fundamental rectangle if u is the variable). For univalence initially, it is necessary to pick initial values of the parameters so that z attains no value more than once in the subregion $\rho \leq |\zeta| \leq 1$. That such a choice is possible is far from obvious and is illustrated by explicit construction later in this paper.

Remark 4. The reason for the restriction on M_0 mentioned after (53) is due to the well-known result that a nontrivial loxodromic function (or an elliptic function in the variable u) must be at least of order two.

We now examine how to derive the evolution equations for the finite set of time-evolving parameters appearing in the solution (103). To do this, we consider the line integral quantities defined by

$$J^j_{k_j}(t) = \oint_{\partial C} K_j(\zeta, t; k_j)\bar{z}(\bar{\zeta}, t) z_\zeta(\zeta, t)\, d\zeta, \tag{108}$$

where

$$K_j(\zeta, t; k_j) \equiv \left[P(\zeta\bar{\zeta}_j)\right]^{k_j} \prod_{\substack{p=1\\p\neq j}}^{N} \left[P(\zeta\bar{\zeta}_p)\right]^{\gamma_p}, \tag{109}$$

and $k_j = 0, 1, 2 \ldots$.

Theorem 4.6. *For the class of initial shapes considered, the following property of the line integral quantities defined in (94)–(95) holds for all $j = 1 \ldots N$:*

$$J^j_{k_j}(t) = 0 \qquad \text{for} \quad k_j \geq \gamma_j. \tag{110}$$

Proof. We use the loxodromic property of $z(\zeta, t)$ (and hence of $\bar{z}(\zeta, t)$) to reduce $J^j_{k_j}(t)$ to the integral around ∂C of the following function of ζ:

$$\left[P(\zeta\bar{\zeta}_j)\right]^{k_j} \left(\prod_{\substack{p=1\\p\neq j}}^{N} \left[P(\zeta\bar{\zeta}_p)\right]^{\gamma_p} \right) \bar{z}(\zeta^{-1}, t) z_\zeta(\zeta, t), \tag{111}$$

which, by use of the form (103) for $z(\zeta, t)$, can be seen to be analytic in C for all $k_j \geq \gamma_j$, and the theorem follows by Cauchy's Theorem. □

Theorem 4.7. *For the class of initial shapes considered, the $J^j_{k_j}(t)$ satisfies the following equation for $k_j = 0, 1, \ldots \gamma_j - 1$ and $j = 1 \ldots N$:*

$$\begin{aligned} \dot{J}^j_{k_j}(t) = \oint_{\partial C} K_j \Bigg[& k_j \zeta \frac{P'(\zeta\bar{\zeta}_j)}{P(\zeta\bar{\zeta}_j)} \left(\frac{d\bar{\zeta}_j}{dt} - \bar{\zeta}_j I \right) + k_j \dot{\rho} \frac{P_\rho(\zeta\bar{\zeta}_j)}{P(\zeta\bar{\zeta}_j)} \\ & + \sum_{\substack{p=1\\p\neq j}}^{N} \gamma_p \zeta \frac{P'(\zeta\bar{\zeta}_j)}{P(\zeta\bar{\zeta}_j)} \left(\frac{d\bar{\zeta}_p}{dt} - \bar{\zeta}_p I(\zeta, t) \right) + \gamma_p \dot{\rho} \frac{P_\rho(\zeta\bar{\zeta}_p)}{P(\zeta\bar{\zeta}_p)} \Bigg] \bar{z} z_\zeta \, d\zeta. \end{aligned} \tag{112}$$

Proof. Applying Theorem 4.1, with the substitution $K(\zeta,t) = K_j(\zeta,t;k_j)$, it follows that

$$\dot{J}^j_{k_j}(t) = \oint_{\partial C} K_j 2G(\zeta,t) z_\zeta + K_j\Bigg[k_j\zeta\frac{P'(\zeta\bar{\zeta}_j)}{P(\zeta\bar{\zeta}_j)}\left(\frac{d\bar{\zeta}_j}{dt} - \bar{\zeta}_j I\right) + k_j\dot{\rho}\frac{P_\rho(\zeta\bar{\zeta}_j)}{P(\zeta\bar{\zeta}_j)} + \sum_{\substack{p=1\\ p\neq j}}^{N}\gamma_p\zeta\frac{P'(\zeta\bar{\zeta}_j)}{P(\zeta\bar{\zeta}_j)}\left(\frac{d\bar{\zeta}_p}{dt} - \bar{\zeta}_p I(\zeta,t)\right) + \gamma_p\dot{\rho}\frac{P_\rho(\zeta\bar{\zeta}_p)}{P(\zeta\bar{\zeta}_p)}\Bigg]\bar{z}z_\zeta\, d\zeta. \quad (113)$$

The analyticity of $K_j G z_\zeta$ in C implies that the first term within the integrand on the right-hand side of (113) does not contribute anything and the result (112) follows. □

Remark 5. Note by inspection that, for $k_j \geq \gamma_j$, the pole singularity of $\bar{z}$ at $\zeta = \bar{\zeta}_j^{-1}$ is cancelled out by the zeroes of K_j at the same point; further, when ζ_j satisfies (57), there is only a *removable* singularity at $\zeta = \bar{\zeta}_j^{-1}$. From this, it is easy to see that the integrand in (112) is analytic in such cases and therefore $\dot{J}^j_{k_j} = 0$ for $k_j \geq \gamma_j$, which is consistent with Theorem 4.6.

Since the evolution of the N poles $\{\zeta_j \mid j = 1\ldots N\}$ is given by the N equations (99), it only remains to deduce the evolution of the $M_0 + 1$ time-evolving parameters $S(t), \{\eta_m(t) \mid m = 1\ldots M_0\}$. Note however that the $\{\eta_m(t)\}$ satisfy the constraint that $\prod_{m=1}^{M_0}[-\eta_m(t)] = \prod_{j=1}^{N}[-\zeta_j(t)]^{\gamma_j}$. Thus, there remains precisely M_0 (generally complex) parameters to determine. Note however that there are M_0 nonzero line integral quantities, namely,

$$\{J^j_{k_j}(t) \mid k_j = 0\ldots\gamma_j - 1\}, \qquad j = 1\ldots N, \quad (114)$$

which can be determined from (112). However, this requires us to invoke the following conjecture that we believe to be true, but which is so far supported only by the numerical evidence (see later):

Conjecture. For a given set of $\zeta_j(t)$, the M_0 quantities in (114) at any time t determine uniquely the $M_0 + 1$ parameters $S(t), \{\eta_m(t) \mid m = 1\ldots M_0\}$ satisfying the constraint (101).

Remark 6. The conjecture above, if true, implies that for a given set of $\zeta_j(t)$, (112) can be viewed as a differential equation to determine $J^j_{k_j}$ since quantities appearing in the integrands, such as z, $\bar{z}$, and I, are completely determined by the parameters characterizing z in (103), which in turn is known for given $J^j_{k_j}$.

Remark 7. We are not asserting a *globally* unique relation between the quantities in (114) and the parameters appearing in (103), only a *locally* unique relation.

It is noted however that the "counting" in the statement of the conjecture is consistent, and the validity of the conjecture is supported by the sample calculation later in the paper, even though this is but a single and very special case. It is expected, however, to be true in general.

Before carrying out a numerical computation to provide verification for the above conjecture, we now state an important theorem that proves extremely useful in simplifying the calculations for certain classes of exact solutions. The following "theorem of invariants" allows us immediately to write down an invariant of motion for the class of initial conditions consistent with (103) for each ζ_j that is a simple pole. An analogous theorem was identified in the case of a simply connected blob [7], [8].

5. A Theorem of Invariants

Theorem 5.1. *(Theorem of Invariants) Suppose that the conformal map $z(\zeta, t)$ has the form (103) and that the evolution of the poles is given by (99). For each j for which the corresponding $\gamma_j = 1$, there exists an invariant of the motion given by*

$$B_j = \frac{J_0^j(t)}{\prod_{\substack{p=1\\ p\neq j}}^{N} \left[P(\bar{\zeta}_p \bar{\zeta}_j^{-1}) \right]^{\gamma_p}}. \tag{115}$$

Proof. Consider the time derivative of J_0^j. From (112), it follows that

$$\frac{dJ_0^j}{dt} = \oint_{\partial C} K_j \left[\sum_{\substack{p=1\\ p\neq j}}^{N} \gamma_p \left(\frac{d\bar{\zeta}_p}{dt} - \bar{\zeta}_p I(\zeta, t) \right) \zeta \frac{P'(\zeta \bar{\zeta}_p)}{P(\zeta \bar{\zeta}_p)} + \gamma_p \dot{\rho} \frac{P_\rho(\zeta \bar{\zeta}_p)}{P(\zeta \bar{\zeta}_p)} \right] \bar{z} z_\zeta \, d\zeta. \tag{116}$$

By inspection it can be seen that the only contribution to the integral in (116) comes from the simple pole in $\bar{z}$ at $\bar{\zeta}_j^{-1}$, since the rest of the integrand is analytic if the evolution of the poles is given by (99). Since the pole at $\bar{\zeta}_j^{-1}$ is *simple*, it is easily seen that (116) can be written as

$$\frac{dJ_0^j}{dt} = \left[\sum_{\substack{p=1\\ p\neq j}}^{N} \gamma_p \left(\frac{d\bar{\zeta}_p}{dt} - \bar{\zeta}_p I(\bar{\zeta}_j^{-1}, t) \right) \bar{\zeta}_j^{-1} \frac{P'(\bar{\zeta}_p \bar{\zeta}_j^{-1})}{P(\bar{\zeta}_p \bar{\zeta}_j^{-1})} + \gamma_p \dot{\rho} \frac{P_\rho(\bar{\zeta}_p \bar{\zeta}_j^{-1})}{P(\bar{\zeta}_p \bar{\zeta}_j^{-1})} \right] J_0^j. \tag{117}$$

Using the fact that

$$I(\bar{\zeta}_j^{-1}, t) = -\frac{d \log(\bar{\zeta}_j^{-1})}{dt}, \tag{118}$$

(117) can be written as

$$\frac{d \log J_0^j(t)}{dt} = \sum_{\substack{p=1\\ p\neq j}}^{N} \frac{d}{dt} \log \left(\left[P(\bar{\zeta}_p \bar{\zeta}_j^{-1}) \right]^{\gamma_p} \right), \tag{119}$$

which clearly can be integrated directly with respect to time to give the required result. □

Remark 8. Note that each invariant B_j is determined from initial conditions alone. If $\gamma_j = 1$ for all j, between 1 and N, then there will be N invariants B_1 through B_N.

A special class of exact solutions having a particularly appealing mathematical structure was found in [7] [8] which dealt with the evolution of a simply connected blob under the effects of surface tension. We now write down the analogous solutions for an annular blob. Consider the class of exact solutions where $z(\zeta, t)$ has the form

$$z(\zeta, t) = S(t) \frac{\prod_{m=1}^{N} P(\frac{\zeta}{\eta_m})}{\prod_{j=1}^{N} P(\frac{\zeta}{\zeta_j})}, \tag{120}$$

with

$$\prod_{m=1}^{N} \eta_m(t) = \prod_{j=1}^{N} \zeta_j(t). \tag{121}$$

This corresponds to the special case of the above solutions with $\{\gamma_j = 1 \mid j = 1 \ldots N\}$, where N is an arbitrary positive integer, $N \geq 2$. The evolution equations for maps of this form can be written down as a very concise set. Of course, the evolution of $\rho(t)$ is always given by (35) but by the previous theorems, for a solution of the form (120) the poles must evolve according to

$$\frac{d\bar{\zeta}_j^{-1}}{dt} = -\bar{\zeta}_j^{-1} I\left(\bar{\zeta}_j^{-1}, t\right), \qquad j = 1 \ldots N, \tag{122}$$

which implicitly gives the evolution of the poles. In addition, by Theorem 4.7, there are N invariants of the motion given by

$$B_j = \frac{J_0^j(t)}{\prod_{\substack{p=1 \\ p \neq j}}^{N} P(\bar{\zeta}_p \bar{\zeta}_j^{-1})}, \qquad j = 1 \ldots N, \tag{123}$$

where $\{B_j \mid j = 1 \ldots N\}$ are complex constants determined from initial conditions. Thus, with the simple poles ζ_j determined from the evolution equations (122), the $N+1$ parameters $S(t)$ and $\{\eta_m(t) \mid m = 1 \ldots N\}$ satisfying the constraint (121) are then determined by inverting the N nonlinear algebraic relations (123). Note that in this special case study it is clearly seen that the total area of the fluid region is directly proportional to the sum of the N invariants of motion, which means that it is also conserved (as it should be, given that there are no sources or sinks in the fluid).

6. Example Calculation

Since the purpose of the present paper is to present the mathematical theory, a full investigation of the physical phenomena exhibited by the class of solutions found here is reserved for a future paper. However, it is necessary to include here at least one sample calculation as evidence for the conjecture stated earlier. It is also necessary to produce an example of an initial (loxodromic) conformal map satisfying all the required conformality and univalency conditions in the annulus $\rho < |\zeta| < 1$, thereby demonstrating, by explicit construction, that such functions do indeed exist.

As discussed in the introduction, the example calculation that has been chosen constitutes a basic paradigm in the study of sintering and represents a natural generalization of the study of the coalescence of two (unequal) cylinders, as carried out recently by Richardson [12] [17]. Here we study the evolution of two (unequal) touching near-cylinders that have the additional feature of a small air bubble in the region where they touch. In the case of just two cylinders, such a bubble may perhaps have been trapped by some mechanism as the cylinders came into contact. More usefully, this example (and more sophisticated versions of it) is expected to represent a basic paradigm for the evolution of a general assemblage of cylinders/particles where there inevitably will be small air bubbles trapped between the cylinders. We point out that we have assumed for simplicity that the pressure of the entrapped bubble of air remains constant in time; in reality, this reasonably can be expected to change as the area of the enclosed bubble reduces.

The initial state of the two (almost) cylindrical blobs and an air bubble between them is represented in Figure 6 and corresponds to the case $N = 3$ with $\gamma_1 = \gamma_2 = \gamma_3 = 1$ so that

$$z(\zeta, t) = S(t)\frac{P(\frac{\zeta}{\eta_1})P(\frac{\zeta}{\eta_2})P(\frac{\zeta}{\eta_3})}{P(\frac{\zeta}{\zeta_1})P(\frac{\zeta}{\zeta_2})P(\frac{\zeta}{\zeta_3})}, \tag{124}$$

with initial parameters given by

$$\begin{aligned} \zeta_1 &= 2.2, \qquad \zeta_2 = -1.25, \qquad \zeta_3 = 1.29039, \qquad \eta_1 = 2.08, \\ \eta_2 &= 1.4, \qquad \eta_3 = -1.21861, \qquad \rho = 0.13, \qquad S = 1.0. \end{aligned} \tag{125}$$

Such a map represents a univalent map from the annulus C to the region shown in Figure 6. Note that there are three simple poles of this mapping, which means, by the theorem of invariants, that we automatically can find three invariants of the motion. Since ρ is initially relatively small, and since it is known that $\rho(t) \to 0$ for the class of solutions found here, certain approximations were made which greatly facilitated the computations to the point where Mathematica could reasonably be used to carry them out. Since no numerical pathologies were expected, an obvious and elementary first method was used in which a simple forward-Euler method was employed to time step the evolution of $\rho(t)$, $\zeta_1(t)$, $\zeta_2(t)$, and $\zeta_3(t)$ (with $h = 0.0002$), while a high-accuracy Newton's method was then used at each time step to invert the three invariants of motion for $\eta_1(t),\ \eta_2(t),\ S(t)$ (with $\eta_3 = \frac{\zeta_1\zeta_2\zeta_3}{\eta_1\eta_2}$). Mathematica coped well with all the calculations, once both the function $P(\zeta)$ and the integrands in (31) and (32) were expanded for small ρ and approximated to within $O(\rho^6)$ (the errors due to this approximation are therefore of order 10^{-5}—i.e., smaller than the global error of the simple time-stepping scheme). Given the existence of points of large curvature in the initial configuration, the blob evolved quickly and the configuration after just 30 time steps is shown relative to the initial configuration in Figure 3.

It is clear that the global momentum of the blob is not conserved and there is an overall translation of the blob in the positive x-direction. As mentioned earlier, this is unimportant because it is only the geometrical evolution of the blob boundary that is of interest. (If

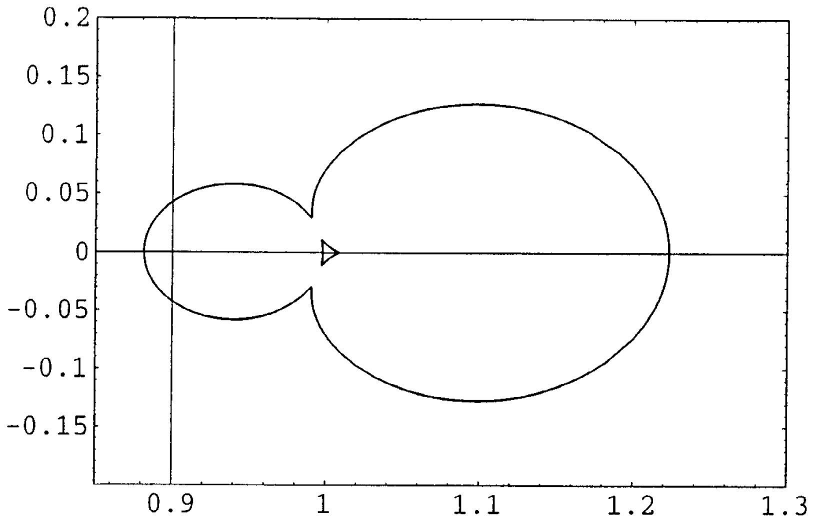

Fig. 2. Initial configuration.

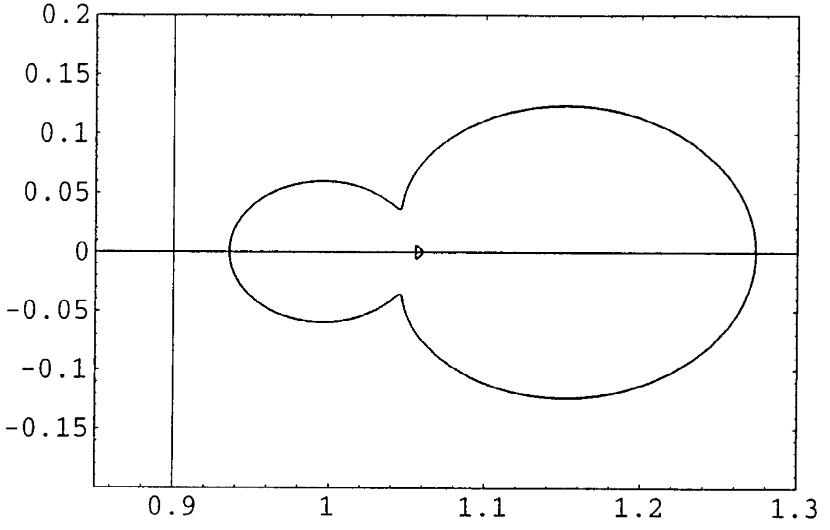

Fig. 3. Configuration after 30 time steps (h = 0.0002).

desired, these physically irrelevant translations can be removed by a straightforward shift of the centres of area to a common point before plotting.) Note also that the near-cusps of the initial configuration (observe that the initial enclosed air bubble has three points of very high curvature) became smoother under evolution, and the enclosed bubble grew smaller, as expected. Note that the points of high curvature in the initial configuration are smoothed out by the effects of surface tension. Clearly, the enclosed bubble is already quite small after only 30 time steps, and although no further integration was carried out, it is expected that as more time evolves, the inner bubble will simply continue to get smaller (probably tending to a circular shape), while the outer boundary of the blob will evolve into a circular shape, as already observed by Richardson [17].

In any event, the principal purpose of including this simple numerical calculation is to verify that the finite set of nonlinear evolution equations for the line integral quantities as derived in the theory above can indeed be solved (at least locally) to provide the time evolution of the parameters in the exact solution (124). In this limited example, the physical behaviour was much as expected. Studies of the physical properties of the mathematical solutions presented here will be left for the future.

7. Discussion

A new class of exact solutions representing the evolution of annular viscous blobs driven by surface tension has been presented. It is expected that this new class of exact solutions will facilitate the study of the physical phenomena exhibited by doubly connected fluid regions with surface tension. One of the principal problems in the study of the new solutions is that of finding initial values of the parameters in the conformal map such that, initially, it is a *univalent* map from C to the fluid domain. At the time of writing, the present authors know of no systematic method of constructing such functions, and the initial configuration found above for the sample calculation was identified after much trial and error. A natural question that arises is whether the boundary of *any* doubly connected fluid domain of given conformal modulus $\rho(t)$ can be approximated as closely as required (in some norm) by a map $z(\zeta, t)$ satisfying the loxodromic property $z(\rho^2\zeta, t) = z(\zeta, t)$, which is a univalent map from C in the complex ζ-plane. If this is true, then a further question that arises is whether the subsequent evolution of the approximating loxodromic function (as given in this paper) will remain a good approximation to the true evolution of that fluid region under Stokes flow driven purely by surface tension. Such questions require further investigation.

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