Pergamon

PII: S0020-7225(96)00080-8

Int. J. Engng Sci. Vol. 35, No. 2, pp. 141-149, 1997

0020-7225/97 $17.00 + 0.00

GENERAL SOLUTIONS TO THE 2D LIOUVILLE EQUATION

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Abstract—This paper presents the most general solutions to the two dimensional elliptic and hyperbolic Liouville equations using elementary techniques. It is demonstrated how previously known general solutions are special cases of these most general solutions. Copyright © 1997 Elsevier Science Ltd

1. INTRODUCTION

This paper presents the most general exact solutions of the quasi-linear partial differential equations

$$\psi_{xx} + \psi_{yy} = \tilde{c}\mathrm{e}^{d\psi} \tag{1}$$

$$\psi_{xx} - \psi_{yy} = \tilde{c}\mathrm{e}^{d\psi} \tag{2}$$

where $\tilde{c}$ and d are real non-zero constants. Equations (1) and (2) are both generally recognised as being forms of the two-dimensional Liouville equation [1], and throughout this paper they will be referred to as the elliptic and hyperbolic Liouville equations respectively. The importance of these equations in various areas of mathematical physics from plasma physics and field theoretical modelling to fluid dynamics has made them the topic of many investigations for solution. A variety of exact solutions have been reported in the literature, many derived using highly sophisticated mathematical techniques [1–16]. For example, most recently Popov [16] employed a geometrical method on a Lobachevskii plane to obtain some general solutions to (1) from solutions of the two-dimensional Laplace equation, while Bhutani *et al.* [2] recently reported a new general solution of (2), and retrieved all previously known general solutions, using a direct method based on the formalism devised by Clarkson and Kruskal [17]. Neither of these two solution methods provide the *most* general solutions to (1) and (2) and solutions of the generality presented in this paper have not, to the best of the author's knowledge, been reported before. This paper therefore serves as a unification of many disparate results spread throughout the literature, and also provides many previously unknown exact solutions.

The methods employed here are essentially elementary yet the solutions obtained are shown to be the most general. It is indicated how to retrieve currently known solutions as special cases of these most general solutions. The general solution to (1) is shown to depend on two arbitrary analytic functions and some constants while the general solution to (2) depends on four arbitrary real functions and some constants. For clarity, the development is presented as a series of theorems and proofs but since the purpose of this paper is to disseminate these formal solutions to the general scientific community where certain solutions might be recognised as having particular physical significance or application, the exposition is non-rigorous. Accordingly, any deeper mathematical implications of the results will not be treated here.

2. THE ELLIPTIC CASE

To illustrate the method of solution for the elliptic case we solve the elliptic Liouville equation in the (x, y)-plane given by

$$\psi_{xx} + \psi_{yy} = \tilde{c}\mathrm{e}^{d\psi} \tag{3}$$

where $\tilde{c}$, d are real constants which are assumed to be non-zero. By shifting to characteristic coordinates, $z = x + iy$ and $\bar{z} = x - iy$, we can equivalently solve

$$\psi_{z\bar{z}} = c e^{d\psi} \tag{4}$$

for real solutions ψ where $c = \tilde{c}/4$. It is noted that by the linear change of dependent variable

$$\phi = d\psi + \log(|cd|) \tag{5}$$

(4) can be written in the canonical form

$$\phi_{z\bar{z}} = \operatorname{sgn}[cd] e^{\phi}. \tag{6}$$

THEOREM 1. Any function $\psi(z, \bar{z})$ that is twice differentiable with respect to z and $\bar{z}$ and is a solution to

$$\psi_{z\bar{z}} = c e^{d\psi} \tag{7}$$

also satisfies

$$\psi_{\bar{z}\bar{z}} - \frac{d}{2}\psi_{\bar{z}}^2 = \bar{E}(\bar{z}) \tag{8}$$

where $\bar{E}$ is some analytic function of $\bar{z}$.

Proof. Integrating (7) with respect to z gives

$$\psi_{\bar{z}} = c \int_{z_0}^{z} e^{d\psi}\, dz + \bar{F}(\bar{z})$$

for some arbitrary analytic function $\bar{F}(\bar{z})$. On differentiating with respect to $\bar{z}$ and using (7), we obtain

$$\psi_{\bar{z}\bar{z}} = \frac{d}{2}\psi_{\bar{z}}^2 + \bar{E}(\bar{z}) \tag{9}$$

where $\bar{E}(\bar{z}) = \bar{F}'(\bar{z})$. Hence Theorem 1 follows. □

THEOREM 2. Any real-valued solution $\psi(z, \bar{z})$ to (8) that is sufficiently differentiable with respect to $\bar{z}$ and z is also a solution to (7) for some real constant c.

Proof. A direct proof of this is possible—the general real solution of (8) can be found directly (see Theorem 3) and it can be checked by substitution that the resulting solutions satisfy (7) for some value of c. An alternative approach is to take the second derivative of (8) with respect to z giving

$$\psi_{\bar{z}\bar{z}zz} - d\psi_{\bar{z}}\psi_{\bar{z}zz} - d\psi_{\bar{z}z}^2 = 0. \tag{10}$$

Taking the complex conjugate of (10) gives

$$\psi_{zz\bar{z}\bar{z}} - d\psi_{z}\psi_{z\bar{z}\bar{z}} - d\psi_{z\bar{z}}^2 = 0. \tag{11}$$

We now define $\omega(z, \bar{z}) = \psi_{\bar{z}z}$. Subtracting (10) from (11) we obtain

$$\psi_{\bar{z}}\omega_z - \psi_z\omega_{\bar{z}} = 0. \tag{12}$$

It follows from (12) that $\omega = f(\psi)$ for some real-valued function f. Differentiating (8) once with respect to z yields

$$\psi_{\bar{z}\bar{z}z} - d\psi_{\bar{z}z}\psi_{\bar{z}} = 0. \tag{13}$$

Using $\omega(z, \bar{z}) = f(\psi)$ then implies

$$\psi_{\bar{z}}(f' - df) = 0 \tag{14}$$

from which it is concluded that any non-trivial real-valued solution $\psi(z, \bar{z})$ of (8) satisfies (7) for some constant c. □

THEOREM 3. Every real valued solution to (7) is of the form

$$\psi = -\frac{2}{d}\log[c_1 y_1(z)\bar{y}_1(\bar{z}) + c_4 y_2(z)\bar{y}_2(\bar{z}) + c_2 y_1(z)\bar{y}_2(\bar{z}) + \bar{c}_2\bar{y}_1(\bar{z})y_2(z)] \tag{15}$$

where $\bar{y}_1(\bar{z})$ and $\bar{y}_2(\bar{z})$ are two independent solutions to

$$y_{\bar{z}\bar{z}} + \frac{d}{2}\bar{E}(\bar{z})y = 0 \tag{16}$$

for some analytic $\bar{E}(\bar{z})$ while c_1 and c_4 are real constants and c_2 is some complex constant.

Remark 1. Real solutions are only defined in regions of the (x, y)-plane where the argument of the logarithm in (15) is positive.

Remark 2. The conjugate function $\bar{f}(z)$ is defined as

$$\bar{f}(z) = \overline{f(\bar{z})}.$$

Proof. From Theorem 1, it follows that a solution to (7) is also a solution to (8) for some $\bar{E}(\bar{z})$. Note that (8) is in the form of a Ricatti equation and can be made into the linear second order differential equation (16) for $y = e^{-d\psi/2}$. Therefore, it follows that

$$y(z, \bar{z}) = E_1(z)\bar{y}_1(\bar{z}) + E_2(z)\bar{y}_2(\bar{z}) \tag{17}$$

for some functions $E_1(z)$ and $E_2(z)$. Since ψ (and therefore y) is real, by taking the complex conjugate of (17), it follows that

$$y(z, \bar{z}) = \bar{E}_1(\bar{z})y_1(z) + \bar{E}_2(\bar{z})y_2(z). \tag{18}$$

Now, since (18) is a solution to (16), it follows that

$$\bar{E}_1(\bar{z}) = \bar{c}_1\bar{y}_1(\bar{z}) + \bar{c}_2\bar{y}_2(\bar{z}) \tag{19}$$

$$\bar{E}_2(\bar{z}) = \bar{c}_3\bar{y}_1(\bar{z}) + \bar{c}_4\bar{y}_2(\bar{z}) \tag{20}$$

for some constants c_1, c_2, c_3 and c_4. On substituting (19) and (20) back into (17) and (18) and equating the two different expressions for y, we obtain the condition that c_1 and c_4 are each real and that $c_2 = \bar{c}_3$. Thus,

$$y(z, \bar{z}) = c_1 y_1(z)\bar{y}_1(\bar{z}) + c_4 y_2(z)\bar{y}_2(\bar{z}) + c_2 y_1(z)\bar{y}_2(\bar{z}) + \bar{c}_2\bar{y}_1(\bar{z})y_2(z). \tag{21}$$

Thus, from the definition of y in terms of ψ, (15) follows. □

Remark 3. Since $\bar{E}(\bar{z})$ is some *arbitrary* analytic function, the requirement that $\bar{y}_1(\bar{z})$ and $\bar{y}_2(\bar{z})$ are independent solutions to (16) can be replaced by choosing $\bar{y}_1$ to be an arbitrary analytic function of $\bar{z}$ while determining $\bar{y}_2(\bar{z})$ from the condition that the wronskian $\bar{w}(\bar{z}) \equiv \bar{y}_1(\bar{z})\bar{y}_2'(\bar{z}) - \bar{y}_1'(\bar{z})\bar{y}_2(\bar{z}) = 1$ (this can be done without any loss of generality). Clearly, once $\bar{y}_1(\bar{z})$ is chosen, an expression for $\bar{E}(\bar{z})$ follows from (16). This unwieldy method of determining $\bar{y}_2(\bar{z})$ from the wronskian can be avoided by use of the following theorem:

THEOREM 4. Let $Y_1(z)$ and $Y_2(z)$ be two arbitrary but independent analytic functions of z. Denote their wronskian by $W(z) \equiv Y_1(z)Y_2'(z) - Y_1'(z)Y_2(z)$. Then $y_1(z) = Y_1(z)/\sqrt{W(z)}$ and $y_2(z) = Y_2(z)/\sqrt{W(z)}$ are two independent analytic functions with unit wronskian.

Proof. Since Y_1 and Y_2 are independent, then $W(z)$ is not zero. The relation $c_1 y_1 + c_2 y_2 = 0$ clearly implies $c_1 Y_1 + c_2 Y_2 = 0$ in some open set; from the independence of Y_1 and Y_2, this implies $c_1 = 0$, $c_2 = 0$; i.e. y_1 and y_2 are independent. On substituting for y_1 and y_2 in terms of Y_1

and Y_2, it follows that the wronskian $w(z)$ of y_1 and y_2 is

$$w(z)=\frac{Y_1(z)Y_2'(z)-Y_2(z)Y_1'(z)}{W(z)}=1. \tag{22}$$

Hence Theorem 4 is proved. □

THEOREM 5. Any real solution to (7) is of the form

$$\psi=-\frac{2}{d}\log[c_1Y_1(z)\bar{Y}_1(\bar{z})+c_4Y_2(z)\bar{Y}_2(\bar{z})+c_2Y_1(z)\bar{Y}_2(\bar{z})+\bar{c}_2\bar{Y}_1(\bar{z})Y_2(z)]+\frac{1}{d}\log[W(z)\bar{W}(\bar{z})] \tag{23}$$

for some independent analytic functions $Y_1(z)$ and $Y_2(z)$, where c_1 and c_4 are real constants and c_2 is a complex constant, while $W(z)$ is the wronskian of $Y_1(z)$ and $Y_2(z)$.

Proof. This follows by substituting for $y_1(z)$ and $y_2(z)$ in terms of functions $Y_1(z)$, $Y_2(z)$ and $W(z)$, as defined in Theorem 4, into (15). □

THEOREM 6. The most general real solution to (7) is given by (23), where $Y_1(z)$ and $Y_2(z)$ are any independent analytic functions of z, $W(z)$ is their wronskian, with real constants c_1 and c_4 and complex constant c_2 satisfying the constraint

$$cd=-2(c_1c_4-|c_2|^2) \tag{24}$$

but which are otherwise arbitrary.

Proof. Since we know any solution of (7) is of the form (23), by directly substituting (23) into equation (7), it is found that (7) is satisfied if and only if the constraint (24) is satisfied. □

According to Theorem 6 it should be possible to retrieve all known solutions of the elliptic Liouville equation as special cases of (23). The well-known general solution given by Liouville [1, 14, 15, 18] is trivially retrieved as a special case of this most general solution. Liouville's solution of (7) when $cd<0$, can be written

$$e^{d\psi}=-\frac{2}{cd}\frac{(u_x^2+u_y^2)}{(u^2+v^2+1)^2} \tag{25}$$

where u and v are arbitrary conjugate functions. This corresponds to the choice $Y_1(z)=f(z)=u+\mathrm{i}v$ where $f(z)$ is an arbitrary analytic function and $Y_2(z)=1$ with $c_1=c_4=\sqrt{-(cd/2)}$ and $c_2=0$. The resulting solution [using (23)] is

$$\psi=-\frac{2}{d}\log\left[\sqrt{-\frac{cd}{2}}(f(z)\bar{f}(\bar{z})+1)\right]+\frac{1}{d}\log[f'(z)\bar{f}'(\bar{z})]. \tag{26}$$

Observing that $f(z)\bar{f}(\bar{z})=u^2+v^2$ and $f'(z)\bar{f}'(\bar{z})=u_x^2+u_y^2$ we retrieve Liouville's solution (25) as a special case of (23). Stuart [15] lists a number of exact solutions of (7) including one that is similar to Liouville's solution for the case $cd>0$ in the form

$$e^{d\psi}=\frac{2}{cd}\frac{(u_x^2+u_y^2)}{(u^2+v^2-1)^2}. \tag{27}$$

This corresponds to $Y_1(z)=f(z)=u+\mathrm{i}v$ ($f(z)$ arbitrary) and $Y_2(z)=1$ with $c_1=\sqrt{cd/2}$, $c_4=-\sqrt{cd/2}$ and $c_2=0$. Stuart [15] also reports a class of solutions (attributed to Varley) for the case $cd<0$ in the form

$$e^{-d\psi/2}=\alpha_1(z)\bar{\alpha}_1(\bar{z})+\alpha_2(z)\bar{\alpha}_2(\bar{z}) \tag{28}$$

where $\alpha_1(z)$, $\alpha_2(z)$ are independent analytic functions of z satisfying the equation

$$f_{zz}-G(z)f=0 \tag{29}$$

with $\alpha_1(z)\alpha_2'(z) - \alpha_2(z)\alpha_1'(z) = \lambda$ and $|\lambda|^2 = -cd/2$ and $G(z)$ is an arbitrary analytic function of z. In fact, using the theorems in this paper, it can now be demonstrated that this general solution is equivalent to the *most* general solution for $cd < 0$. To see this, by combining and rewriting the various results of Theorems 1–6, it has now been established that the *most* general solution of (7) can be written

$$\psi = -\frac{2}{d}\log\left[c_1\left(y_1 + \frac{c_2}{c_1}y_2\right)\left(\bar{y}_1 + \frac{\bar{c}_2}{c_1}\bar{y}_2\right) - \frac{cd}{2c_1}y_2\bar{y}_2\right] \tag{30}$$

where y_1 and y_2 are independent solutions of (16) for some $E(z)$. If $cd < 0$ it is clear that in order for the argument of the logarithm in (30) to be positive then necessarily $c_1 > 0$. Identifying $\alpha_1(z) = \sqrt{c_1}(y_1 + (c_2/c_1)y_2)$ and $\alpha_2(z) = \sqrt{-cd/2c_1}\,y_2(z)$ it is seen that (28) is in fact equivalent to the *most* general solution for $cd < 0$. This very important and significant fact is not stated in Stuart [15], nor does it seem to have been acknowledged elsewhere in the literature. The three types of solution of $\nabla^2\psi = e^{\psi}$ (corresponding to $c = 1/4$, $d = 1$ in our notation) recently identified by Popov [16] using geometrical methods can be written

$$\psi = \log\left[\frac{2(v_x^2 + v_y^2)}{v^2}\right] \tag{31}$$

$$\psi = \log\left[\frac{2(v_x^2 + v_y^2)}{\sinh^2 v}\right] \tag{32}$$

$$\psi = \log\left[\frac{2(v_x^2 + v_y^2)}{\sin^2 v}\right] \tag{33}$$

where $v(x, y) = \mathrm{Re}[f(z)]$ and $f(z)$ is a general analytic function of $z = x + iy$. To retrieve (31) take $c_1 = c_4 = 0$, $c_2 = 1/\sqrt{8}$ with $Y_1(z) = f(z)$, $Y_2(z) = 1$ in (23). Noting that $v_x^2 + v_y^2 = f'(z)\bar{f}'(\bar{z})$ we retrieve the required result. To obtain (32) we take $c_1 = c_4 = 0$, $c_2 = 1/\sqrt{8}$ with $Y_1(z) = \sinh[f(z)/2]$, $Y_2(z) = \cosh[f(z)/2]$. Result (33) is obtained by taking $c_1 = c_4 = 0$, $c_2 = 1/\sqrt{8}$ with $Y_1(z) = \sin[f(z)/2]$, $Y_2(z) = \cos[f(z)/2]$.

Note that from the canonical form (6) it is clear that there are essentially two distinct types of elliptic Liouville equation depending on sgn$[cd]$ which, by (24), is the same as the sign of the determinant-like quantity $|c_2|^2 - c_1c_4$. The solutions in each case have somewhat different behaviours. In particular it is known from more general analysis [19, 20] that when $cd > 0$ the elliptic Liouville equation possesses no solution valid in the entire plane, while for $cd < 0$ it does possess such solutions. These properties can now be demonstrated explicitly for the elliptic Liouville equation using the above general representation of the solutions. For example, we briefly sketch a direct proof of the fact that for $cd > 0$, (7) has *no* solutions valid in the entire complex z-plane. The proof is by contradiction. Suppose there exists a solution of (7) for $cd > 0$ valid in the entire plane. Then by Theorems 1–6, the solution *necessarily* has the form (30) where $y_1(z)$ and $y_2(z)$ are independent solutions of (16) for some $E(z)$ and, without loss of generality, $c_1 > 0$. Since the solution is valid everywhere, $y_1(z)$ and $y_2(z)$ must be entire functions. Also, in order that the argument of the logarithm in (30) is strictly positive, the following inequality must hold *everywhere* in the finite z-plane:

$$\left|\sqrt{c_1}\left(y_1 + \frac{c_2}{c_1}y_2\right)\right| > \sqrt{\frac{cd}{2c_1}}\,|y_2|. \tag{34}$$

In addition, $y_1 + (c_2/c_1)y_2$ can have no zeros in the finite z-plane because the argument of the logarithm in (30) would fail to be strictly positive at any zero of $y_1 + (c_2/c_1)y_2$. Equation (34) thus implies that

$$\left|\frac{y_2(z)}{y_1(z) + (c_2/c_1)y_2(z)}\right| < \sqrt{\frac{2c_1^2}{cd}}. \tag{35}$$

However, since $y_1 + (c_2/c_1)y_2$ has no zeros and since $y_1(z)$ and $y_2(z)$ are entire then the function

$y_2(z)/[y_1(z)+(c_2/c_1)y_2(z)]$ is also entire. But (35) states that it is a *bounded* entire function which implies (by the Liouville theorem) that it must be a constant function. Finally, this then implies that $y_1(z)$ and $y_2(z)$ are linearly *dependent*, which is the required contradiction. It is a nice feature that the Liouville *theorem* proves to be the result from analytic function theory needed to prove this result on solutions to the Liouville *equation*.

Finally, we remark that the classical Dirichlet boundary value problem in a bounded domain with finite boundary values always has a unique solution when $cd>0$ (but not when $cd<0$—see for example [21]). Thus, as one example of the utility of the solutions presented here in solving real physical problems, it can be envisaged that the above representation of the most general solution, combined perhaps with conformal mapping techniques, might be used to solve such classical Dirichlet boundary value problems. The form given in (23) would seem to be the most convenient for such purposes.

3. THE HYPERBOLIC CASE

We now extend this analysis to the two-dimensional hyperbolic Liouville equation in the $(\tilde{x},\tilde{y})$-plane given by

$$\psi_{\tilde{x}\tilde{x}}-\psi_{\tilde{y}\tilde{y}}=\tilde{c}\mathrm{e}^{d\psi} \tag{36}$$

where $\tilde{c}$, d are again real constants, assumed to be non-zero. By shifting to characteristic coordinates (x,t) where $x=\tilde{x}+\tilde{y}$ and $t=\tilde{x}-\tilde{y}$ we can equivalently solve

$$\psi_{xt}=c\mathrm{e}^{d\psi} \tag{37}$$

where x and t are real coordinates, and $c=\tilde{c}/4$. Note that by the linear change of dependent variable (5) the canonical form for the hyperbolic Liouville equation (37) can be written

$$\phi_{xt}=\mathrm{sgn}[cd]\mathrm{e}^{\phi}. \tag{38}$$

We now demonstrate that the general solution of the hyperbolic Liouville equation depends on four arbitrary real functions, in contrast to two arbitrary analytic functions as in the elliptic case in Section 2.

THEOREM 7. Any function $\psi(x,t)$ that is twice differentiable with respect to both x and t and is a real solution to

$$\psi_{xt}=c\mathrm{e}^{d\psi} \tag{39}$$

simultaneously satisfies the two equations

$$\psi_{xx}-\frac{d}{2}\psi_x^2=E(x) \tag{40}$$

$$\psi_{tt}-\frac{d}{2}\psi_t^2=F(t) \tag{41}$$

for some choice of functions $E(x)$ and $F(t)$.

Proof. Integrating (39) with respect to t gives

$$\psi_x=c\int_{t_0}^{t}\mathrm{e}^{d\psi}\,\mathrm{d}t+G(x)$$

for some arbitrary real function $G(x)$. Differentiating this equation with respect to x and using (39) gives

$$\psi_{xx}-\frac{d}{2}\psi_x^2=E(x) \tag{42}$$

where $E(x) = G'(x)$. Hence ψ satisfies equation (40). By the symmetry of (39) in x and t, the same manipulations imply that ψ also satisfies equation (41) for some $F(t)$. Thus Theorem 7 follows. □

THEOREM 8. Any sufficiently differentiable solution of both (40) and (41) satisfies equation (39) for some choice of the constant c.

Proof. A direct proof of this is possible—general simultaneous solutions to equations (40) and (41) can be found directly (see Theorem 9), and it can be checked by substitution that these are solutions of (39). An alternative approach is to differentiate (40) twice with respect to t giving

$$\psi_{xxtt} - d\psi_{xtt}\psi_x - d\psi_{xt}^2 = 0. \tag{43}$$

Similarly, differentiating (41) twice with respect to x yields

$$\psi_{ttxx} - d\psi_{txx}\psi_t - d\psi_{tx}^2 = 0. \tag{44}$$

Subtracting (43) from (44) implies

$$\psi_{xtt}\psi_x - \psi_{txx}\psi_t = 0 \tag{45}$$

which implies

$$\psi_{xt} = f(\psi) \tag{46}$$

for some real function $f(\psi)$. Differentiating (40) once with respect to t and using (46) gives

$$\psi_x(f' - df) = 0 \tag{47}$$

thus implying that any non-trivial simultaneous solution of equations (40) and (41) satisfies (39) for some value of c. □

THEOREM 9. Every solution of (39) is of the form

$$\psi = -\frac{2}{d}\log[c_1 y_1(t)w_1(x) + c_2 y_1(t)w_2(x) + c_3 y_2(t)w_1(x) + c_4 y_2(t)w_2(x)]. \tag{48}$$

There $y_1(t)$, $y_2(t)$ are two independent solutions of

$$y_{tt} + \frac{d}{2}F(t)y = 0 \tag{49}$$

and $w_1(x)$, $w_2(x)$ are two independent solutions of

$$w_{xx} + \frac{d}{2}E(x)w = 0 \tag{50}$$

and c_1, c_2, c_3, c_4 are real constants.

Remark 4. Real solutions are defined in regions of the (x, t)-plane where the argument of the logarithm in (48) is positive.

Proof. From Theorem 7, solutions of (39) simultaneously satisfy (40) and (41) for some $E(x)$ and $F(t)$. Note that (40) is of the Ricatti form and can be made into a linear second order equation for $M(x, t) = e^{-d\psi/2}$. Using this transformation the resulting equation for $M(x, t)$ is (50) i.e.

$$M_{xx} + \frac{d}{2}E(x)M = 0. \tag{51}$$

Therefore,

$$M(x, t) = E_1(t)w_1(x) + E_2(t)w_2(x) \tag{52}$$

for some functions $E_1(t)$ and $E_2(t)$. Now since ψ is also a solution of (41) then $M(x, t)$ is also a solution to (49) and we deduce that

$$E_1(t) = c_1 y_1(t) + c_3 y_2(t) \tag{53}$$

and

$$E_2(t) = c_2 y_1(t) + c_4 y_2(t) \tag{54}$$

for some real constants c_1, c_2, c_3, c_4. Substituting (53) and (54) into (52) gives the result (48).

□

THEOREM 10. Any real solution to (39) is of the form

$$\psi = -\frac{2}{d}\log[c_1 Y_1(t)W_1(x) + c_2 Y_1(t)W_2(x) + c_3 Y_2(t)W_1(x) + c_4 Y_2(t)W_2(x)] + \frac{1}{d}\log[Y(t)W(x)] \tag{55}$$

where $Y_1(t)$, $Y_2(t)$ are independent sufficiently differentiable functions with wronskian $Y(t)$, $W_1(x)$, $W_2(x)$ are independent sufficiently differentiable functions with wronskian $W(x)$ and c_1, c_2, c_3, c_4 are real constants.

Proof. Analogous to proof of Theorems 4 and 5.

THEOREM 11. The most general real solution to (39) is given by (55), where $Y_1(t)$, $Y_2(t)$ are any independent functions with wronskian $Y(t)$, $W_1(x)$, $W_2(x)$ are any independent functions of x with wronskian $W(x)$ and c_1, c_2, c_3, c_4 are real constants satisfying the constraint

$$cd = -2(c_1c_4 - c_2c_3) \tag{56}$$

but which are otherwise arbitrary.

Proof. Since we know any solution of (39) is of the form (55), by substituting (55) into equation (39), we find that (39) is satisfied if and only if constraint (56) is satisfied. □

Theorem 11 implies that all known solutions of (39) should be retrievable as special cases of (55). For example, the choice

$$Y_1(t) = \sigma(t) + z_0, \qquad Y_2(t) = 1, \qquad W_1(x) = \theta(x), \qquad W_2(x) = 1 \tag{57}$$

with $c_1 = c_4 = 0$, $c_2 = c_3 = 1/\sqrt{2}$ and with $\sigma(t)$ and $\theta(x)$ arbitrary real functions and z_0 a real constant, represents a well-known general solution to the hyperbolic Liouville equation (39) (with $c = d = 1$)

$$\psi(x, t) = \log\left[\frac{2\theta'(x)\sigma'(t)}{(\theta(x) + \sigma(t) + z_0)^2}\right] \tag{58}$$

which coincides with the one obtained by Ibragimov [5] using Backlund transformation techniques, by Tamizhmani and Lakshmanan [6] using a Painleve analysis, and by Bhutani *et al.* [2] using a direct method for finding similarity solutions following the Clarkson and Kruskal formalism [17]. It is also the general solution which normally appears in text-books [22, 23]. The choice

$$Y_1(t) = \cosh\left(\frac{-\sqrt{C}(\sigma(t) + z_0)}{2}\right), \qquad Y_2(t) = \sinh\left(\frac{-\sqrt{C}(\sigma(t) + z_0)}{2}\right)$$

$$W_1(x) = \sinh\left(\frac{-\sqrt{C}\theta(x)}{2}\right), \qquad W_2(x) = \cosh\left(\frac{-\sqrt{C}\theta(x)}{2}\right) \tag{59}$$

with $c_2 = c_3 = 1/\sqrt{2}$ and $c_1 = c_4 = 0$ and where C is a real constant gives the solution

$$\psi = \log\left[-\left(\frac{C}{2}\right)\theta'(x)\sigma'(t)\mathrm{sech}^2\left(\frac{-\sqrt{C}}{2}(\theta(x) + \sigma(t) + z_0)\right)\right] \tag{60}$$

which is the solution of (39) (for $c = d = 1$) discovered recently by Bhutani *et al.* [2] using different methods and which can be related to that found in [3] using an isovector approach. The choice

$$Y_1(t) = \cos\left(\frac{\sqrt{C}(\sigma(t) + z_0)}{2}\right), \qquad Y_2(t) = -\sin\left(\frac{\sqrt{C}(\sigma(t) + z_0)}{2}\right)$$

$$W_1(x) = \sin\left(\frac{\sqrt{C}\theta(x)}{2}\right), \qquad W_2(x) = \cos\left(\frac{\sqrt{C}\theta(x)}{2}\right) \tag{61}$$

with $c_2 = c_3 = 1/\sqrt{2}$ and $c_1 = c_4 = 0$ gives the solution

$$\psi = \log\left[\left(\frac{C}{2}\right)\theta'(x)\sigma'(t)\sec^2\left(\frac{\sqrt{C}}{2}(\theta(x) + \sigma(t) + z_0)\right)\right] \tag{62}$$

which corresponds to a solution of (39) (for $c = d = 1$) reported in Ibragimov [5] (when $C = 4$) and which was also retrieved by Bhutani *et al.* [2].

Acknowledgements—The author would especially like to thank Professor S. Tanveer of Ohio State University for help and advice in the preparation of this paper. The author also thanks David Hill for useful discussions and Professor P. G. Saffman for his encouragement, and acknowledges support from Department of Energy, Grant No. DE-FG03-89ER25073.

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(Received 20 November 1995; accepted 11 July 1996)