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Explicit integral solutions for the plane elastostatic semi-strip

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A new method for solving the biharmonic equation in an arbitrary convex polygon with arbitrary linear boundary conditions is applied to a class of mixed boundary-value problems involving a semi-infinite strip. Emphasis is placed on boundary-value problems for which an explicit solution can be constructed. A variety of mixed boundary-value problems are shown to admit explicit solutions. This class includes, but is not limited to, the canonical problems of the elastostatic semi-strip. New integral representations of generalizations of these canonical problems are derived where the sidewall boundary conditions are inhomogeneous.

Keywords: Riemann–Hilbert; biharmonic; elastostatics; semi-strip

1. Introduction

A new method for solving linear partial differential equations in a plane convex polygon is presented in Fokas (2001). Examples of the partial differential equations considered include evolution equations as well as elliptic-type equations such as the Laplace (Fokas & Kapaev 2003) and modified Helmholtz equations (Antipov & Fokas 2004; ben-Avraham & Fokas 2001). Applications of the method to physical problems include solutions of the modified Helmholtz equation in triangular regions—a problem of relevance to the study of diffusion-limited coalescence (ben-Avraham & Fokas 2001). Recently, Crowdy & Fokas (2004) have shown how this method can be implemented for problems involving the biharmonic equation in a convex polygon with linear boundary conditions. The latter mathematical problem appears in a wide range of physical applications, such as plane elastostatics and slow viscous flows of Newtonian fluids.

Although the new method applies to biharmonic problems in an *arbitrary* simply connected convex polygon, various elastostatics problems involving the semi-infinite strip have received such wide and varied attention over the last century that it seems appropriate to present, in detail, the application of the new general method to this particular class of problems. Boundary-value problems for the elastostatic semi-strip are relevant to the bending of plates, the stretching, bending and flexure of cantilever beams, and the estimation of boundary-layer effects in plate and shell

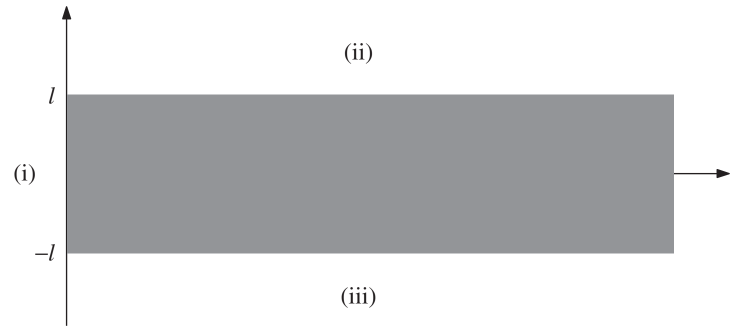

Figure 1. Domain and boundary conditions: (i) $\sin\beta_2 q_x + \cos\beta_2 q_{xx} = f_2(y)$, $\sin B_2 q_{xxx} + \cos B_2 q_{yy} = F_2(y)$; (ii) $\cos\beta_3 q_{xx} + \sin\beta_3 q_{yy} = f_3(y)$, $\cos B_3 q_{yx} + \sin B_3 q_{yyy} = F_3(y)$; (iii) $\cos\beta_1 q_{xx} + \sin\beta_1 q_{yy} = f_1(y)$, $\cos B_1 q_{yx} + \sin B_1 q_{yyy} = F_1(y)$.

theories. Previous treatments of this general class of problem can be found in the literature (Benthem 1963; Bogy 1975; Gregory 1979, 1980; Gupta 1973; Johnson & Little 1965; Joseph & Sturges 1978; Smith 1952; Spence 1978, 1982, 1983; Vorovich & Kopasenko 1966; Williams 1952).

The new method presented here involves the following.

(1) Given an arbitrary convex polygon in the complex z-plane, construct an integral representation in the complex k-plane of the solution $q(z,\bar{z})$ of the biharmonic equation. The representation involves certain functions $\rho(k)$ and $\tilde{\rho}(k)$, which we call the *spectral functions*.

(2) The spectral functions are expressed as integrals in the complex z-plane over the boundary of the polygon involving derivatives of q.

(3) The spectral functions are not independent but they satisfy two relations known as the *global relations*.

It is emphasized that the construction is valid for an arbitrary polygon and for arbitrary linear boundary conditions. However, for a given boundary-value problem only a subset of the derivatives of q on the boundary are known. Thus, to compute $\{\rho(k), \tilde{\rho}(k)\}$ one must analyse the global relations in order to express the unknown part of $\{\rho(k), \tilde{\rho}(k)\}$ in terms of the given boundary conditions. This analysis is, in general, complicated and leads to a *matrix differential Riemann–Hilbert problem.* However, for some simple polygons and boundary conditions this Riemann–Hilbert problem can be solved in closed form. In the present paper, to illustrate the new method in its simplest form, we concentrate on precisely this class of 'exact' elastostatic problems in the case where the polygon is the semi-infinite strip.

In § 2, we review the key elements of the general method presented in Crowdy & Fokas (2004). In § 3, we review some elements of the theory of plane elasticity and describe certain classes of boundary-value problems arising therein. A typical class of such problems is illustrated in figure 1, where $\{\beta_j, B_j \mid j = 1, 2, 3\}$ are arbitrary constants and $\{f_j, F_j \mid j = 1, 2, 3\}$ are arbitrary functions with appropriate smoothness and decay. In § 4, we show how the general method described in § 2 can be applied to this class of problems. In particular, we demonstrate how to compute the associated spectral functions.

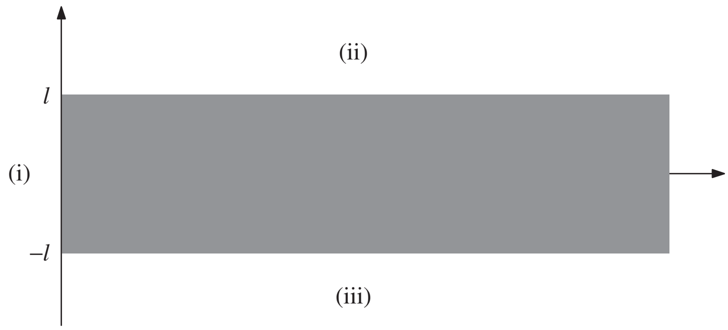

Figure 2. Boundary-value problem for proposition 1.1: (i) $q_x = f_2(y)$, $q_{xxx} = F_2(y)$; (ii) $\cos\beta q_{xx} + \sin\beta q_{yy} = f_1(y)$, $\cos B q_{yx} + \sin B q_{yyy} = -F_1(y)$; (iii) $\cos\beta q_{xx} + \sin\beta q_{yy} = f_1(y)$, $\cos B q_{yx} + \sin B q_{yyy} = F_1(y)$.

Finally, § 5 isolates a particular subclass of the boundary-value problems considered in § 3 having the special property that the spectral functions can be determined in closed form. An explicit integral representation of the solution can then be derived. The main result of § 5 is summarized in the following proposition.

Proposition 1.1. *Let $q(x,y)$ be a real-valued function satisfying the biharmonic equation in $x > 0$ and $|y| \leqslant l$ and let $q(x,y)$ decay sufficiently fast as $x \to \infty$. Let $q(x,y)$ satisfy the boundary conditions depicted in figure 2, where B and β are constants, $f_1(x)$ and $F_1(x)$ decay sufficiently fast as $x \to \infty$ and $f_2(y)$ and $F_2(y)$ are even functions of y. Then*

$$q_{zz} = Q_1(z) + \bar{z} Q_2(z), \tag{1.1}$$

where $Q_1(z)$ and $Q_2(z)$ are defined by

$$\left.\begin{aligned} Q_1(z) &= \frac{1}{8}\int_{-\infty}^{\infty} \frac{g(k)}{\mathcal{E}(k)} \frac{\Phi_1(\mathrm{i}k)\mathrm{e}^{\mathrm{i}kz}}{\sinh kl}\,\mathrm{d}k + \int_{l_1} \mathrm{e}^{\mathrm{i}kz}\tilde{G}_1(k)\,\mathrm{d}k + \int_{l_3} \mathrm{e}^{\mathrm{i}kz}\tilde{G}_3(k)\,\mathrm{d}k \\ &\quad + \int_{l_2} \mathrm{e}^{\mathrm{i}kz}(G_2(k) - (\bar{G}_1(k) + \bar{G}_2(k) + \bar{G}_3(k)))\,\mathrm{d}k, \\ Q_2(z) &= \int_{l_1} \mathrm{e}^{-kl+\mathrm{i}kz}\mathcal{R}_1(k)\,\mathrm{d}k + \int_{l_3} \mathrm{e}^{kl+\mathrm{i}kz}\mathcal{R}_3(k)\,\mathrm{d}k \\ &\quad + \int_{l_1} \mathrm{e}^{\mathrm{i}kz}\mathcal{G}_1(k)\,\mathrm{d}k + \int_{l_2} \mathrm{e}^{\mathrm{i}kz}\mathcal{G}_2(k)\,\mathrm{d}k + \int_{l_3} \mathrm{e}^{\mathrm{i}kz}\mathcal{G}_3(k)\,\mathrm{d}k, \end{aligned}\right\} \tag{1.2}$$

where an overbar denotes complex conjugation and the complex conjugate function $\bar{f}$ of a given function $f(k)$ is defined as

$$\bar{f}(k) = \overline{f(\bar{k})}. \tag{1.3}$$

Note that

(i) the contours l_j are the rays given in figure 3;

(ii) the functions $\{G_j, \mathcal{G}_j\}$ can be computed in terms of the given boundary conditions to within a finite set of unknown parameters, which, however, do *not*

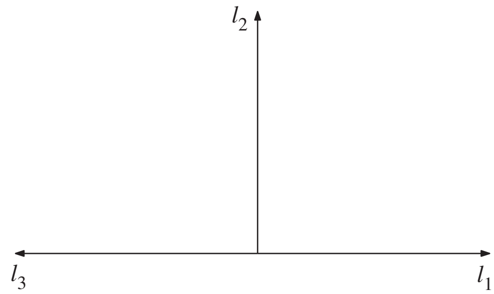

Figure 3. Rays of integration

contribute to the solution (expressions for these functions are given in § 5 using functions defined in the appendix);

(iii) the function $\Phi_1(\mathrm{i}k)$ solves the following scalar Riemann–Hilbert problem:

(a) $\Phi_1(\mathrm{i}k)$ is holomorphic for $\mathrm{Im}[k] > 0$;

(b) $\Phi_1(\mathrm{i}k) = \mathcal{O}(k)$ as $k \to \infty$;

(c) $\Phi_1(\mathrm{i}k)$ satisfies

$$\bar{\mathcal{E}}(k)\Phi_1(\mathrm{i}k) - \mathcal{E}(k)\Phi_1(-\mathrm{i}k) = G_6(k), \quad k \in \mathbb{R}, \tag{1.4}$$

where

$$\mathcal{E}(k) \equiv \mathcal{C}(k) + \mathrm{i}\mathcal{D}(k)\left(\frac{\bar{\alpha}(k)}{k\Delta}\right)\coth kl, \tag{1.5}$$

with

$$\left.\begin{aligned}\mathcal{C}(k) &\equiv -(kl\sin B + \mathrm{i}l\cos B)\sinh kl + 2\sin B\cosh kl,\\ \mathcal{D}(k) &\equiv -\mathrm{i}\Sigma\sinh kl - \mathrm{i}kl\Delta\cosh kl,\end{aligned}\right\} \tag{1.6}$$

and

$$\Sigma = \cos\beta + \sin\beta, \qquad \Delta = \cos\beta - \sin\beta, \qquad \alpha(k) = k\sin B - \mathrm{i}\cos B; \tag{1.7}$$

(iv) the functions $\mathcal{R}_1(k)$ and $\mathcal{R}_3(k)$ are given by

$$\left.\begin{aligned}\mathcal{R}_1(k) &= \left(\frac{2\mathrm{i}\sin B\bar{\mathcal{E}}(k) - \Sigma\mathcal{F}(k)}{4\mathcal{E}(k)} - \frac{\mathrm{i}lA_1(k)}{8} - \frac{\mathrm{i}A_1'(k)}{8} - \frac{\mathrm{i}\sin B}{2}\right)\Phi_1(\mathrm{i}k)\\ &\qquad - \frac{\mathrm{i}A_1(k)}{8}\frac{\mathrm{d}\Phi_1(\mathrm{i}k)}{\mathrm{d}k},\\ \mathcal{R}_3(k) &= \left(\frac{2\mathrm{i}\sin B\bar{\mathcal{E}}(k) + \Sigma\mathcal{F}(k)}{4\mathcal{E}(k)} + \frac{\mathrm{i}A_3(k)l}{8} - \frac{\mathrm{i}A_3'(k)}{8} - \frac{\mathrm{i}\sin B}{2}\right)\Phi_1(\mathrm{i}k)\\ &\qquad - \frac{\mathrm{i}A_3(k)}{8}\frac{\mathrm{d}\Phi_1(\mathrm{i}k)}{\mathrm{d}k},\end{aligned}\right\} \tag{1.8}$$

where

$$A_1(k) = (1 + \coth kl)\frac{g(k)}{\mathcal{E}(k)}, \qquad A_3(k) = (1 - \coth kl)\frac{g(k)}{\mathcal{E}(k)}, \tag{1.9}$$

with

$$g(k) = \alpha(k)\mathcal{C}(k) - \bar{\alpha}(k)\overline{\mathcal{C}}(k) \tag{1.10}$$

and

$$\mathcal{F}(k) = -\frac{\mathrm{i}g(k)\coth kl}{k\Delta}. \tag{1.11}$$

2. The new method

Let the real-valued function $q(x, y)$ satisfy the biharmonic equation

$$\nabla^4 q = 0, \tag{2.1}$$

in a domain $D \subset \mathbb{R}^2$. If $z = x + \mathrm{i}y$ and $\bar{z} = x - \mathrm{i}y$, then (2.1) becomes

$$\frac{\partial^4 q}{\partial z^2 \partial \bar{z}^2} = q_{zz\bar{z}\bar{z}} = 0, \tag{2.2}$$

where subscripts denote differentiation. Let $W(z, \bar{z}, k)$ be the following differential 1-form,

$$W(z, \bar{z}, k) = \mathrm{e}^{-\mathrm{i}kz}(q_{zz\bar{z}} + \lambda(q_{zz} - \bar{z}q_{zz\bar{z}}))\,\mathrm{d}z, \quad k \in \mathbb{C}, \tag{2.3}$$

where λ is an arbitrary constant. Then

$$\mathrm{d}W = \mathrm{e}^{-\mathrm{i}kz}[q_{zz\bar{z}} + \lambda(q_{zz} - \bar{z}q_{zz\bar{z}})]_{\bar{z}}\,\mathrm{d}\bar{z} \wedge \mathrm{d}z. \tag{2.4}$$

Therefore, if q satisfies the biharmonic equation (2.2), then $\mathrm{d}W = 0$, i.e. W is a *closed* 1-form.

Suppose D is a simply connected domain. Then because W is a closed form,

$$\oint_{\partial D} W = 0, \tag{2.5}$$

where ∂D denotes the boundary of D. Therefore, since λ is arbitrary, it follows that

$$\oint_{\partial D} \mathrm{e}^{-\mathrm{i}kz} q_{zz\bar{z}}\,\mathrm{d}z = 0, \quad k \in \mathbb{C}, \tag{2.6}$$

and

$$\oint_{\partial D} \mathrm{e}^{-\mathrm{i}kz}(q_{zz} - \bar{z}q_{zz\bar{z}})\,\mathrm{d}z = 0, \quad k \in \mathbb{C}. \tag{2.7}$$

Equations (2.6) and (2.7) will be referred to henceforth as the *global relations*.

We now concentrate on the case where D is a bounded, convex *polygon* with corners at the points $z_1, z_2, \ldots, z_{n+1} = z_1$. The straight line segment joining z_j and z_{j+1} will be called side j (see figure 4). The global relations (2.6) and (2.7) become

$$\sum_{j=1}^{n} \rho_j(k) = 0, \quad \sum_{j=1}^{n} \tilde{\rho}_j(k) = 0, \quad k \in \mathbb{C}, \tag{2.8}$$

where

$$\rho_j(k) = \int_{z_{j+1}}^{z_j} \mathrm{e}^{-\mathrm{i}kz} q_{zz\bar{z}}\,\mathrm{d}z, \qquad \tilde{\rho}_j(k) = \int_{z_{j+1}}^{z_j} \mathrm{e}^{-\mathrm{i}kz}(q_{zz} - \bar{z}q_{zz\bar{z}})\,\mathrm{d}z. \tag{2.9}$$

The functions $\{\rho_j(k), \tilde{\rho}_j(k)\}$ will be called the *spectral functions*.

Since the global relations involve the spectral functions, it is desirable to express the solution $q(x, y)$ in terms of $\{\rho_j(k), \tilde{\rho}_j(k)\}$ $(j = 1, 2, \ldots, n)$. Such an expression is given in proposition 2.1.

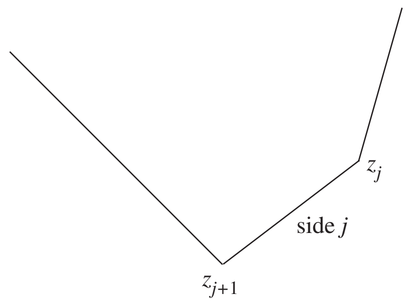

Figure 4. Definition of the jth side.

Proposition 2.1. *Let q be a real, biharmonic function in an arbitrary convex bounded n-sided polygon D. Then, $q_{zz} \equiv Q_1(z) + \bar{z}Q_2(z)$, where $Q_1(z)$ and $Q_2(z)$ are given explicitly as functions of z by the integral representations*

$$Q_1(z) = \frac{1}{2\pi}\sum_{j=1}^{n}\int_{l_j} \mathrm{e}^{\mathrm{i}kz}\tilde{\rho}_j(k)\,\mathrm{d}k, \qquad Q_2(z) = \frac{1}{2\pi}\sum_{j=1}^{n}\int_{l_j} \mathrm{e}^{\mathrm{i}kz}\rho_j(k)\,\mathrm{d}k, \tag{2.10}$$

where $l_j,\ j = 1,\dots,n$, are the rays in the complex k-plane

$$l_j = \{k \in \mathbb{C} : \arg(k) = -\arg(z_j - z_{j+1})\}, \quad z_{n+1} = z_1, \tag{2.11}$$

oriented from zero to infinity.

In essence, the result stated in proposition 2.1 is obtained by employing a generalized form of the standard Fourier transform (usually applied to functions analytic in an infinite strip) applied to the case of functions which are analytic in a more general convex polygon. Details of the derivation of this generalized Fourier transform can be found in Fokas (2001), where a transform method for functions harmonic in a simply connected convex polygon is derived. Proposition 2.1 generalizes this result to the case of functions which are *bi*harmonic in a convex polygon (see also Crowdy & Fokas 2004).

The integral representation for q_{zz} contains explicit dependence on z. Thus, if q_{zz} is known, q can be obtained by direct integration with respect to z, where all the unknown functions of $\bar{z}$ can be specified by the requirement that q is real. This determines q up to an inconsequential arbitrary real linear function which does not affect the physical stresses (which are given by second derivatives of q) and simply corresponds to an arbitrary uniform displacement of points in the elastic material.

3. Elastostatics in a semi-infinite strip

The boundary-value problems for the semi-infinite strip considered in this paper have been motivated by various mixed boundary-value problems arising in plane elasticity. Two such problems are that of plane strain (Spence 1983) and the bending of a thin plate (Smith 1952). For additional background material on what follows, we refer the reader to Love (1944). It is assumed that q is a real function in the semi-infinite strip $x > 0$, $|y| < l$, where l is some positive real constant. In plane strain, $q(x, y)$ corresponds to the *Airy stress function*, the physical stresses σ_{ij} being given by

$$\sigma_{11} = q_{yy}, \qquad \sigma_{12} = -q_{xy}, \qquad \sigma_{22} = q_{xx}. \tag{3.1}$$

In the bending problem, the biharmonic function $q(x,y)$ is the *deflection*. In either application, two boundary conditions are required on each of the sidewalls and on the end-strip. The two sidewall conditions are usually taken to be identical and homogeneous. Two common choices are *clamped* and *traction-free* sidewalls. In plane strain, these correspond respectively to

$$(1-\nu)q_{yy} - \nu q_{xx} = 0, \quad (1-\nu)q_{yyy} + (2-\nu)q_{xxy} = 0, \quad \text{on } y = \pm l \tag{3.2}$$

(here ν is a material parameter (Love 1944)) and

$$q_{xy} = q_{yy} = 0 \quad \text{on } y = \pm l. \tag{3.3}$$

In the bending problem, the clamped and traction-free sidewall conditions correspond to

$$q = q_y = 0 \quad \text{on } y = \pm l \tag{3.4}$$

and

$$q_{yy} + \nu q_{xx} = 0, \quad q_{yyy} + (2-\nu)q_{xxy} = 0, \quad \text{on } y = \pm l, \tag{3.5}$$

respectively.

Motivated by these, we now present the particular class of boundary conditions to be considered in detail in this paper.

Sidewall boundary conditions. The sidewall boundary conditions on the two semi-infinite boundaries $y = \pm l$, $x > 0$, are taken to be

$$\left.\begin{aligned} \cos\beta_1\ q_{xx} + \sin\beta_1\ q_{yy} &= f_1(x), \\ \cos B_1\ q_{yx} + \sin B_1\ q_{yyy} &= F_1(x), \end{aligned}\right\} \quad \text{on } y = -l,\ x > 0, \tag{3.6}$$

where β_1 and B_1 are real constants and $f_1(x)$, $F_1(x)$ are given real functions which decay appropriately as $x \to \infty$, and

$$\left.\begin{aligned} \cos\beta_3\ q_{xx} + \sin\beta_3\ q_{yy} &= f_3(x), \\ \cos B_3\ q_{yx} + \sin B_3\ q_{yyy} &= F_3(x), \end{aligned}\right\} \quad \text{on } y = +l,\ x > 0, \tag{3.7}$$

where β_3 and B_3 are real constants and $f_3(x)$, $F_3(x)$ are given real functions which decay appropriately as $x \to \infty$.

End-strip boundary conditions. The end-strip boundary conditions we consider are

$$\left.\begin{aligned} \sin\beta_2\ q_x + \cos\beta_2\ q_{xx} &= f_2(y), \\ \sin B_2\ q_{xxx} + \cos B_2\ q_{yy} &= F_2(y), \end{aligned}\right\} \quad \text{on } x = 0,\ |y| < l, \tag{3.8}$$

where β_2 and B_2 are real constants and $f_2(y)$, $F_2(y)$ are given real functions.

To examine these choices, consider first the sidewall conditions. Observe that the choices $\beta_j = \pi/2, B_j = 0$ and $f_j(x) = F_j(x) = 0$ ($j = 1$ and 3) yield the traction-free sidewall conditions in the case of plane strain. Furthermore, with the choices $\beta_j = B_j = 0$ and $f_j(x) = F_j(x) = 0$ ($j = 1$ and 3) we retrieve the case of clamped sidewalls in the bending problem since (3.6) reduces to

$$q_{xx} = 0, \qquad q_{xy} = 0, \tag{3.9}$$

which are relevant when conditions (3.4) hold.

The following end-strip boundary conditions have a particular importance:

$$q_{yy} = f_2(y), \quad q_{xx} = F_2(y), \quad \text{on } x = 0, \ |y| \leqslant l, \tag{3.10}$$

and

$$q_x = f_2(y), \quad q_{xxx} = F_2(y), \quad \text{on } x = 0, \ |y| \leqslant l. \tag{3.11}$$

The two special cases (3.10) and (3.11) are associated with two particular boundary-value problems, which will be referred to as problems I and II, respectively.

Problem I. $\beta_j = B_j = 0$; $\beta_2 = B_2 = 0$; $f_j(x) = F_j(x) = 0$, $j = 1, 3$.

Problem II. $\beta_j = B_j = 0$; $\beta_2 = B_2 = \pi/2$; $f_j(x) = F_j(x) = 0$, $j = 1, 3$.

In the elastostatics literature, problems I and II are referred to as *canonical* (see, for example, Spence 1983) because the general solutions can be written down explicitly by exploiting certain biorthogonality relations between the Papkovich–Fadle eigenfunctions of the strip. This fact appears to have been first discovered by Smith (1952) in his consideration of the bending problem where the sidewalls are clamped and the end-strip is loaded by moments—a problem that is mathematically identical to problem I. Problems for which the biorthogonality conditions are not sufficient to determine the solution in closed form are called non-canonical.

Given the nature of the sidewall conditions (3.2) and (3.5), it would seem natural to consider the class of generalized sidewall conditions given by

$$\left.\begin{aligned} \cos\beta_j \ q_{xx} + \sin\beta_j \ q_{yy} &= f_j(x), \\ \cos B_j \ q_{xxy} + \sin B_j \ q_{yyy} &= F_j(x), \end{aligned}\right\} \quad \text{on } y = +l, \ x > 0, \tag{3.12}$$

for $j = 1$ and 3. The general method to be presented here certainly applies to the choice (3.12). However, it is our aim here to show that when the two choices of end-strip conditions (3.10) and (3.11) are made, there are large classes of sidewall boundary conditions which admit explicit solutions. If, instead, the class of sidewall conditions (3.12) is considered, then inspection reveals that the mathematical problem is invariant to the transformation $x \mapsto -x$, which implies that a combination of standard transform methods (e.g. a cosine transform) could be applied to obtain closed-form solutions. While we might have used the new method presented here to retrieve these solutions, we would have presented nothing new mathematically. Therefore, to illustrate the mathematical possibilities associated with the new method, a deliberate modification of the second boundary condition in (3.12) is made. The q_{xxy} term has been replaced by q_{xy}, thus leading to our choice of sidewall conditions (3.7). This choice eliminates the invariance of the problem under the transformation $x \mapsto -x$ and thus precludes use of standard transform techniques such as the cosine transform. Nevertheless, when the new method is applied to this problem (explicit details are given later) it is found that the problem admits closed-form solutions. Since the purpose of this paper is to present a new mathematical method, we do not here offer details of any specific physical problems in which these modified mathematical problems might arise.

In writing down all the above boundary conditions, it is assumed that the physical problem has been suitably non-dimensionalized. For example, suppose that a physical problem gives rise, on the sidewall, to

$$\gamma q_{\tilde{x}\tilde{y}} + \delta q_{\tilde{y}\tilde{y}\tilde{y}} = \tilde{g}(\tilde{x}), \quad 0 < \tilde{x} < \infty, \tag{3.13}$$

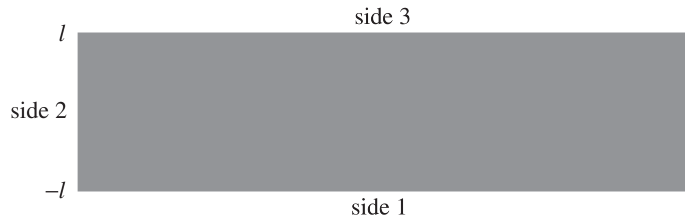

Figure 5. Semi-infinite strip and definition of sides 1, 2 and 3.

where $\tilde{x}$ and $\tilde{y}$ are dimensional coordinates. For dimensional consistency, it is clear that at least one of the parameters γ and δ must be dimensional. However, by non-dimensionalizing, (3.13) can be written

$$q_{xy} + \eta q_{yyy} = g(x),$$

where η is some non-dimensional parameter and x and y are non-dimensional lengths. Now letting

$$\frac{1}{\sqrt{1+\eta^2}} = \cos\beta, \qquad \frac{\eta}{\sqrt{1+\eta^2}} = \sin\beta, \qquad \frac{g(x)}{\sqrt{1+\eta^2}} \equiv f(x),$$

we produce the general form of the boundary conditions given above.

The authors are not aware of any general theorems on existence and uniqueness for general boundary-value problems in the semi-strip. Another concern is that in the case of general boundary-value problems involving a semi-strip, the solution q can be singular at the corners. A general methodology for treating corner singularities is described in respect of the Laplace equation in Fokas & Kapaev (2003). There exist two 'sources' of singularities: one is 'external' and is due to a lack of smoothness of the given boundary conditions; the other is 'internal' and depends on the angle of the corner and on the type of boundary conditions given on the neighbouring edges. For example, in the case of the Laplace equation it is known that if Dirichlet or Neumann boundary conditions are prescribed on both sides of a corner of angle ϕ, then an 'internal' singularity is possible only if $\phi \geqslant \pi$ (which cannot happen for a convex polygon). However, if a Neumann condition is applied on one edge and a Dirichlet condition on the other, then an internal singularity is possible if $\phi \geqslant \pi/2$. The occurrence and nature of the singularities can be determined by a *local* analysis, but the coefficients of the leading terms depend on the *global* analysis. The situation with the biharmonic equation is expected to be similar and, in this paper, we assume that there exists no such corner singularities.

Since the purpose of this paper is to present an example of a new constructive method, in what follows it is assumed that a solution to any particular boundary-value problem considered here exists and is unique. An advantage of the new method is that it can both address the question of existence and provide an algorithmic approach to characterizing the coefficients of the leading terms in the expansion of the solution at the singularities. For the special class of problems considered in § 5, any global information deriving from the corners cancels out in the final solution. It is therefore not necessary to perform any local analysis at the corners in order to derive the solution representations. This is consistent with our *a priori* assumption that no such corner singularities arise in the class of problems considered here.

4. Application of the method to the semi-strip

Consider the case of the semi-infinite strip depicted in figure 5. Let the corners of the polygon be given by $z_1 = -\infty - \mathrm{i}l$, $z_2 = -\mathrm{i}l$, $z_3 = \mathrm{i}l$ and $z_4 = \infty + \mathrm{i}l$, so that sides 1, 2 and 3 are defined as in figure 5. Although the results of proposition 2.1 are stated for the case of a bounded polygon, a similar result is valid for an unbounded polygon with $z_4 = z_1 = \infty$. For the semi-infinite strip shown in figure 5, (2.10) yields the following integral representation of the solution

$$q_{zz} = \frac{\bar{z}}{2\pi} \sum_{j=1}^{3} \int_{l_j} \mathrm{e}^{\mathrm{i}kz} \tilde{\rho}_j(k)\,\mathrm{d}k + \frac{1}{2\pi} \sum_{j=1}^{3} \int_{l_j} \mathrm{e}^{\mathrm{i}kz} \rho_j(k)\,\mathrm{d}k, \tag{4.1}$$

where l_j are the rays depicted in figure 3.

Spectral functions. Equations (2.9) yield

$$\left.\begin{aligned} \rho_1(k) &= \int_{-\mathrm{i}l}^{-\mathrm{i}l+\infty} (q_{zz} - \bar{z}q_{zz\bar{z}})\mathrm{e}^{-\mathrm{i}kz}\,\mathrm{d}z, \\ \rho_2(k) &= \int_{\mathrm{i}l}^{-\mathrm{i}l} (q_{zz} - \bar{z}q_{zz\bar{z}})\mathrm{e}^{-\mathrm{i}kz}\,\mathrm{d}z, \\ \rho_3(k) &= \int_{\mathrm{i}l+\infty}^{\mathrm{i}l} (q_{zz} - \bar{z}q_{zz\bar{z}})\mathrm{e}^{-\mathrm{i}kz}\,\mathrm{d}z, \end{aligned}\right\} \tag{4.2}$$

and

$$\left.\begin{aligned} \tilde{\rho}_1(k) &= \int_{-\mathrm{i}l}^{-\mathrm{i}l+\infty} q_{zz\bar{z}}\mathrm{e}^{-\mathrm{i}kz}\,\mathrm{d}z, \\ \tilde{\rho}_2(k) &= \int_{\mathrm{i}l}^{-\mathrm{i}l} q_{zz\bar{z}}\mathrm{e}^{-\mathrm{i}kz}\,\mathrm{d}z, \\ \tilde{\rho}_3(k) &= \int_{\mathrm{i}l+\infty}^{\mathrm{i}l} q_{zz\bar{z}}\mathrm{e}^{-\mathrm{i}kz}\,\mathrm{d}z. \end{aligned}\right\} \tag{4.3}$$

It is noted that $\{\rho_2(k), \tilde{\rho}_2(k)\}$ are entire functions of k, while $\{\rho_1(k), \tilde{\rho}_1(k)\}$ and $\{\rho_3(k), \tilde{\rho}_3(k)\}$ are analytic in $\mathrm{Im}[k] < 0$.

Using the facts that $\bar{z} = z + 2\mathrm{i}l$ on side 1, $\bar{z} = -z$ on side 2 and $\bar{z} = z - 2\mathrm{i}l$ on side 3, equations (4.2) and (4.3) imply the following differential relations

$$\left.\begin{aligned} \rho_1(k) &= \int_{-\mathrm{i}l}^{-\mathrm{i}l+\infty} q_{zz}\mathrm{e}^{-\mathrm{i}kz}\,\mathrm{d}z - \mathrm{i}\frac{\mathrm{d}\tilde{\rho}_1(k)}{\mathrm{d}k} - 2\mathrm{i}l\tilde{\rho}_1(k), \\ \rho_2(k) &= \int_{\mathrm{i}l}^{-\mathrm{i}l} q_{zz}\mathrm{e}^{-\mathrm{i}kz}\,\mathrm{d}z + \mathrm{i}\frac{\mathrm{d}\tilde{\rho}_2(k)}{\mathrm{d}k}, \\ \rho_3(k) &= \int_{\mathrm{i}l+\infty}^{\mathrm{i}l} q_{zz}\mathrm{e}^{-\mathrm{i}kz}\,\mathrm{d}z - \mathrm{i}\frac{\mathrm{d}\tilde{\rho}_3(k)}{\mathrm{d}k} + 2\mathrm{i}l\tilde{\rho}_3(k). \end{aligned}\right\} \tag{4.4}$$

Boundary conditions. Let $q(x, y)$ satisfy the following boundary conditions.

Side 1. $\cos\beta_1 q_{xx} + \sin\beta_1 q_{yy} = f_1(x)$, $\cos B_1 q_{xy} + \sin B_1 q_{yyy} = F_1(x)$.

Side 2. $\sin\beta_2 q_x + \cos\beta_2 q_{xx} = f_2(y)$, $\sin B_2 q_{xxx} + \cos B_2 q_{yy} = F_2(y)$.

Side 3. $\cos\beta_3 q_{xx} + \sin\beta_3 q_{yy} = f_3(x)$, $\cos B_3 q_{xy} + \sin B_3 q_{yyy} = F_3(x)$.

With this choice of boundary conditions, it is convenient to define some associated functions which will greatly facilitate the subsequent analysis of the global relations. We consider each boundary separately.

Side 1. The first boundary condition on side 1 is

$$\cos\beta_1 q_{xx} + \sin\beta_1 q_{yy} = f_1(x), \tag{4.5}$$

which involves the unit vector $(\cos\beta_1, \sin\beta_1)$. To this boundary condition it is natural to associate the *unknown* function $q_1(x)$ given by

$$q_1(x) = -\sin\beta_1 q_{xx} + \cos\beta_1 q_{yy}, \tag{4.6}$$

involving the orthogonal vector $(-\sin\beta_1, \cos\beta_1)$. Solving (4.5) and (4.6) for q_{xx} and q_{yy}, we obtain

$$q_{xx} = \cos\beta_1 f_1(x) - \sin\beta_1 q_1(x), \qquad q_{yy} = \sin\beta_1 f_1(x) + \cos\beta_1 q_1(x). \tag{4.7}$$

Similarly, associated with the second boundary condition on side 1 we define the unknown function $Q_1(x)$,

$$Q_1(x) = \sin B_1 q_{xy} + \cos B_1 q_{yyy}, \tag{4.8}$$

which implies the following relations for q_{xy} and q_{yyy}:

$$q_{xy} = \cos B_1 F_1(x) - \sin B_1 Q_1(x), \qquad q_{yyy} = \sin B_1 F_1(x) + \cos B_1 Q_1(x). \tag{4.9}$$

Side 2. In the same way, we introduce unknown functions $q_2(y)$ and $Q_2(y)$ by means of the relations

$$q_{xx} = \cos\beta_2 f_2(y) - \sin\beta_2 q_2(y), \qquad q_x = \sin\beta_2 f_2(y) + \cos\beta_2 q_2(y). \tag{4.10}$$

$$q_{yy} = \cos B_2\ F_2(y) - \sin B_2 Q_2(y), \quad q_{xxx} = \sin B_2\ F_2(y) + \cos B_2 Q_2(y). \tag{4.11}$$

Side 3. Similarly to the treatment of side 1, we introduce $q_3(x)$ and $Q_3(x)$ by means of

$$q_{xx} = \cos\beta_3\ f_3(x) - \sin\beta_3\ q_3(x), \qquad q_{yy} = \sin\beta_3\ f_3(x) + \cos\beta_3\ q_3(x), \tag{4.12}$$

$$q_{xy} = \cos B_3\ F_3(x) - \sin B_3\ Q_3(x), \quad q_{yyy} = \sin B_3\ F_3(x) + \cos B_3\ Q_3(x). \tag{4.13}$$

Spectral functions for the given boundary conditions. The spectral functions (4.2) and (4.3) involve the quantities q_{zz} and $q_{zz\bar{z}}$, which are

$$\left.\begin{aligned} q_{zz\bar{z}} &= \tfrac{1}{8}(q_{xxx} + q_{xyy} - \mathrm{i}q_{yyy} - \mathrm{i}q_{xxy}),\\ q_{zz} &= \tfrac{1}{4}(q_{xx} - q_{yy} - 2\mathrm{i}q_{xy}). \end{aligned}\right\} \tag{4.14}$$

Consider first side 1. The values of q_{xx}, q_{yy}, q_{xy} and q_{yyy} on side 1 are given by (4.7) and (4.9). Also, on this side $q_{xxx} = \partial_x(q_{xx})$, $q_{xyy} = \partial_x(q_{yy})$ and $q_{xxy} = \partial_x(q_{xy})$. Therefore, using integration by parts and equations (4.7) and (4.9), all the boundary values needed for the determination of the spectral functions can be expressed in terms of $f_1(x)$, $F_1(x)$, $q_1(x)$ and $Q_1(x)$. Side 3 can be analysed similarly.

Now consider side 2. The boundary values of q_x, q_{xx}, q_{yy} and q_{xxx} are given by (4.10) and (4.11). Also, on this side, $q_{xy} = \partial_y(q_x)$, $q_{xyy} = \partial_y^2(q_x)$ and $q_{yyy} = \partial_y(q_{yy})$.

Thus, using (4.10) and (4.11) as well as integration by parts, the spectral functions can be written in terms of $f_2(y)$, $F_2(y)$, $q_2(y)$ and $Q_2(y)$.

In this way, the spectral functions can be written as follows:

$$\left.\begin{aligned}
\tilde{\rho}_1(k) &= \tfrac{1}{8}\mathrm{e}^{-kl}(-(k\sin B_1 + \mathrm{i}\cos B_1)\varPhi_1(-\mathrm{i}k) + \mathrm{i}k(\cos\beta_1 - \sin\beta_1)\phi_1(-\mathrm{i}k)) \\
&\qquad + G_1(k), \\
\tilde{\rho}_2(k) &= \tfrac{1}{8}((k\sin B_2 + \mathrm{i}\cos B_2)\varPhi_2(k) + k(\sin\beta_2 + \mathrm{i}k\cos\beta_2)\phi_2(k)) + G_2(k), \\
\tilde{\rho}_3(k) &= \tfrac{1}{8}\mathrm{e}^{kl}(-(k\sin B_3 + \mathrm{i}\cos B_3)\varPhi_3(-\mathrm{i}k) + \mathrm{i}k(\cos\beta_3 - \sin\beta_3)\phi_3(-\mathrm{i}k)) \\
&\qquad + G_3(k),
\end{aligned}\right\} \tag{4.15}$$

and

$$\left.\begin{aligned}
\rho_1(k) &= \tfrac{1}{4}\mathrm{e}^{-kl}(-(\sin\beta_1 + \cos\beta_1)\phi_1(-\mathrm{i}k) + 2\mathrm{i}\sin B_1\varPhi_1(-\mathrm{i}k)) \\
&\qquad - \mathrm{i}\frac{\mathrm{d}\tilde{\rho}_1(k)}{\mathrm{d}k} - 2\mathrm{i}l\tilde{\rho}_1(k) + g_1(k), \\
\rho_2(k) &= \tfrac{1}{4}(\mathrm{i}\sin B_2\varPhi_2(k) - (2k\cos\beta_2 + \mathrm{i}\sin\beta_2)\phi_2(k)) + \mathrm{i}\frac{\mathrm{d}\tilde{\rho}_2(k)}{\mathrm{d}k} + g_2(k), \\
\rho_3(k) &= \tfrac{1}{4}\mathrm{e}^{kl}(-(\sin\beta_3 + \cos\beta_3)\phi_3(-\mathrm{i}k) + 2\mathrm{i}\sin B_3\varPhi_3(-\mathrm{i}k)) \\
&\qquad - \mathrm{i}\frac{\mathrm{d}\tilde{\rho}_3(k)}{\mathrm{d}k} + 2\mathrm{i}l\tilde{\rho}_3(k) + g_3(k),
\end{aligned}\right\} \tag{4.16}$$

where the *unknown* functions $\{\phi_j, \varPhi_j\}$ are given by

$$\left.\begin{aligned}
\phi_1(-\mathrm{i}k) &\equiv \int_0^\infty q_1(x)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x, & \varPhi_1(-\mathrm{i}k) &\equiv \int_0^\infty Q_1(x)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x, \\
\phi_2(k) &\equiv \int_l^{-l} q_2(y)\mathrm{e}^{ky}\,\mathrm{d}y, & \varPhi_2(k) &\equiv \int_l^{-l} Q_2(y)\mathrm{e}^{ky}\,\mathrm{d}y, \\
\phi_3(-\mathrm{i}k) &\equiv \int_\infty^0 q_3(x)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x, & \varPhi_3(-\mathrm{i}k) &\equiv \int_\infty^0 Q_3(x)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x,
\end{aligned}\right\} \tag{4.17}$$

and the functions $\{g_j(k)\}$ and $\{G_j(k)\}$ are given explicitly in the appendix. The latter functions are *known* (from the given boundary conditions) to within a finite set of real parameters $q(0, \pm l)$, $q_x(0, \pm l)$, $q_{xx}(0, \pm l)$, $q_{yy}(0, \pm l)$ and $q_{xy}(0, \pm l)$.

For *unbounded* polygons the global relations (2.8) are not valid for all $k \in \mathbb{C}$ but are valid in the region of the complex k-plane for which ρ_j and $\tilde{\rho}_j$ are defined. In the case of the semi-infinite strip, the global relations are

$$\left.\begin{aligned}
\tilde{\rho}_1(k) + \tilde{\rho}_2(k) + \tilde{\rho}_3(k) &= 0, \\
\rho_1(k) + \rho_2(k) + \rho_3(k) &= 0,
\end{aligned}\right\} \quad \mathrm{Im}[k] \leqslant 0. \tag{4.18}$$

Substituting (4.15) and (4.16) into the above equations provides two equations for the six unknown functions $\varPhi_j(k)$ and $\phi_j(k)$. For arbitrary choices of B_j and β_j, the equations obtained can be manipulated to yield a matrix differential Riemann–Hilbert problem for the unknown spectral functions. In what follows, we will concentrate on those particular choices of boundary conditions for which it is possible to solve this Riemann–Hilbert problem in closed form.

5. A class of problems with explicit solutions

We now consider a subclass of the above class of boundary-value problems for which the spectral functions can be found explicitly. In many physical applications the sidewall boundary conditions on $y = \pm l$ are of the same type, i.e.

$$\beta_3 = \beta_1 \equiv \beta, \qquad B_3 = B_1 \equiv B, \tag{5.1}$$

where these equations define the new constants β and B. In such a situation, it is natural to restrict attention to solutions to the problem which are even or odd with respect to reflection in the x-axis. Indeed, any set of end-strip data can be decomposed into a sum of parts which are *even* and *odd* with respect to reflection in the x-axis. We now assume that we seek solutions which are symmetric with respect to reflection across the x-axis, or, equivalently, solutions which are *even functions of* y. It will further be assumed that any inhomogeneous functions appearing in the boundary conditions are compatible with solutions possessing such symmetries. This implies

$$\left.\begin{aligned} q(x,-l) &= q(x,l), & q_y(x,-l) &= -q_y(x,l), \\ q_{yy}(x,-l) &= q_{yy}(x,l), & q_{yyy}(x,-l) &= -q_{yyy}(x,l). \end{aligned}\right\} \tag{5.2}$$

Using these facts, together with the definitions of $q_j(x)$ and $Q_j(x)$, it can be deduced that

$$\phi_3(-\mathrm{i}k) = -\phi_1(-\mathrm{i}k) \quad \text{and} \quad \Phi_3(-\mathrm{i}k) = \Phi_1(-\mathrm{i}k). \tag{5.3}$$

(*a*) *Analysis of global relations*

Using (5.3), the global relations (4.18) reduce to the pair of equations

$$\begin{aligned} &-\cosh kl(k\sin B + \mathrm{i}\cos B)\Phi_1(-\mathrm{i}k) - \mathrm{i}k\sinh kl(\cos\beta - \sin\beta)\phi_1(-\mathrm{i}k) \\ &\qquad + \tfrac{1}{2}(k\sin B_2 + \mathrm{i}\cos B_2)\Phi_2(k) + \tfrac{1}{2}(\sin\beta_2 + \mathrm{i}k\cos\beta_2)k\phi_2(k) = G_4(k) \end{aligned} \tag{5.4}$$

and

$$\begin{aligned} &\mathcal{C}(k)\Phi_1(-\mathrm{i}k) + \mathcal{D}(k)\phi_1(-\mathrm{i}k) + \sin B_2\Phi_2(k) + 2\mathrm{i}k\cos\beta_2\phi_2(k) \\ &\qquad + \tfrac{1}{2}(k\sin B_2 + \mathrm{i}\cos B_2)\frac{\mathrm{d}\Phi_2(k)}{\mathrm{d}k} + \tfrac{1}{2}k(\sin\beta_2 + \mathrm{i}k\cos\beta_2)\frac{\mathrm{d}\phi_2(k)}{\mathrm{d}k} = G_5(k), \end{aligned} \tag{5.5}$$

both valid for $\mathrm{Im}[k] \leqslant 0$, where the coefficient functions $\mathcal{C}(k)$ and $\mathcal{D}(k)$ are defined by

$$\left.\begin{aligned} \mathcal{C}(k) &\equiv -(kl\sin B + \mathrm{i}l\cos B)\sinh kl + 2\sin B\cosh kl, \\ \mathcal{D}(k) &\equiv -\mathrm{i}(\cos\beta + \sin\beta)\sinh kl - \mathrm{i}kl(\cos\beta - \sin\beta)\cosh kl, \end{aligned}\right\} \tag{5.6}$$

and the functions $G_4(k)$ and $G_5(k)$ by

$$G_4(k) \equiv -4(G_1(k) + G_2(k) + G_3(k)), \tag{5.7}$$

$$G_5(k) \equiv 2\mathrm{i}(g_1(k) + g_2(k) + g_3(k)) + 4l(G_1(k) - G_3(k)) - 4\frac{\mathrm{d}G_2(k)}{\mathrm{d}k}. \tag{5.8}$$

Note that in deriving (5.4) and (5.5) we have used the fact that

$$\frac{\mathrm{d}(\tilde{\rho}_1(k) + \tilde{\rho}_3(k))}{\mathrm{d}k} = -\frac{\mathrm{d}\tilde{\rho}_2(k)}{\mathrm{d}k}. \tag{5.9}$$

Expressions for $G_4(k)$ and $G_5(k)$ are given explicitly in the appendix. $G_4(k)$ and $G_5(k)$ are again known up to a finite set of parameters.

Equations (5.4) and (5.5) are two differential equations, valid in the lower-half k-plane, relating the lower analytic functions $\Phi_1(-\mathrm{i}k)$ and $\phi_1(-\mathrm{i}k)$ and the entire functions $\Phi_2(k)$ and $\phi_2(k)$.

It is noted that $\mathcal{D}(k)$ satisfies the following relation:

$$\overline{\mathcal{D}}(k) = -\mathcal{D}(k). \tag{5.10}$$

(b) *Case* $\beta_2 = B_2 = \pi/2$

This choice corresponds to taking the end-strip boundary conditions of canonical problem II. In this case, (5.4) and (5.5) become

$$\begin{aligned}-\cosh kl(k\sin B + \mathrm{i}\cos B)\Phi_1(-\mathrm{i}k) - \mathrm{i}k\sinh kl(\cos\beta - \sin\beta)\phi_1(-\mathrm{i}k)\\ + \tfrac{1}{2}k(\Phi_2(k) + \phi_2(k)) = G_4(k), \quad \mathrm{Im}[k] \leqslant 0,\end{aligned} \tag{5.11}$$

and

$$\begin{aligned}\mathcal{C}(k)\Phi_1(-\mathrm{i}k) + \mathcal{D}(k)\phi_1(-\mathrm{i}k) + \Phi_2(k) + \tfrac{1}{2}k\frac{\mathrm{d}}{\mathrm{d}k}(\Phi_2(k) + \phi_2(k))\\ = G_5(k), \quad \mathrm{Im}[k] \leqslant 0.\end{aligned} \tag{5.12}$$

The definitions of the spectral functions (4.15) and (4.16), equation (5.3) and the fact that $\cos\beta_2 = \cos B_2 = 0$, $\sin\beta_2 = 1$, $\sin B_2 = 1$, imply that the spectral functions depend on

$$\{\Phi_1(-\mathrm{i}k), \phi_1(-\mathrm{i}k)\}, \quad k \in \mathbb{R} \quad \text{and} \quad \{\Phi_2(k) + \phi_2(k), \Phi_2(k) - \phi_2(k)\}, \quad \arg[k] = \pi/2. \tag{5.13}$$

By analysing the global relations (5.4) and (5.5) it will be shown that the unknown functions (5.13) can be determined in closed form.

Proposition 5.1. *Let the functions $\Phi_1(-\mathrm{i}k), \phi_1(-\mathrm{i}k), \Phi_2(k), \phi_2(k)$ satisfy (5.4) and (5.5). Then these functions can be determined in closed form as follows.*

(1) *The sectionally holomorphic function $\{\Phi_1(\mathrm{i}k), \Phi_1(-\mathrm{i}k)\}$ satisfies the scalar Riemann–Hilbert problem:*

  (i) *$\Phi_1(\mathrm{i}k)$ is holomorphic in* $\mathrm{Im}[k] > 0$*;*

  (ii) *$\Phi_1(\mathrm{i}k) \sim \mathcal{O}(k^{-1})$ as $k \to \infty$;*

  (iii)
$$\bar{\mathcal{E}}(k)\Phi_1(\mathrm{i}k) - \mathcal{E}(k)\Phi_1(-\mathrm{i}k) = G_6(k), \quad k \in \mathbb{R}, \tag{5.14}$$
  where $\mathcal{E}(k), \Delta$ and $\alpha(k)$ are defined in (1.5) and (1.7) and
$$G_6(k) \equiv \bar{G}_5(k) - G_5(k) + \frac{\mathrm{i}\mathcal{D}(k)}{k\Delta\sinh kl}(G_4(k) - \bar{G}_4(k)). \tag{5.15}$$

(2) *The sectionally holomorphic functions $\{\phi_1(\mathrm{i}k), \phi_1(-\mathrm{i}k)\}$ satisfies the scalar Riemann–Hilbert problem*

  (i) *$\phi_1(\mathrm{i}k)$ is holomorphic in* $\mathrm{Im}[k] > 0$*;*

(ii) $\phi_1(\mathrm{i}k) \sim \mathcal{O}(k^{-1})$ *as* $k \to \infty$*;*

(iii)
$$\mathcal{E}(k)[\phi_1(\mathrm{i}k) + \phi_1(-\mathrm{i}k)] = \mathcal{F}(k)\Phi_1(\mathrm{i}k) + G_7(k), \quad k \in \mathbb{R}, \tag{5.16}$$
where $\mathcal{F}(k)$ *is defined in (1.11) and*
$$G_7(k) \equiv \frac{G_6(k)\mathcal{C}(k)}{\mathcal{D}(k)} + \frac{\mathcal{E}(k)}{\mathcal{D}(k)}(G_5(k) - \bar{G}_5(k)). \tag{5.17}$$

(3) *The entire functions* $\{\Phi_2(k), \phi_2(k)\}$ *satisfy the algebraic relations*
$$\Phi_2(k) + \phi_2(k) = -2\mathrm{i}\Delta \sinh kl \phi_1(\mathrm{i}k) + \frac{2\alpha(k)}{k}\cosh kl \Phi_1(\mathrm{i}k) + 2\frac{\bar{G}_4(k)}{k} \tag{5.18}$$
and
$$\begin{aligned}\Phi_2(k) - \phi_2(k) = {} & 2\mathrm{i}\Delta \sinh kl\phi_1(\mathrm{i}k) - 2\mathrm{i}\Sigma \sinh kl\phi_1(\mathrm{i}k) + 2\mathrm{i}k\Delta \sinh kl \frac{\mathrm{d}\phi_1(\mathrm{i}k)}{\mathrm{d}k} \\ & - \left(4\sin B\cosh kl + 2\cosh kl\frac{\mathrm{d}\alpha(k)}{\mathrm{d}k}\right)\Phi_1(\mathrm{i}k) \\ & - 2\alpha(k)\cosh kl\frac{\mathrm{d}\Phi_1(\mathrm{i}k)}{\mathrm{d}k} + G_8(k), \quad \mathrm{Im}[k] \geqslant 0,\end{aligned} \tag{5.19}$$
where Σ *is defined in (1.7) and*
$$G_8(k) = 2\bar{G}_5(k) - 2\frac{\mathrm{d}\bar{G}_4(k)}{\mathrm{d}k}. \tag{5.20}$$

Proof. Taking the complex conjugates of (5.11) and (5.12) and using (5.10) we find
$$\begin{aligned}& -\cosh kl(k\sin B - \mathrm{i}\cos B)\Phi_1(\mathrm{i}k) + \mathrm{i}k\sinh kl(\cos\beta - \sin\beta)\phi_1(\mathrm{i}k) \\ & \qquad + \tfrac{1}{2}k(\Phi_2(k) + \phi_2(k)) = \bar{G}_4(k), \quad \mathrm{Im}[k] \geqslant 0,\end{aligned} \tag{5.21}$$
and
$$\begin{aligned}& \overline{\mathcal{C}}(k)\Phi_1(\mathrm{i}k) - \mathcal{D}(k)\phi_1(\mathrm{i}k) + \Phi_2(k) + \tfrac{1}{2}k\frac{\mathrm{d}}{\mathrm{d}k}(\Phi_2(k) + \phi_2(k)) \\ & \qquad = \bar{G}_5(k), \quad \mathrm{Im}[k] \geqslant 0.\end{aligned} \tag{5.22}$$
Subtracting (5.21) from (5.11), and (5.12) from (5.22), we obtain
$$\begin{aligned}& \cosh kl[(k\sin B - \mathrm{i}\cos B)\Phi_1(\mathrm{i}k) - (k\sin B + \mathrm{i}\cos B)\Phi_1(-\mathrm{i}k)] \\ & \qquad - \mathrm{i}(\cos\beta - \sin\beta)k\sinh kl[\phi_1(\mathrm{i}k) + \phi_1(-\mathrm{i}k)] = G_4(k) - \bar{G}_4(k),\end{aligned} \tag{5.23}$$
and
$$\overline{\mathcal{C}(\bar{k})}\Phi_1(\mathrm{i}k) - \mathcal{C}(k)\Phi_1(-\mathrm{i}k) - \mathcal{D}(k)[\phi_1(\mathrm{i}k) + \phi_1(-\mathrm{i}k)] = \bar{G}_5(k) - G_5(k), \tag{5.24}$$
both valid only for $k \in \mathbb{R}$.

This case is distinguished in that both (5.23) and (5.24) contain the same combination $\phi_1(\mathrm{i}k) + \phi_1(-\mathrm{i}k)$. Eliminating this combination between (5.23) and (5.24) yields (5.14). Then (5.24) and (5.14) imply (5.16). Using the notation (1.7), (5.21) becomes (5.18). Finally, replacing $\Phi_2(k) + \phi_2(k)$ in (5.22) by the right-hand side of (5.18), multiplying the resulting equation by two and subtracting (5.21) we find (5.19). ∎

Proposition 5.1 shows that the spectral functions can be expressed in terms of the solution of two scalar Riemann–Hilbert problems with the jump conditions (5.14) and (5.16). It turns out that the function $\phi_1(\mathrm{i}k)$ does not contribute to the solution q_{zz}. Thus, it is possible to avoid solving the Riemann–Hilbert problem for $\phi_1(\pm\mathrm{i}k)$.

With this observation in mind, it is seen from (5.15) that $G_6(k)$ depends on a combination of known functions as well as $G_4(k)$ and $G_5(k)$. In the appendix, explicit formulae for $G_4(k)$ and $G_5(k)$ are given. It is important to note that these functions are completely determined from the boundary data. In particular, from the formulae in the appendix it is seen that only the quantities $q_x(0,l)$ and $q_{xy}(0,l)$ appear. With $\beta_2 = B_2 = \pi/2$, the function $q_x(0,y)$ (and hence $q_x(0,l)$) is known explicitly from the given boundary conditions, as is $q_{xy}(0,y)$ (and hence $q_{xy}(0,l)$) by differentiation. The function $\Phi_1(\mathrm{i}k)$ is therefore completely determined by the boundary data.

(c) *The computation of $Q_1(z)$*

$Q_1(z)$ depends on the spectral functions $\tilde\rho_j(k)$, $j = 1,2,3$, defined in (4.15) with $\phi_3(-\mathrm{i}k) = -\phi_1(-\mathrm{i}k)$ and $\Phi_3(-\mathrm{i}k) = \Phi_1(-\mathrm{i}k)$ with $\beta_2 = B_2 = \pi/2$. $\tilde\rho_2(k)$ involves $\Phi_2(k) + \phi_2(k)$; thus, using (5.18), we find

$$\begin{aligned}\tilde\rho_2(k) = \mathrm{e}^{-kl}(\tfrac{1}{8}\mathrm{i}k\Delta\phi_1(\mathrm{i}k) + \tfrac{1}{8}\alpha(k)\Phi_1(\mathrm{i}k))&\\ + \mathrm{e}^{kl}(-\tfrac{1}{8}\mathrm{i}k\Delta\phi_1(\mathrm{i}k) + \tfrac{1}{8}\alpha(k)\Phi_1(\mathrm{i}k))&\\ + \tfrac{1}{4}\bar G_4(k) + G_2(k), \quad \mathrm{Im}[k] \geqslant 0.&\end{aligned} \tag{5.25}$$

The function $\tilde\rho_1(k)$ involves $\Phi_1(-\mathrm{i}k)$ and $\phi_1(-\mathrm{i}k)$ but using (5.14) and (5.16) these can be rewritten in terms of $\Phi_1(\mathrm{i}k)$ and $\phi_1(\mathrm{i}k)$. The coefficient of $\phi_1(\mathrm{i}k)$ is $-\frac{1}{8}\mathrm{i}k\Delta\mathrm{e}^{-kl}$ while the coefficient of $\Phi_1(\mathrm{i}k)$ is

$$-\frac{\mathrm{e}^{-kl}}{8}\left(\frac{\bar\alpha(k)\bar{\mathcal{E}}(k) - \mathrm{i}k\Delta\mathcal{F}(k)}{\mathcal{E}(k)}\right). \tag{5.26}$$

The expression in brackets in (5.26) can be simplified using the definition of $\mathcal{E}(k)$ and $\mathcal{F}(k)$. Indeed,

$$\frac{\mathcal{E}(k) - \mathcal{C}(k)}{\bar\alpha(k)} = \frac{\overline{\mathcal{E}}(k) - \overline{\mathcal{C}}(k)}{\alpha(k)}, \tag{5.27}$$

so that

$$\bar\alpha(k)\overline{\mathcal{E}(k)} = \mathcal{E}(k)\alpha(k) - g(k), \tag{5.28}$$

where $g(k)$ is defined in (1.10). It follows that the term in brackets in (5.26) equals

$$\alpha(k) - (1 + \coth kl)\frac{g(k)}{\mathcal{E}(k)}. \tag{5.29}$$

Thus $\tilde{\rho}_1(k)$ is given by

$$\tilde{\rho}_1(k) = -\mathrm{e}^{-kl}(\tfrac{1}{8}\mathrm{i}k\Delta\phi_1(\mathrm{i}k) + \tfrac{1}{8}\alpha(k)\Phi_1(\mathrm{i}k)) + \tfrac{1}{8}A_1(k)\mathrm{e}^{-kl}\Phi_1(\mathrm{i}k) + \tilde{G}_1(k) \tag{5.30}$$

for $k \in \mathbb{R}$, where

$$A_1(k) = (1 + \coth kl)\frac{g(k)}{\mathcal{E}(k)} \tag{5.31}$$

and

$$\tilde{G}_1(k) = G_1(k) + \frac{\mathrm{e}^{-kl}}{8\mathcal{E}(k)}(\mathrm{i}k\Delta G_7(k) + \bar{\alpha}(k)G_6(k)). \tag{5.32}$$

Similarly,

$$\tilde{\rho}_3(k) = \mathrm{e}^{kl}(\tfrac{1}{8}\mathrm{i}k\Delta\phi_1(\mathrm{i}k) - \tfrac{1}{8}\alpha(k)\Phi_1(\mathrm{i}k)) + \tfrac{1}{8}A_3(k)\mathrm{e}^{kl}\Phi_1(\mathrm{i}k) + \tilde{G}_3(k) \tag{5.33}$$

for $k \in \mathbb{R}$, where

$$A_3(k) = \frac{(1 - \coth kl)}{\mathcal{E}(k)}g(k) \tag{5.34}$$

and

$$\tilde{G}_3(k) = G_3(k) + \frac{\mathrm{e}^{kl}}{8\mathcal{E}(k)}(-\mathrm{i}k\Delta G_7(k) + \bar{\alpha}(k)G_6(k)). \tag{5.35}$$

Equations (5.25), (5.30) and (5.33) provide the equations needed for the computation of $Q_1(z)$. It is remarkable that all the terms involving $\phi_1(\mathrm{i}k)$ as well as some of the terms involving $\Phi_1(\mathrm{i}k)$ give zero contribution to $Q_1(z)$. Indeed, consider first the terms in the brackets on the right-hand side of the expression for $\tilde{\rho}_1(k)$ in (5.30) as well as the corresponding term on the right-hand side of (5.25). These terms, multiplied by $\mathrm{e}^{\mathrm{i}kz}$, are analytic and bounded in the first quadrant of the complex k-plane and thus by Cauchy's theorem these terms give zero contribution. Similar considerations apply in the second quadrant of the complex k-plane for the terms in the bracket on the right-hand side of (5.33) and the corresponding terms of $\tilde{\rho}_2(k)$ in (5.25). Thus, the only terms which give a non-trivial contribution are those involving the forcing functions $G_j(k)$, and $A_1(k)$ and $A_3(k)$.

(*d*) *The computation of* $Q_2(z)$

$Q_2(z)$ depends on $\{\rho_j(k)\}$ defined by (4.16) with $\phi_3(-\mathrm{i}k) = -\phi_1(-\mathrm{i}k)$ and $\Phi_3(-\mathrm{i}k) = \Phi_1(-\mathrm{i}k)$ with $\beta_2 = B_2 = \pi/2$. The functions $\{\rho_j(k)\}$ depend on $\{\tilde{\rho}_j(k)\}$ as well as $\phi_1(-\mathrm{i}k), \Phi_1(-\mathrm{i}k)$ and the particular combination $\Phi_2(k) - \phi_2(k)$. The aim is to write the functions $\{\rho_j(k)\}$ purely in terms of $\Phi_1(\mathrm{i}k)$ and $\phi_1(\mathrm{i}k)$. The functions $\{\tilde{\rho}_j(k)\}$ are given by (5.25), (5.30) and (5.33) in terms of $\Phi_1(\mathrm{i}k)$ and $\phi_1(\mathrm{i}k)$; the combination $\Phi_2(k) - \phi_2(k)$ is given by (5.19) in terms of $\Phi_1(\mathrm{i}k)$ and $\phi_1(\mathrm{i}k)$ and the functions $\Phi_1(-\mathrm{i}k)$ and $\phi_1(-\mathrm{i}k)$ are readily expressed in terms of $\Phi_1(\mathrm{i}k)$ and $\phi_1(\mathrm{i}k)$ using (5.14) and (5.16). In this way, straightforward algebraic manipulations yield the following expressions for $\{\rho_j(k)\}$ in terms of $\Phi_1(\mathrm{i}k)$ and $\phi_1(\mathrm{i}k)$:

$$\rho_1(k) = \mathrm{e}^{-kl}(\tilde{\mathcal{R}}_1(k) + \mathcal{R}_1(k)) + \mathcal{G}_1(k), \quad k \in \mathbb{R}, \tag{5.36}$$

$$\rho_2(k) = -\mathrm{e}^{-kl}\tilde{\mathcal{R}}_1(k) - \mathrm{e}^{kl}\tilde{\mathcal{R}}_3(k) + \mathcal{G}_2(k), \quad \mathrm{Im}[k] \geqslant 0, \tag{5.37}$$

$$\rho_3(k) = \mathrm{e}^{kl}(\tilde{\mathcal{R}}_3(k) + \mathcal{R}_3(k)) + \mathcal{G}_3(k), \quad k \in \mathbb{R}, \tag{5.38}$$

where

$$\left.\begin{aligned}\tilde{\mathcal{R}}_1(k) &= \left(\frac{\Sigma}{4} - \frac{\Delta kl}{8} - \frac{\Delta}{8}\right)\phi_1(\mathrm{i}k) - \frac{\Delta k}{8}\frac{\mathrm{d}\phi_1(\mathrm{i}k)}{\mathrm{d}k} \\ &\quad + \left(\frac{\mathrm{i}l\alpha(k)}{8} + \frac{\mathrm{i}}{8}\frac{\mathrm{d}\alpha(k)}{\mathrm{d}k} + \frac{\mathrm{i}\sin B}{2}\right)\Phi_1(\mathrm{i}k) + \frac{\mathrm{i}\alpha(k)}{8}\frac{\mathrm{d}\Phi_1(\mathrm{i}k)}{\mathrm{d}k}, \\ \tilde{\mathcal{R}}_3(k) &= \left(-\frac{\Sigma}{4} - \frac{\Delta kl}{8} + \frac{\Delta}{8}\right)\phi_1(\mathrm{i}k) + \frac{\Delta k}{8}\frac{\mathrm{d}\phi_1(\mathrm{i}k)}{\mathrm{d}k} \\ &\quad + \left(-\frac{\mathrm{i}l\alpha(k)}{8} + \frac{\mathrm{i}}{8}\frac{\mathrm{d}\alpha(k)}{\mathrm{d}k} + \frac{\mathrm{i}\sin B}{2}\right)\Phi_1(\mathrm{i}k) + \frac{\mathrm{i}\alpha(k)}{8}\frac{\mathrm{d}\Phi_1(\mathrm{i}k)}{\mathrm{d}k},\end{aligned}\right\} \tag{5.39}$$

while $\mathcal{R}_1(k)$ and $\mathcal{R}_3(k)$ are defined in (1.8) and

$$\mathcal{G}_1(k) = g_1(k) - \frac{\mathrm{e}^{-kl}}{4\mathcal{E}(k)}(\Sigma G_7(k) + 2\mathrm{i}\sin B G_6(k)) - \mathrm{i}\tilde{G}_1'(k) - 2\mathrm{i}l\tilde{G}_1(k), \tag{5.40}$$

$$\mathcal{G}_2(k) = g_2(k) + \frac{\mathrm{i}G_8(k)}{4} + \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}k}\left(\frac{\bar{G}_4(k)}{4} + G_2(k)\right), \tag{5.41}$$

$$\mathcal{G}_3(k) = g_3(k) - \frac{\mathrm{e}^{kl}}{4\mathcal{E}(k)}(\Sigma G_7(k) - 2\mathrm{i}\sin B G_6(k)) - \mathrm{i}\tilde{G}_3'(k) + 2\mathrm{i}l\tilde{G}_3(k). \tag{5.42}$$

Using arguments identical to those used to find $Q_1(z)$, it follows that $\tilde{\mathcal{R}}_1(k)$ and $\tilde{\mathcal{R}}_3(k)$ do not contribute to $Q_2(z)$. Substituting these expressions into the integral formulae (2.10) yields the results of proposition 1.1.

The Riemann–Hilbert problem for $\Phi_1(\pm\mathrm{i}k)$ is completely determined by the boundary data so $\Phi_1(\mathrm{i}k)$ can be found explicitly. The index of the Riemann–Hilbert problem (5.14) can be determined in the standard way (Ablowitz & Fokas 1997) by considering the function

$$\frac{\mathcal{E}(k)}{\bar{\mathcal{E}}(k)} = \frac{\mathcal{C}(k) + \mathrm{i}\mathcal{D}(k)(\bar{\alpha}(k)/k\Delta)\coth kl}{\bar{\mathcal{C}}(k) - \mathrm{i}\bar{\mathcal{D}}(k)(\alpha(k)/k\Delta)\coth kl}. \tag{5.43}$$

It can be shown that this expression tends to unity as $k \to \infty$ so the Riemann–Hilbert problem is continuous at infinity. As an example, assume the parameters of the problem are such that the index is zero. Then $\Phi_1(\mathrm{i}k)$ can be written in the form of the Cauchy integral

$$\Phi_1(\mathrm{i}k) = \frac{X(\mathrm{i}k)}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\frac{G_6(k')}{X(\mathrm{i}k')\bar{\mathcal{E}}(k')}\frac{\mathrm{d}k'}{k'-k}, \tag{5.44}$$

where $X(\mathrm{i}k)$ is the solution of the homogeneous RH problem

$$X(\mathrm{i}k) = \frac{\mathcal{E}(k)}{\bar{\mathcal{E}}(k)}X(-\mathrm{i}k). \tag{5.45}$$

The solution of this problem is given by

$$X(\mathrm{i}k) = \mathrm{e}^{\Gamma(\mathrm{i}k)}, \qquad X(-\mathrm{i}k) = \mathrm{e}^{\Gamma(-\mathrm{i}k)} \tag{5.46}$$

where

$$\Gamma(\mathrm{i}k) = \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\log\left(\frac{G_6(k')}{\bar{\mathcal{E}}(k')}\right)\frac{\mathrm{d}k'}{k'-k}. \tag{5.47}$$

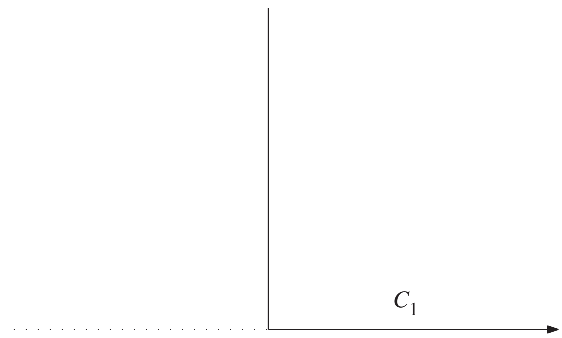

Figure 6. The contour C_1.

The integral expressions for $Q_1(z)$ and $Q_2(z)$ also depend on $G_j(k)$, which, in turn, can be seen from (A 2)–(A 4) to depend on the constants $q_{xx}(0, \pm l)$ and $q_{yy}(0, \pm l)$, which are not known from the boundary conditions (note that $q_x(0, \pm l)$ and $q_{xy}(0, \pm l)$ can be determined from the boundary conditions as pointed out earlier). However, their contribution to the final solution is zero. Indeed, consider first the integral expression (1.2) for $Q_1(z)$. It is found that the terms in the expression for $Q_1(z)$ involving the unknown constants $q_{xx}(0, \pm l)$ and $q_{yy}(0, \pm l)$ has the form

$$(q_{xx}(0,-l) + q_{yy}(0,-l))\left[-\int_{l_1} \frac{\mathrm{e}^{-kl+\mathrm{i}kz}}{8}\,\mathrm{d}k + \int_{l_2} \frac{\mathrm{e}^{-kl+\mathrm{i}kz}}{8}\,\mathrm{d}k\right] \\ - (q_{xx}(0,l) + q_{yy}(0,l))\left[-\int_{l_2} \frac{\mathrm{e}^{kl+\mathrm{i}kz}}{8}\,\mathrm{d}k + \int_{l_3} \frac{\mathrm{e}^{kl+\mathrm{i}kz}}{8}\,\mathrm{d}k\right]. \quad (5.48)$$

However, the first integral term in square brackets is an integral around the closed contour C_1 shown in figure 6 of a function analytic everywhere in the first quadrant. It is therefore zero by Cauchy's theorem. A similar argument applies to the second integral term in square brackets.

Now consider the integral expression for $Q_2(z)$. The functions $\mathcal{G}_j(k)$ depend on $g_j(k)$ (which by inspection of the defining formulae in the appendix are found to be completely determined from the boundary data), the functions $G_4(k)$, $G_6(k)$, $G_7(k)$ and $G_8(k)$ (which, by inspection of the defining formulae are also seen to be completely determined by the given boundary data) and finally the functions $G_1(k)$, $G_2(k)$ and $G_3(k)$, which are *not* completely determined by the boundary data because they depend on the unknown constants $q_{xx}(0, \pm l)$ and $q_{yy}(0, \pm l)$. The contributions from $G_1(k)$, $G_2(k)$ and $G_3(k)$ in the integral expression for $Q_2(z)$ is

$$\int_{l_1} \mathrm{e}^{\mathrm{i}kz}(-\mathrm{i}\bar{G}_1'(k) - 2\mathrm{i}l\bar{G}_1(k))\,\mathrm{d}k + \int_{l_2} \mathrm{e}^{\mathrm{i}kz}\mathrm{i}\bar{G}_2'(k)\,\mathrm{d}k \\ + \int_{l_3} \mathrm{e}^{\mathrm{i}kz}(-\mathrm{i}\bar{G}_3'(k) + 2\mathrm{i}l\bar{G}_3(k))\,\mathrm{d}k. \quad (5.49)$$

Using the formulae (A 2)–(A 4) for $G_1(k)$, $G_2(k)$ and $G_3(k)$, it is straightforward to show that the premultiplying terms of the unknown quantities $q_{xx}(0, \pm l) + q_{yy}(0, \pm l)$ are again integrals which turn out to be zero. In summary, $Q_1(z)$ and $Q_2(z)$ are completely determined by the given boundary data and the solution does not depend on any *a priori* unknown corner data.

6. The canonical problems of elastostatics

The class of explicitly solvable boundary-value problems just described includes (but is not restricted to) the two canonical problems of plane elastostatics. In these canonical problems, the sidewalls are assumed to be clamped so that

$$q_x(x, \pm l) = q_y(x, \pm l) = 0. \tag{6.1}$$

In our notation, this corresponds to the choice $B = \beta = 0$ with the homogeneous choice $F_1(x) = f_1(x) = 0$. The only forcing terms in this case are the specified end-strip functions $f_2(y)$ and $F_2(y)$. It is well known (Smith 1952) that this problem can be solved using infinite sums of Papkovich–Fadle eigenfunctions which automatically satisfy the homogeneous sidewall conditions. Proposition 1.1 generalizes this result and shows that explicit solutions of this general class of end-strip problems are also possible when the sidewall boundary conditions are *inhomogeneous*. Physically, this corresponds to making arbitrary specifications of the displacements on the sidewalls of the elastostatic strip. Such solutions are not captured by a series expansion of Papkovich–Fadle eigensolutions. The new method produces explicit integral representations for the solutions and thus generalizes the Papkovich–Fadle solution method.

It is instructive to examine how to retrieve the Papkovich–Fadle eigenfunction expansions from our integral representations. This also provides an important check on the preceding analysis. With $B = \beta = 0$,

$$\left.\begin{aligned} \mathcal{C}(k) &= -\mathrm{i}l \sinh kl, \\ \mathcal{D}(k) &= -\mathrm{i} \sinh kl - \mathrm{i}kl \cosh kl, \\ \mathcal{E}(k) &= \frac{\mathrm{i}(\cosh kl \sinh kl + kl)}{k \sinh kl}. \end{aligned}\right\} \tag{6.2}$$

It can also be shown that $g(k) = 0$, which means that $A_1(k) = A_3(k) = \mathcal{R}_1(k) = \mathcal{R}_2(k) = 0$. Thus, all contributions to the integral representations of $Q_1(z)$ and $Q_2(z)$ which depend on $\Phi_1(\mathrm{i}k)$ vanish and the integral representations (1.2) become explicit and *do not require the solution of a scalar Riemann–Hilbert problem.*

Note that $\mathcal{E}(k)$ vanishes when

$$\sinh kl \cosh kl + kl = 0, \tag{6.3}$$

or, letting $\mathrm{i}\lambda = kl$, this becomes $\sin\lambda\cos\lambda + \lambda = 0$, which is the well-known eigenvalue condition for the even Papkovich–Fadle eigenfunctions of the semi-infinite strip (Spence 1983). In the case of *homogeneous* sidewall conditions (and ignoring the corner contributions which are known to give zero total contribution to the integrals), we have

$$G_1(k) = g_1(k) = G_3(k) = g_3(k) = 0, \tag{6.4}$$

while the functions $G_2(k)$ and $g_2(k)$ can be seen to be entire functions of k. All other functions arising in the analysis depend only on these two entire functions. Indeed,

$$\left.\begin{aligned} G_4(k) &= -4G_2(k), \\ G_5(k) &= 2\mathrm{i}g_2(k) - 4G_2'(k), \\ G_6(k) &= -4\mathrm{i}g_2(k) + 8G_2'(k) - \frac{8G_2(k)}{k}(1 + kl\coth kl), \\ G_7(k) &= (-4\mathrm{i}g_2(k) + 8G_2'(k))\frac{\coth kl}{k} - \frac{8G_2(k)l}{k}, \\ G_8(k) &= -2\mathrm{i}\bar{g}_2(k) + 4\bar{G}_2'(k). \end{aligned}\right\} \tag{6.5}$$

To within unimportant corner contributions, $g_2(k)$ and $G_2(k)$ are given in terms of the end-strip data as follows:

$$\left.\begin{aligned} g_2(k) &= -\frac{k}{2}\int_l^{-l} \mathrm{e}^{ky} f_2(y)\,\mathrm{d}y, \\ G_2(k) &= \frac{\mathrm{i}}{8}\int_l^{-l} \mathrm{e}^{ky} F_2(y)\,\mathrm{d}y + \frac{\mathrm{i}k^2}{8}\int_l^{-l} \mathrm{e}^{ky} f_2(y)\,\mathrm{d}y. \end{aligned}\right\} \tag{6.6}$$

It also follows that $\alpha(k) = -\mathrm{i}$, $\Delta = 1 = \Sigma$, $\bar{G}_2(k) = -G_2(k)$ and $\bar{G}_5(k) = -G_5(k)$.

Using all the above known functions in the expressions (1.2) we find explicit integral representations for the solution. For example, consider $Q_1(z)$. Some algebra reveals that

$$\left.\begin{aligned} \tilde{G}_1(k) &= -G_2(k) + \frac{\sinh^2 kl G_2(k)}{kl + \sinh kl\cosh kl} - \frac{G_5(k)k}{4(kl + \sinh kl\cosh kl)}, \\ \tilde{G}_3(k) &= -G_2(k) - \frac{\sinh^2 kl G_2(k)}{kl + \sinh kl\cosh kl} + \frac{G_5(k)k}{4(kl + \sinh kl\cosh kl)}, \end{aligned}\right\} \tag{6.7}$$

so that substituting these into the integral expression (1.2) for $Q_1(z)$ gives

$$\begin{aligned} Q_1(z) = \int_0^\infty \mathrm{e}^{\mathrm{i}kz}\left(-G_2(k) + \frac{\sinh^2 kl\; G_2(k)}{kl + \sinh kl\cosh kl} - \frac{G_5(k)k}{4(kl + \sinh kl\cosh kl)}\right)\mathrm{d}k \\ + \int_0^{\mathrm{i}\infty} \mathrm{e}^{\mathrm{i}kz} G_2(k)\,\mathrm{d}k + \int_0^{\mathrm{i}\infty} \mathrm{e}^{\mathrm{i}kz} G_2(k)\,\mathrm{d}k \\ + \int_0^{-\infty} \mathrm{e}^{\mathrm{i}kz}\left(-G_2(k) - \frac{\sinh^2 kl\; G_2(k)}{kl + \sinh kl\cosh kl} + \frac{G_5(k)k}{4(kl + \sinh kl\cosh kl)}\right)\mathrm{d}k, \end{aligned} \tag{6.8}$$

or, equivalently,

$$\begin{aligned} Q_1(z) = -\oint_{C_1} \mathrm{e}^{\mathrm{i}kz} G_2(k)\,\mathrm{d}k + \oint_{C_2} \mathrm{e}^{\mathrm{i}kz} G_2(k)\,\mathrm{d}k \\ + \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kz}\left(\frac{\sinh^2 kl\; G_2(k)}{kl + \sinh kl\cosh kl} - \frac{G_5(k)k}{4(kl + \sinh kl\cosh kl)}\right)\mathrm{d}k, \end{aligned} \tag{6.9}$$

where C_1 is the contour shown in figure 6 and C_2 is the contour shown in figure 7. A straightforward exercise in residue calculus reveals that this integral representation

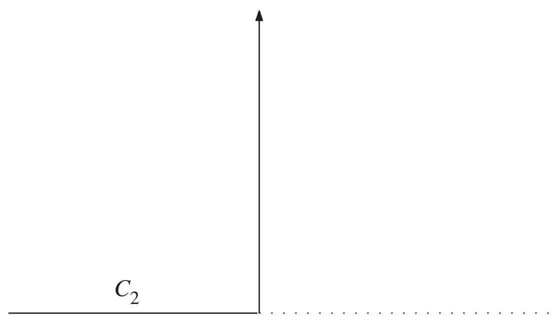

Figure 7. The contour C_2.

can be reduced to a pure residue sum over the Papkovich–Fadle zeros in the upper half-plane, as indeed can the integral representation for $Q_2(z)$.

7. Discussion

The detailed analysis above has shown that the second canonical problem (problem II) of elastostatics is but one of a much wider class of mixed boundary-value problems which are solvable in closed form. The solutions to this class of problems have been derived in the form of explicit integrals. This broader class of boundary-value problems has the same end-strip boundary conditions as the first canonical problem but allows more general sidewall conditions. To the best of our knowledge, it has not previously been demonstrated (e.g. by use of previously known methods) that this particular class of problems is explicitly solvable.

By an analogous analysis, the first canonical problem of elastostatics can also be shown to be but one of a much broader class of explicitly solvable problems. This generalized class of problems possesses the same end-strip conditions as problem I but allows for the same non-trivial sidewall conditions considered above.

It is interesting that Spence (1982) used a special Fourier integral form for the even solutions of canonical problem I of elastostatics (with homogeneous sidewall conditions) to prove the long-conjectured completeness of the Papkovich–Fadle eigenfunction expansion. This Fourier integral form of the solution was originally constructed in Spence (1978). The Papkovich–Fadle expansion was shown to be equivalent to a residue sum of this Fourier integral and this fact proved crucial in Spence's completeness proofs. An essentially equivalent, but substantially different, integral form of the solution to the same problem can be constructed using a similar analysis to that shown above. This method is very different to that employed by Spence (1978). It is expected that the integral representations will similarly provide a valuable tool for future analysis of the solutions.

Finally, even in problems where an explicit solution is not available, the general method of Crowdy & Fokas (2004) provides a systematic way of producing integral representations of the solutions which are explicit in the physical variables but where the spectral data are the solution of a differential matrix Riemann–Hilbert problem. This Riemann–Hilbert problem is essentially the global relation and this can be constructed algorithmically. The same general method (Crowdy & Fokas 2004) is applicable not just to the semi-strip but to arbitrary polygons.

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Appendix A. Expressions for inhomogeneous functions

$$\left.\begin{aligned} g_1(k) &\equiv \frac{\mathrm{e}^{-kl}}{4}\bigg((\cos\beta_1-\sin\beta_1)\int_0^\infty \mathrm{e}^{-\mathrm{i}kx} f_1(x)\,\mathrm{d}x - 2\mathrm{i}\cos B_1\int_0^\infty \mathrm{e}^{-\mathrm{i}kx}F_1(x)\,\mathrm{d}x\bigg),\\ g_2(k) &\equiv \frac{1}{4}\bigg((\mathrm{i}\cos\beta_2-2k\sin\beta_2)\int_l^{-l}\mathrm{e}^{ky}f_2(y)\,\mathrm{d}y-\mathrm{i}\cos B_2\int_l^{-l}\mathrm{e}^{ky}F_2(y)\,\mathrm{d}y\bigg)\\ &\quad+\left[\frac{q_x\mathrm{e}^{ky}}{2}\right]_l^{-l},\\ g_3(k) &\equiv \frac{\mathrm{e}^{kl}}{4}\bigg((\cos\beta_3-\sin\beta_3)\int_\infty^0 \mathrm{e}^{-\mathrm{i}kx} f_3(x)\,\mathrm{d}x - 2\mathrm{i}\cos B_3\int_\infty^0 \mathrm{e}^{-\mathrm{i}kx}F_3(x)\,\mathrm{d}x\bigg), \end{aligned}\right\} \tag{A 1}$$

$$\begin{aligned} G_1(k) \equiv{}& \frac{\mathrm{e}^{-kl}}{8}[\mathrm{i}q_{xy}(0,-l)-q_{xx}(0,-l)-q_{yy}(0,-l)]\\ &+\frac{\mathrm{i}k\mathrm{e}^{-kl}}{8}(\cos\beta_1+\sin\beta_1)\int_0^\infty \mathrm{e}^{-\mathrm{i}kx}f_1(x)\,\mathrm{d}x\\ &+\frac{\mathrm{e}^{-kl}}{8}(k\cos B_1-\mathrm{i}\sin B_1)\int_0^\infty \mathrm{e}^{-\mathrm{i}kx}F_1(x)\,\mathrm{d}x, \end{aligned} \tag{A 2}$$

$$\begin{aligned} G_2(k) \equiv{}& \tfrac{1}{8}[(q_{xx}+q_{yy}+\mathrm{i}q_{xy}-\mathrm{i}kq_x)\mathrm{e}^{ky}]_l^{-l}\\ &+\frac{(\mathrm{i}\sin B_2-k\cos B_2)}{8}\int_l^{-l}\mathrm{e}^{ky}F_2(y)\,\mathrm{d}y\\ &+\frac{(\mathrm{i}k^2\sin\beta_2-k\cos\beta_2)}{8}\int_l^{-l}\mathrm{e}^{ky}f_2(y)\,\mathrm{d}y, \end{aligned} \tag{A 3}$$

$$\begin{aligned} G_3(k) \equiv{}& -\frac{\mathrm{e}^{kl}}{8}[\mathrm{i}q_{xy}(0,l)-q_{xx}(0,l)-q_{yy}(0,l)]\\ &+\frac{\mathrm{i}k\mathrm{e}^{kl}}{8}(\cos\beta_3+\sin\beta_3)\int_\infty^0 \mathrm{e}^{-\mathrm{i}kx}f_3(x)\,\mathrm{d}x\\ &+\frac{\mathrm{e}^{-kl}}{8}(k\cos B_3-\mathrm{i}\sin B_3)\int_\infty^0 \mathrm{e}^{-\mathrm{i}kx}F_3(x)\,\mathrm{d}x, \end{aligned} \tag{A 4}$$

$$\begin{aligned} G_4(k) &\equiv -4(G_1(k)+G_2(k)+G_3(k))\\ &= -2\mathrm{i}q_{xy}(0,l)\cosh kl+\mathrm{i}kq_x(0,l)\sinh kl\\ &\quad-\frac{\mathrm{i}k\mathrm{e}^{-kl}}{2}(\cos\beta_1+\sin\beta_1)\int_0^\infty \mathrm{e}^{-\mathrm{i}kx}f_1(x)\,\mathrm{d}x\\ &\quad-\frac{\mathrm{e}^{-kl}}{2}(k\cos B_1-\mathrm{i}\sin B_1)\int_0^\infty \mathrm{e}^{-\mathrm{i}kx}F_1(x)\,\mathrm{d}x\\ &\quad-\frac{(\mathrm{i}\sin B_2-k\cos B_2)}{2}\int_l^{-l}\mathrm{e}^{ky}F_2(y)\,\mathrm{d}y \end{aligned}$$

$$
\begin{aligned}
&- \frac{(\mathrm{i}k^2\sin\beta_2 - k\cos\beta_2)}{2}\int_l^{-l} \mathrm{e}^{ky} f_2(y)\,\mathrm{d}y \\
&- \frac{\mathrm{i}k\mathrm{e}^{kl}}{2}(\cos\beta_3 + \sin\beta_3)\int_\infty^0 \mathrm{e}^{-\mathrm{i}kx} f_3(x)\,\mathrm{d}x \\
&- \frac{\mathrm{e}^{-kl}}{2}(k\cos B_3 - \mathrm{i}\sin B_3)\int_\infty^0 \mathrm{e}^{-\mathrm{i}kx} F_3(x)\,\mathrm{d}x,
\end{aligned}
$$

$$
\begin{aligned}
G_5(k) &\equiv 2\mathrm{i}(g_1(k)+g_2(k)+g_3(k)) + 4l(G_1(k)-G_3(k)) - 4\frac{\mathrm{d}G_2(k)}{\mathrm{d}k} \\
&= -\mathrm{i}q_x(0,l)(3\sinh kl + kl\cosh kl) + 2\mathrm{i}lq_{xy}(0,l)\sinh kl \\
&\quad + \frac{\mathrm{i}\mathrm{e}^{-kl}}{2}\bigg[[(1+kl)\cos\beta_1 - (1-kl)\sin\beta_1]\int_0^\infty \mathrm{e}^{-\mathrm{i}kx} f_1(x)\,\mathrm{d}x \\
&\qquad\qquad + [k\cos B_1 - \mathrm{i}\sin B_1 - 2\mathrm{i}\cos B_1]\int_0^\infty \mathrm{e}^{-\mathrm{i}kx} F_1(x)\,\mathrm{d}x\bigg] \\
&\quad - (\cos\beta_2 + 2\mathrm{i}k\sin\beta_2)\int_l^{-l} \mathrm{e}^{ky} f_2(y)\,\mathrm{d}y + \cos B_2\int_l^{-l} \mathrm{e}^{ky} F_2(y)\,\mathrm{d}y \\
&\quad + \frac{\mathrm{i}\mathrm{e}^{kl}}{2}\bigg[[(1-kl)\cos\beta_1 - (1+kl)\sin\beta_1]\int_\infty^0 \mathrm{e}^{-\mathrm{i}kx} f_3(x)\,\mathrm{d}x \\
&\qquad\qquad + [-k\cos B_1 + \mathrm{i}\sin B_1 - 2\mathrm{i}\cos B_1]\int_\infty^0 \mathrm{e}^{-\mathrm{i}kx} F_3(x)\,\mathrm{d}x\bigg] \\
&\quad - \left(\frac{\mathrm{i}\sin B_2 - k\cos B_2}{2}\right)\int_l^{-l} \mathrm{e}^{ky} yF_2(y)\,\mathrm{d}y \\
&\quad - \left(\frac{\mathrm{i}k^2\sin\beta_2 - k\cos\beta_2}{2}\right)\int_l^{-l} \mathrm{e}^{ky} yf_2(y)\,\mathrm{d}y.
\end{aligned}
\tag{A 5}
$$

Note that in (A 5) symmetry has been used to reduce the number of unknowns, e.g. $q(0,l) = q(0,-l)$, etc. It is important to note that $G_4(k)$ and $G_5(k)$ depend only on the *two* unknowns $q_{xy}(0,l)$ and $q_x(0,l)$.

References

Ablowitz, M. J. & Fokas, A. S. 1997 *Complex variables.* Cambridge University Press.

Antipov, Y. & Fokas, A. S. 2004 The modified Helmholtz equation in a semi-strip. (Submitted.)

ben-Avraham, D. & Fokas, A. S. 2001 The solution of the modified Helmholtz equation in a triangular region and application to diffusion-limited coalescence. *Phys. Rev.* E **64**, 016114.

Benthem, J. P. 1963 A Laplace transform method for the solution of semi-infinite and finite strip problems in stress analysis. *Q. Appl. Math.* **16**, 413–429.

Bogy, D. B. 1975 Solution of the plane end problem for a semi-infinite elastic strip. *J. Appl. Math. Phys.* **26**, 749–769.

Crowdy, D. G. & Fokas, A. S. 2004 On a transform method for the biharmonic equation in a convex polygon. (In preparation.)

Fokas, A. S. 2001 Two-dimensional linear partial differential equations in a convex polygon. *Proc. R. Soc. Lond.* A **457**, 371–393.

Fokas, A. S. & Kapaev, A. 2003 On a transform method for the Laplace equation in a convex polygon. *IMA J. Appl. Math.* **68**, 1–55.

Gregory, R. D. 1979 Green's functions, bi-linear forms, and completeness of the eigenfunctions for the elastostatic strip and wedge. *J. Elastic.* **9**, 283–309.

Gregory, R. D. 1980 The semi-infinite strip $x \geqslant 0$, $-1 \leqslant y \leqslant +1$; completeness of the Papkovich–Fadle eigenfunctions when ϕ_{xx} and ϕ_{yy} are presribed. *J. Elastic.* **10**, 57–80.

Gupta, G. D. 1973 An integral equation approach to the semi-infinite strip problem. *J. Appl. Mech.* **40**, 948–954.

Johnson, M. W. & Little, R. W. 1965 The semi-infinite elastic strip. *Q. Appl. Math.* **22**, 335–344.

Joseph, D. D. & Sturges, L. 1978 The convergence of bi-orthogonal series for biharmonic and Stokes flow edge problems. *SIAM J. Appl. Math.* **34**, 7–26.

Love, A. E. H. 1944 *A treatise on the mathematical theory of elasticity*. New York: Dover.

Smith, R. T. 1952 The bending of a semi-infinite strip. *J. Austral. Scient. Res.* **5**, 227–237.

Spence, D. A. 1978 Mixed boundary value problems for the elastic strip: the eigenfunction expansion. Research Report TSR1863, Mathematics Research Center, University of Wisconsin Madison.

Spence, D. A. 1982 A note on the eigenfunction expansion for the elastic strip. *SIAM J. Appl. Math.* **42**, 155–173.

Spence, D. A. 1983 A class of biharmonic end-strip problems arising in elasticity and Stokes flow. *IMA J. Appl. Math.* **30**, 107–139.

Vorovich, I. I. & Kopasenko, V. V. 1966 Some problems in the theory of elasticity for a semi-infinite strip. *J. Appl. Math. Mech.* **30**, 128–136.

Williams, M. L. 1952 Stress singularities resulting from various boundary conditions in angular corners of plates in extension. *J. Appl. Mech.* **19**, 526–528.