# 3 M3-4-5A16 Assessed Problems # 3 Do at least five of these eight problems. Study the rest of them!

**Exercise 3.1** In Euclidean $\mathbb{R}^3$ vector components, the Jacobi–Lie bracket of two divergence-free vector fields $v$ and $w$ is expressed as

$$\big[v,\, w\big]_i = w_{i,j}v_j - v_{i,j}w_j \quad \text{with} \quad i,j = 1,2,3\,. \tag{1}$$

Here, a subscript comma denotes partial derivative, e.g., $v_{i,j} = \partial v_i/\partial x_j$, and one sums repeated indices over their range.

(a) Show that $[v,w]_{i,i} = 0$ for the expression in (1), so the commutator of two divergenceless vector fields yields another one.

(b) Verify the Jacobi identity using streamlined notation

$$[v,\, w] = v(w) - w(v)\,,$$

and invoking bilinearity of the Jacobi–Lie bracket.

(c) Show that the vector field

$$X_G = \nabla C \times \nabla G \tag{2}$$

is divergence-free for all smooth functions $C, G \in C^\infty(\mathbb{R}^3)$.

(d) We say that a volume form $\Lambda$ on a Poisson manifold $P$ is *Hamiltonian*, if

$$0 = \pounds_{X_G}\Lambda = d(X_G \,\lrcorner\, \Lambda) + X_G \,\lrcorner\, d\Lambda = d(X_G \,\lrcorner\, \Lambda)\,,$$

for all smooth functions $G \in C^\infty(P)$.

Show that for the volume form $\Lambda = \mathbb{R}^3$ this Hamiltonian condition implies,

$$\mathrm{div}(FX_G)\Lambda = \{F, G\}\Lambda\,,$$

with Poisson bracket

$$\{F, H\} := -\,\nabla C \cdot \nabla F \times \nabla H\,. \tag{3}$$

Hint: For $\Lambda = d^3x$ we have $X_G \,\lrcorner\, d^3x = \nabla C \times \nabla G \cdot d\mathbf{S} = dC \wedge dG$.

(e) Show that the $\mathbb{R}^3$ bracket (3) may be identified with the divergenceless vector fields in (2) by computing

$$[X_G, X_H] = -X_{\{G,H\}}\,, \tag{4}$$

where $[X_G, X_H]$ is the Jacobi–Lie bracket of vector fields $X_G$ and $X_H$.

$\square$

**Exercise 3.2** The dynamical system for the divergence-free motion $\mathbf{x} = (x_1, x_2, x_3)^T \in \mathbb{R}^3$ along the intersection of two orthogonal circular cylinders is given by

$$\dot{x}_1 = x_2x_3\,, \quad \dot{x}_2 = -\,x_1x_3\,, \quad \dot{x}_3 = x_1x_2\,.$$

(a) Write this system in three-dimensional vector $\mathbb{R}^3$-bracket notation as

$$\dot{\mathbf{x}} = \nabla H_1 \times \nabla H_2\,,$$

where $H_1$ and $H_2$ are two conserved functions, whose level sets are circular cylinders oriented, respectively, along the $x_3$-direction ($H_1$) and $x_1$-direction ($H_2$).

(b) Show that the velocity $\dot{\mathbf{x}} \in T\mathbb{R}^3$ is divergence-free.

(c) Restrict the equations and their $\mathbb{R}^3$ Poisson bracket to a level set of $H_1$ by defining cylindrical coordinates so that

$$x_1 = r\cos\theta\,, \quad x_2 = r\sin\theta\,, \quad x_3 = p\,.$$

Show that the Poisson bracket on the cylinder $H_1 = const$ is canonical with a symplectic form given by its area 2-form $\omega = rd\theta \wedge dp$. To check, show that $X_{H_2} = \{\,\cdot\,, H_2\}$ satisfies $X_{H_2} \rfloor rd\theta\wedge dp = dH_2$.

(d) Derive the equations of motion on a level set of $H_1$ and express them in the form of Newton's Law for the planar motion of a simple pendulum. This means planar pendulum motion is isomorphic to the divergence-free motion in $\mathbb{R}^3$ along the intersection of two orthogonal circular cylinders.

$\square$

**Exercise 3.3** Consider the divergence-free motion in $\mathbb{R}^3$ along the intersections of a vertically oriented circular cylinder and a sphere off-set by an amount $s$ along the $x_2$-axis, given respectively by

$$C = \frac{1}{2}(x_1^2 + x_2^2)\,, \quad S = \frac{1}{2}(x_1^2 + (x_2 - s)^2 + x_3^2)$$

(a) Write the corresponding equations of motion in three-dimensional vector $\mathbb{R}^3$-bracket notation as

$$\dot{\mathbf{x}} = \nabla C \times \nabla S\,.$$

(b) Show that this system preserves the level sets of $C$ and $S$.

(c) Restrict the equations and their $\mathbb{R}^3$ Poisson bracket to a level set of $C$. Show that the Poisson bracket on the circular cylinder $C = const$ is symplectic.

(d) Derive the equations of motion on a level set of $C$ by defining cylindrical coordinates so that

$$x_1 = r\cos\theta\,, \quad x_2 = r\sin\theta\,, \quad x_3 = p\,, \quad \text{with} \quad r = \sqrt{2C}\,.$$

and express them as Newton's Law for a Duffing oscillator.

$\square$

**Exercise 3.4 Vector notation for differential basis elements:**
One denotes differential basis elements $dx^i$ and $dS_i = \frac{1}{2}\epsilon_{ijk}dx^j \wedge dx^k$, for $i, j, k = 1, 2, 3$, in vector notation as

$$\begin{aligned}
d\mathbf{x} &:= (dx^1, dx^2, dx^3)\,, \\
d\mathbf{S} &= (dS_1, dS_2, dS_3) \\
&:= (dx^2 \wedge dx^3,\, dx^3 \wedge dx^1,\, dx^1 \wedge dx^2)\,, \\
dS_i &:= \frac{1}{2}\epsilon_{ijk}dx^j \wedge dx^k\,, \\
d^{\,3}x &= d\mathrm{Vol} := dx^1 \wedge dx^2 \wedge dx^3\,.
\end{aligned}$$

(a) ***Vector algebra operations***

(i) Show that contraction with the vector field $X = X^j\partial_j =: \mathbf{X}\cdot\nabla$ recovers the following familiar operations among vectors

$$
\begin{aligned}
X \,\lrcorner\, d\mathbf{x} &= \mathbf{X}\,, \\
X \,\lrcorner\, d\mathbf{S} &= \mathbf{X}\times d\mathbf{x}\,, \\
(\text{or,}\quad X \,\lrcorner\, dS_i &= \epsilon_{ijk}X^j dx^k) \\
Y \,\lrcorner\, X \,\lrcorner\, d\mathbf{S} &= \mathbf{X}\times\mathbf{Y}\,, \\
X \,\lrcorner\, d^{\,3}x &= \mathbf{X}\cdot d\mathbf{S} = X^k dS_k\,, \\
Y \,\lrcorner\, X \,\lrcorner\, d^{\,3}x &= \mathbf{X}\times\mathbf{Y}\cdot d\mathbf{x} = \epsilon_{ijk}X^iY^jdx^k\,, \\
Z \,\lrcorner\, Y \,\lrcorner\, X \,\lrcorner\, d^{\,3}x &= \mathbf{X}\times\mathbf{Y}\cdot\mathbf{Z}\,.
\end{aligned}
$$

(ii) Show that these are consistent with

$$X \,\lrcorner\, (\alpha\wedge\beta) = (X \,\lrcorner\, \alpha)\wedge\beta + (-1)^k\alpha\wedge(X \,\lrcorner\, \beta)\,,$$

for a $k$-form $\alpha$.

(iii) Use (ii) to compute $Y \,\lrcorner\, X \,\lrcorner\, (\alpha\wedge\beta)$ and $Z \,\lrcorner\, Y \,\lrcorner\, X \,\lrcorner\, (\alpha\wedge\beta)$.

(b) ***Exterior derivative examples in vector notation***
Show that the exterior derivative and wedge product satisfy the following relations in components and in three-dimensional vector notation

$$
\begin{aligned}
df &= f_{,j}\,dx^j =: \nabla f\cdot d\mathbf{x} \\
0 = d^2f &= f_{,jk}\,dx^k\wedge dx^j \\
df\wedge dg &= f_{,j}\,dx^j\wedge g_{,k}\,dx^k =: (\nabla f\times\nabla g)\cdot d\mathbf{S} \\
df\wedge dg\wedge dh &= f_{,j}\,dx^j\wedge g_{,k}\,dx^k\wedge h_{,l}\,dx^l =: (\nabla f\cdot\nabla g\times\nabla h)\,d^{\,3}x
\end{aligned}
$$

Likewise, show that

$$
\begin{aligned}
d(\mathbf{v}\cdot d\mathbf{x}) &= (\operatorname{curl}\mathbf{v})\cdot d\mathbf{S} \\
d(\mathbf{A}\cdot d\mathbf{S}) &= (\operatorname{div}\mathbf{A})\,d^{\,3}x\,.
\end{aligned}
$$

Verify the compatibility condition $d^2=0$ for these forms as

$$
\begin{aligned}
0 = d^2f = d(\nabla f\cdot d\mathbf{x}) &= (\operatorname{curl}\operatorname{grad}f)\cdot d\mathbf{S}\,, \\
0 = d^2(\mathbf{v}\cdot d\mathbf{x}) = d\big((\operatorname{curl}\mathbf{v})\cdot d\mathbf{S}\big) &= (\operatorname{div}\operatorname{curl}\mathbf{v})\,d^{\,3}x\,.
\end{aligned}
$$

Verify the exterior derivatives of these contraction formulas for $X = \mathbf{X}\cdot\nabla$

(i) $d(X \,\lrcorner\, \mathbf{v}\cdot d\mathbf{x}) = d(\mathbf{X}\cdot\mathbf{v}) = \nabla(\mathbf{X}\cdot\mathbf{v})\cdot d\mathbf{x}$

(ii) $d(X \,\lrcorner\, \boldsymbol{\omega}\cdot d\mathbf{S}) = d(\boldsymbol{\omega}\times\mathbf{X}\cdot d\mathbf{x}) = \operatorname{curl}(\boldsymbol{\omega}\times\mathbf{X})\cdot d\mathbf{S}$

(iii) $d(X \,\lrcorner\, f\,d^{\,3}x) = d(f\mathbf{X}\cdot d\mathbf{S}) = \operatorname{div}(f\mathbf{X})\,d^{\,3}x$

(c) Use Cartan's formula,

$$\pounds_X\alpha = X \,\lrcorner\, d\alpha + d(X \,\lrcorner\, \alpha)$$

for a $k-$form $\alpha$, $k=0,1,2,3$ in $\mathbb{R}^3$ to verify the Lie derivative formulas:

(i) $\pounds_X f = X \,\lrcorner\, df = \mathbf{X}\cdot\nabla f$

(ii) $\pounds_X\,(\mathbf{v}\cdot d\mathbf{x}) = \big(-\,\mathbf{X}\times\operatorname{curl}\mathbf{v} + \nabla(\mathbf{X}\cdot\mathbf{v})\big)\cdot d\mathbf{x}$

(iii) $\pounds_X(\boldsymbol{\omega}\cdot d\mathbf{S}) = \big(\operatorname{curl}(\boldsymbol{\omega}\times\mathbf{X}) + \mathbf{X}\operatorname{div}\boldsymbol{\omega}\big)\cdot d\mathbf{S}$
$\qquad = \big(-\,\boldsymbol{\omega}\cdot\nabla\mathbf{X} + \mathbf{X}\cdot\nabla\boldsymbol{\omega} + \boldsymbol{\omega}\operatorname{div}\mathbf{X}\big)\cdot d\mathbf{S}$

(iv) $\pounds_X(f\,d^{\,3}x) = (\operatorname{div}f\mathbf{X})\,d^{\,3}x$

(v) Derive these formulas from the dynamical definition of Lie derivative.

(d) Verify the following Lie derivative identities both by using Cartan's formula and by using the dynamical definition of Lie derivative:

(i) $\pounds_{fX}\alpha = f\pounds_X\alpha + df \wedge (X \,\lrcorner\, \alpha)$
(ii) $\pounds_X d\alpha = d\big(\pounds_X\alpha\big)$
(iii) $\pounds_X(X \,\lrcorner\, \alpha) = X \,\lrcorner\, \pounds_X\alpha$
(iv) $\pounds_X(Y \,\lrcorner\, \alpha) = (\pounds_X Y) \,\lrcorner\, \alpha + Y \,\lrcorner\, (\pounds_X\alpha)$
(v) $\pounds_X(\alpha \wedge \beta) = (\pounds_X\alpha) \wedge \beta + \alpha \wedge \pounds_X\beta$

□

**Exercise 3.5** ***Operations among vector fields***

*The Lie derivative of one vector field by another is called the* ***Jacobi-Lie bracket****, defined as*

$$\pounds_X Y := [X\,,\,Y] := \nabla Y \cdot X - \nabla X \cdot Y = -\,\pounds_Y X$$

*In components, the Jacobi-Lie bracket is*

$$[X\,,\,Y] = \Big[X^k\frac{\partial}{\partial x^k}\,,\,Y^l\frac{\partial}{\partial x^l}\Big] = \bigg(X^k\frac{\partial Y^l}{\partial x^k} - Y^k\frac{\partial X^l}{\partial x^k}\bigg)\frac{\partial}{\partial x^l}$$

*The Jacobi-Lie bracket among vector fields satisfies the Jacobi identity,*

$$[\,X\,,\,[Y\,,\,Z]\,] + [\,Y\,,\,[Z\,,\,X]\,] + [\,Z\,,\,[X\,,\,Y]\,] = 0$$

*Verify the following formulas*

*(a)* $X \,\lrcorner\, (Y \,\lrcorner\, \alpha) = -\,Y \,\lrcorner\, (X \,\lrcorner\, \alpha)$

*(b)* $[X\,,\,Y] \,\lrcorner\, \alpha = \pounds_X(Y \,\lrcorner\, \alpha) - Y \,\lrcorner\, (\pounds_X\alpha)$*, for zero-forms (functions) and one-forms.*

*(c)* $\pounds_{[X\,,\,Y]}\alpha = \pounds_X\pounds_Y\alpha - \pounds_Y\pounds_X\alpha$*, as a result of part (b). Use part (c) to verify the Jacobi identity.*

*(d) Verify the formula in part (b) for arbitrary* $k-$*forms using the dynamical definition of the Lie derivative.*

*(e) Starting from the formula in part (b) for arbitrary* $k-$*forms prove the following*

$$\begin{aligned}\pounds_X(Y \,\lrcorner\, \alpha) - \pounds_Y(X \,\lrcorner\, \alpha) &= 2[X,Y] \,\lrcorner\, \alpha + Y \,\lrcorner\, \pounds_X\alpha - X \,\lrcorner\, \pounds_Y\alpha \\ &= [X,Y] \,\lrcorner\, \alpha - X \,\lrcorner\, (Y \,\lrcorner\, \alpha) + d(X \,\lrcorner\, (Y \,\lrcorner\, \alpha))\end{aligned}$$

**Exercise 3.6** ***Hamiltonian vector fields***

*Let* $X_f = \{\,\cdot\,,\,f\,\} \in \mathrm{Ham}(T^*M)$ *be a Hamiltonian vector field on phase-space* $T^*M$ *generated by taking the canonical Poisson bracket* $\{\,\cdot\,,\,\cdot\,\}$ *with the smooth phase-space function* $f :\in C^\infty(T^*M,\mathbb{R})$*. Prove that:*

*(i) The relation* $X_f \,\lrcorner\, \omega = df$ *is equivalent to* $X_f = \{\,\cdot\,,\,f\,\}$*.*

*(ii) The following two relations hold:* $X_{f+const} = X_f$ *and* $\pounds_{X_f}\omega = 0$*.*

*(iii) The collection of all Hamiltonian vector fields is closed with respect to linear combinations (over* $\mathbb{R}$*) as well as under their commutator; namely,*

$$X_f + \lambda X_g = X_{f+\lambda X_g} \quad \textit{and} \quad [X_f, X_g] = -\,X_{\{f,g\}}\,.$$

*This means that they constitute an (infinite-dimensional) Lie algebra* $\mathrm{Ham}(T^*M) \subset \mathfrak{X}(T^*M)$*.*

*(iv) The following relations hold*

$$[X_f\,,\,X_g] \,\lrcorner\, \omega = \pounds_{X_f}(X_g \,\lrcorner\, \omega) - \pounds_{X_g}(X_f \,\lrcorner\, \omega) = d(X_f \,\lrcorner\, (X_g \,\lrcorner\, \omega)) = d\{f,g\}$$

**Exercise 3.7** ***Clebsch-Hamilton principle for a cotangent lift momentum map***

Prove the following
**Proposition**[Clebsch-Hamilton principle] The Euler–Poincaré equation

$$\frac{d}{dt}\frac{\delta l}{\delta \xi} = \mathrm{ad}^*_\xi \frac{\delta l}{\delta \xi} \tag{5}$$

on the dual Lie algebra $\mathfrak{g}^*$ is equivalent to the following implicit variational principle,

$$\delta S(\xi, q, \dot{q}, p) = \delta \int_a^b l(\xi, q, \dot{q}, p)\, dt = 0, \tag{6}$$

for an action constrained by the reconstruction formula

$$\begin{aligned} S(\xi, q, \dot{q}, p) &= \int_a^b l(\xi, q, \dot{q}, p)\, dt \\ &= \int_a^b \Big[ l(\xi) + \Big\langle\!\!\Big\langle\, p\,,\, \dot{q} + \pounds_\xi q \,\Big\rangle\!\!\Big\rangle \Big]\, dt\,, \end{aligned} \tag{7}$$

in which the pairing $\langle\!\langle\,\cdot\,,\,\cdot\,\rangle\!\rangle : T^*M \times TM \mapsto \mathbb{R}$ maps an element of the cotangent space (a momentum covector) and an element from the tangent space (a velocity vector) to a real number.[1]
For the proof, here are two convenient definitions.

**Definition**[The diamond operation $\diamond$] The diamond operation ($\diamond$) is defined as minus the dual of the Lie derivative, namely,

$$\Big\langle\, p \diamond q\,,\, \xi \,\Big\rangle = \Big\langle\!\!\Big\langle\, p\,,\, -\pounds_\xi q \,\Big\rangle\!\!\Big\rangle\,. \tag{8}$$

**Definition**[Transpose of the Lie derivative] The ***transpose*** of the Lie derivative $\pounds_\xi^T p$ is defined via the pairing $\langle\!\langle\,\cdot\,,\,\cdot\,\rangle\!\rangle$ between $(q,p) \in T^*M$ and $(q,\dot{q}) \in TM$ as

$$\Big\langle\!\!\Big\langle\, \pounds_\xi^T p\,,\, q \,\Big\rangle\!\!\Big\rangle = \Big\langle\!\!\Big\langle\, p\,,\, \pounds_\xi q \,\Big\rangle\!\!\Big\rangle\,. \tag{9}$$

The notation in these two definitions distinguishes between two types of pairings,

$$\langle\,\cdot\,,\,\cdot\,\rangle : \mathfrak{g}^* \times \mathfrak{g} \mapsto \mathbb{R} \quad \text{and} \quad \langle\!\langle\,\cdot\,,\,\cdot\,\rangle\!\rangle : T^*M \times TM \mapsto \mathbb{R}\,. \tag{10}$$

(a) Begin the proof by showing that stationarity variations of the constrained action in (7) imply the following set of equations:

$$\frac{\delta l}{\delta \xi} = p \diamond q\,, \quad \dot{q} = -\pounds_\xi q\,, \quad \dot{p} = \pounds_\xi^T p\,. \tag{11}$$

Hint: use the definitions of $\diamond$ and $\pounds_\xi^T$.

(b) Finish by expanding the time derivative

$$\frac{d}{dt}\Big\langle\, \frac{\delta l}{\delta \xi}\,,\, \eta \,\Big\rangle = \frac{d}{dt}\Big\langle\, p \diamond q\,,\, \eta \,\Big\rangle$$

for a fixed Lie algebra element $\eta \in \mathfrak{g}$.

Hint: use the definitions of $\diamond$ and $\pounds_\xi^T$ again.

[1] $\langle\!\langle\,\cdot\,,\,\cdot\,\rangle\!\rangle : T^*M \times TM \mapsto \mathbb{R}$ also occurs in the Legendre transformation.

□

**Exercise 3.8** The point of this exercise is to show that the Legendre transformation of the Clebsch-Hamilton variational principle in variables $q$ and $p$ leads to the ***Lie–Poisson Hamiltonian form*** of these equations.

(a) Show that the Legendre transform takes the Lagrangian

$$l(p,q,\dot q,\xi) = l(\xi) + \Big\langle\!\!\Big\langle\, p\,,\,\dot q + \pounds_\xi q \,\Big\rangle\!\!\Big\rangle$$

in the action (7) to the Hamiltonian,

$$H(p,q) = \Big\langle\!\!\Big\langle\, p\,,\,\dot q \,\Big\rangle\!\!\Big\rangle - l(p,q,\dot q,\xi) = \Big\langle\!\!\Big\langle\, p\,,\,-\pounds_\xi q \,\Big\rangle\!\!\Big\rangle - l(\xi)\,,$$

whose variations are given by

$$\begin{aligned}
\delta H(p,q) &= \Big\langle\!\!\Big\langle\, \delta p\,,\,-\pounds_\xi q \,\Big\rangle\!\!\Big\rangle + \Big\langle\!\!\Big\langle\, p\,,\,-\pounds_\xi \delta q \,\Big\rangle\!\!\Big\rangle \\
&\quad + \Big\langle\!\!\Big\langle\, p\,,\,-\pounds_{\delta\xi} q \,\Big\rangle\!\!\Big\rangle - \Big\langle\, \frac{\delta l}{\delta \xi}\,,\,\delta\xi \,\Big\rangle \\
&= \Big\langle\!\!\Big\langle\, \delta p\,,\,-\pounds_\xi q \,\Big\rangle\!\!\Big\rangle + \Big\langle\!\!\Big\langle\, -\pounds_\xi^T p\,,\,\delta q \,\Big\rangle\!\!\Big\rangle + \Big\langle\, p\diamond q - \frac{\delta l}{\delta \xi}\,,\,\delta\xi \,\Big\rangle\,.
\end{aligned}$$

(b) Show that these variational derivatives recover Equations (11) in canonical Hamiltonian form,

$$\dot q = \delta H/\delta p = -\pounds_\xi q \quad\text{and}\quad \dot p = -\delta H/\delta q = \pounds_\xi^T p\,,$$

and that, moreover, independence of $H$ from $\xi$ yields the momentum relation,

$$\frac{\delta l}{\delta \xi} = p\diamond q\,. \tag{12}$$

(c) The previous exercise showed that

$$\frac{d\mu}{dt} = \{\mu, h\} = \mathrm{ad}^*_{\delta h/\delta\mu}\,\mu\,, \tag{13}$$

where

$$\mu = p\diamond q = \frac{\delta l}{\delta\xi}\,,\quad h(\mu) = \langle \mu, \xi\rangle - l(\xi)\,,\quad \xi = \frac{\delta h}{\delta \mu}\,. \tag{14}$$

The evolution of a smooth real function $f:\mathfrak{g}^*\to\mathbb{R}$ is governed by

$$\frac{df}{dt} = \left\langle \frac{\delta f}{\delta\mu}\,,\,\frac{d\mu}{dt}\right\rangle \tag{15}$$

Show that this equation for $df/dt$ implies the Lie-Poisson bracket

$$\{f,h\} := -\left\langle \mu\,,\,\left[\frac{\delta f}{\delta\mu}\,,\,\frac{\delta h}{\delta\mu}\right]\right\rangle\,.$$

(d) Explain why this result for $\{f,h\}$ satisfies the definition of a Poisson bracket.