Solutions to Assessed Homework 2 **Darryl Holm**

Problems **1(a-c)** are verified easily from Cartan's formula $\pounds_X\alpha = X \lrcorner\, d\alpha + d(X \lrcorner\, \alpha)$ for $\alpha \in \Lambda^k$. Problems **1.d(i-iv)** are answered as follows:

1d *Lie derivative identities by using Cartan's formula*

1.d(i) By its linearity, contraction satisfies $d(fX \lrcorner\, \alpha) = fd(X \lrcorner\, \alpha) + df \wedge (X \lrcorner\, \alpha)$ Hence,

$$
\begin{aligned}
\pounds_{fX}\alpha &= fX \lrcorner\, d\alpha + d(fX \lrcorner\, \alpha) \\
&= fX \lrcorner\, d\alpha + fd(X \lrcorner\, \alpha) + df \wedge (X \lrcorner\, \alpha) \\
&= f\pounds_X\alpha + df \wedge (X \lrcorner\, \alpha)
\end{aligned}
$$

1.d(ii) Cartan's formula implies $\pounds_X d\alpha = d\big(\pounds_X\alpha\big)$ by comparing the definitions:

$$
\begin{aligned}
\pounds_X d\alpha &= X \lrcorner\, d^2\alpha + d(X \lrcorner\, d\alpha) \\
d\big(\pounds_X\alpha\big) &= d(X \lrcorner\, d\alpha) + d^2(X \lrcorner\, \alpha)
\end{aligned}
$$

By $d^2 = 0$, these are both equal to $d(X \lrcorner\, d\alpha)$ and the result follows.

1.d(iii) One also proves $\pounds_X(X \lrcorner\, \alpha) = X \lrcorner\, \pounds_X\alpha$ by comparing the definitions:

$$
\begin{aligned}
\pounds_X(X \lrcorner\, \alpha) &= X \lrcorner\, d(X \lrcorner\, \alpha) + d(X \lrcorner\, (X \lrcorner\, \alpha)) \\
X \lrcorner\, \pounds_X\alpha &= X \lrcorner\, d(X \lrcorner\, \alpha) + X \lrcorner\, (X \lrcorner\, d\alpha)
\end{aligned}
$$

By $X \lrcorner\, (X \lrcorner\, \alpha) = 0$, these are both equal to $X \lrcorner\, d(X \lrcorner\, \alpha)$ and the result follows.

1.d(iv) $\pounds_X(\alpha \wedge \beta) = (\pounds_X\alpha) \wedge \beta + \alpha \wedge \pounds_X\beta$

This important identity (product rule for the Lie derivative) follows when the two defining properties

$$
X \lrcorner\, (\alpha \wedge \beta) = (X \lrcorner\, \alpha) \wedge \beta + (-1)^k \alpha \wedge (X \lrcorner\, \beta)\,,
$$

and

$$
d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^k \alpha \wedge d\beta\,,
$$

are combined with Cartan's formula.
(The corresponding underlined terms cancel below.)

$$
\begin{aligned}
\pounds_X(\alpha \wedge \beta) &= (X \lrcorner\, d\alpha) \wedge \beta + \underline{[(-1)^{k+1} d\alpha \wedge (X \lrcorner\, \beta)]} + \underline{\underline{[(-1)^k (X \lrcorner\, \alpha) \wedge d\beta\,]}} \\
&\quad + (-1)^{2k} \alpha \wedge (X \lrcorner\, d\beta) + d(X \lrcorner\, \alpha) \wedge \beta + \underline{\underline{[(-1)^{k-1} (X \lrcorner\, \alpha) \wedge d\beta\,]}} \\
&\quad + \underline{[(-1)^k d\alpha \wedge (X \lrcorner\, \beta)]} + (-1)^{2k} \alpha \wedge d(X \lrcorner\, \beta) \\
&= (X \lrcorner\, d\alpha + d(X \lrcorner\, \alpha)) \wedge \beta + (-1)^{2k} \alpha \wedge (X \lrcorner\, d\beta + d(X \lrcorner\, \beta)) \\
&= (\pounds_X\alpha) \wedge \beta + \alpha \wedge \pounds_X\beta
\end{aligned}
$$

2 *Verifying the formulas for operations among vector fields*

2(a) By direct substitution

$$\begin{aligned} X \lrcorner (Y \lrcorner \alpha) &= X^l Y^m \alpha_{mli_3\ldots i_k} dx^{i_3} \wedge \cdots \wedge dx^{i_k} \\ &= -X^l Y^m \alpha_{lmi_3\ldots i_k} dx^{i_3} \wedge \cdots \wedge dx^{i_k} \\ &= -Y \lrcorner (X \lrcorner \alpha)\,, \quad \text{by antisymmetry of } \alpha_{mli_3\ldots i_k} \end{aligned}$$

2(b) For zero-forms (functions) all terms in the formula vanish identically. The formula $[X\,,\,Y] \lrcorner \alpha = \pounds_X(Y \lrcorner \alpha) - Y \lrcorner (\pounds_X \alpha)$ is seen to hold for a one-form $\alpha = \mathbf{v}\cdot d\mathbf{x}$ by comparing

$$\begin{aligned} [X\,,\,Y] \lrcorner \alpha &= (X^k Y^l_{,k} - Y^k X^l_{,k}) v_l \\ \text{with} \quad \pounds_X(Y \lrcorner \alpha) - Y \lrcorner (\pounds_X \alpha) &= X^k \partial_k (Y^l v_l) - Y^l (X^k v_{l,k} + v_j X^j_{,l}) \end{aligned}$$

to see that it holds in an explicit calculation.

(By a general theorem Abraham and Marsden [1978], verification for zero-forms and one-forms is sufficient to imply the result for all $k-$forms. Notice that exercise 1.iv(c) is an example for 3-forms. Try writing the formula in vector notation for 2-forms!)

Remark. One may remember this formula by writing it as a ***product rule***:

$$\pounds_X(Y \lrcorner \alpha) = (\pounds_X Y) \lrcorner \alpha + Y \lrcorner (\pounds_X \alpha)\,.$$

■

2(c) Given $[X\,,\,Y] \lrcorner \alpha = \pounds_X(Y \lrcorner \alpha) - Y \lrcorner (\pounds_X \alpha)$ as verified in part **2(b)** we use Cartan's formula to compute

$$\begin{aligned} \pounds_{[X\,,Y]}\alpha &= d([X\,,\,Y] \lrcorner \alpha) + [X\,,\,Y] \lrcorner d\alpha \\ &= d\big(\pounds_X(Y \lrcorner \alpha) - Y \lrcorner (\pounds_X \alpha)\big) + \pounds_X(Y \lrcorner d\alpha) - Y \lrcorner (\pounds_X d\alpha) \\ &= \pounds_X d(Y \lrcorner \alpha) - d(Y \lrcorner (\pounds_X \alpha) + \pounds_X(Y \lrcorner d\alpha) - Y \lrcorner d(\pounds_X \alpha) \\ &= \pounds_X(\pounds_Y \alpha) - \pounds_Y(\pounds_X \alpha)\,, \end{aligned}$$

as required. Thus, answering problem **2(b)** provides the key to solving **2(c)**.
■

Consequently, $\pounds_{[\,Z\,,[X\,,Y]\,]}\alpha = \pounds_Z\pounds_X\pounds_Y\alpha - \pounds_Z\pounds_Y\pounds_X\alpha - \pounds_X\pounds_Y\pounds_Z\alpha + \pounds_Y\pounds_X\pounds_Z\alpha$, and summing over cyclic permutations immediately verifies that

$$\pounds_{[\,Z\,,[X\,,Y]\,]}\,\alpha + \pounds_{[\,X\,,[Y\,,Z]\,]}\,\alpha + \pounds_{[\,Y\,,[Z\,,X]\,]}\,\alpha = 0\,.$$

This is the ***Jacobi identity for the Lie derivative***. ■