# MPA16 Notes: Geometric Mechanics

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## Text for the course MPA16:

*Geometric Mechanics I: Dynamics and Symmetry*, by Darryl D Holm

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**Abstract**

MPA16

Should it be PA or AP? We will do both in this class!

AP = Applications of Pure maths, e.g., Noether's theorem: Lie group symmetry of Hamilton's variational principle implies conservation laws for its equations of motion.

PA = Purifications of Applied maths, e.g., Euler fluid dynamics describes geodesic flow on the manifold of smooth invertible maps acting on the domain of flow.

We will do both in this class!

## Marks

1. ***Assessed Homework:***

    - To help you prepare for the Final Exam, three Assessed Homework sets of 4 or 5 problems each will be handed out, spaced about three weeks apart, e.g., at week 3, week 6 and week 9. Each Assessed Homework set will be due about twelve days after it is assigned, although the due date for the last set may be delayed until immediately after the Winter Break, if desired.

2. ***Final Exam:*** Three of the five questions will be taken from the assessed homework assignments.

3. To help you prepare for the Assessed Homework sets, many Practice Problems and sketches of their solutions will be provided intermittently, written on the board as the lectures progress.

4. Lecture notes will be available online at `http://wwwf.imperial.ac.uk/~dholm/classnotes/`

## Office hours

Arranged individually or in groups by appointment via email.

# Class introduction

This class explains, via many self-contained examples, a systematic framework for using ***Geometry*** in studying ***Mechanics***. Here, these terms mean the following.

- *Geometry* involves linear algebra, transformation theory, differential equations, variational calculus, Lie groups and their actions on manifolds.
- *Mechanics* means "the branch of physics concerned with the motion of bodies in a frame of reference". Usually this means differential equations, e.g., $\dot{X} = F(X, t)$.
- *Study* means "formulate and solve, so as to reveal the geometric meaning of the problem and thereby understand better how to obtain **and interpret** its solution geometrically".

For example, in the language of GM, Euler's rigid body dynamics becomes geodesic motion on the Lie group of 3D rotations $SO(3)$ with respect to the Riemannian metric given by the moment of inertia.

The solution may also be represented as motion by smooth flows parameterised by time $t$ that takes place along the intersections of two-dimensional surfaces in $\mathbb{R}^3$ that are level sets of the conservation laws for energy and angular momentum.

This year is also the centenary of **Noether's theorem** which associates symmetries of Hamilton's variational principle with constants of motion of the Euler-Lagrange equations. We will use Noether's theorem as an organising principle throughout these lectures, on both the Lagrangian and Hamiltonian sides.

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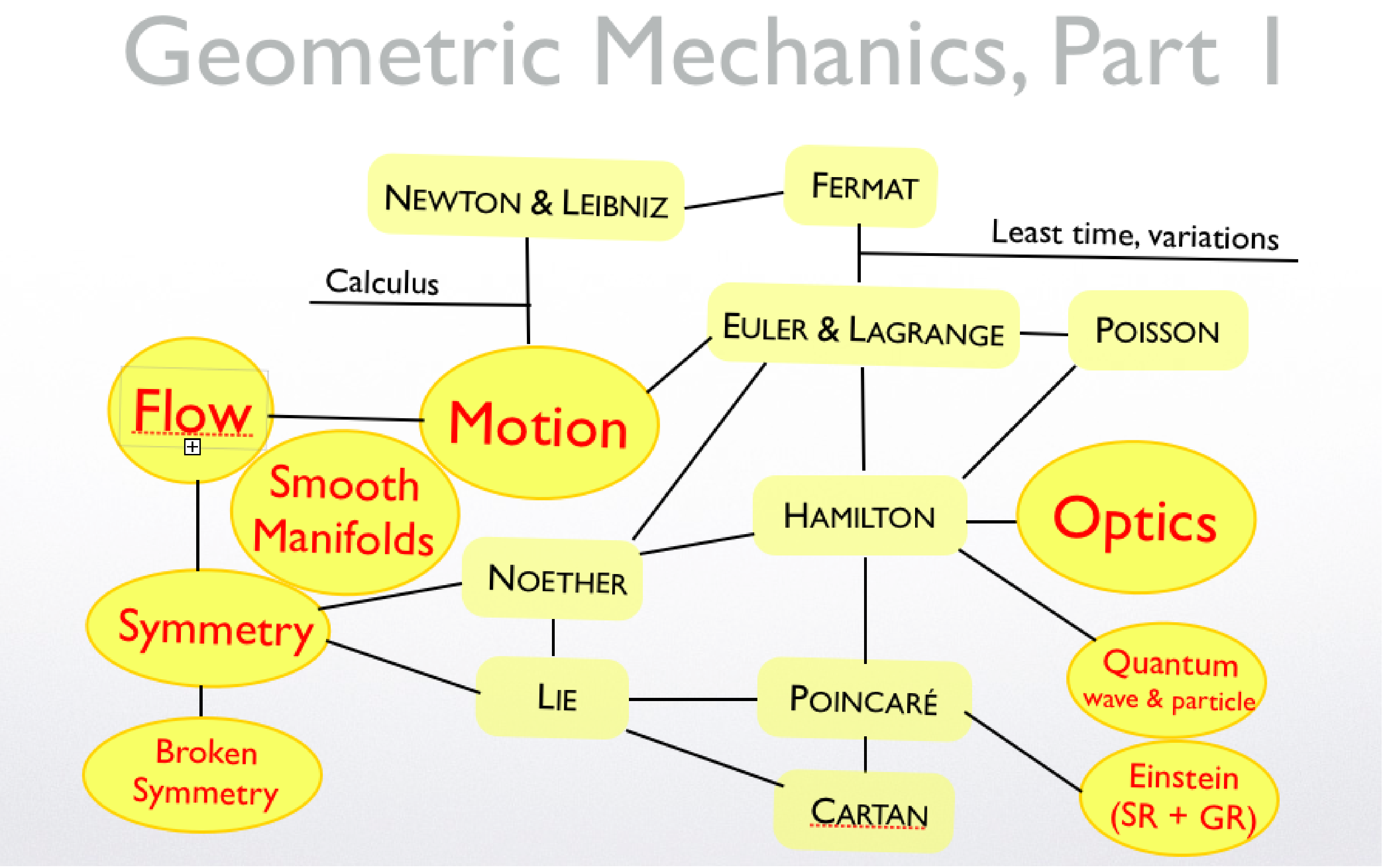


Figure 1: Geometric Mechanics has involved many great mathematicians!

# 1 Classical mechanics: Definitions for motion on smooth manifolds

## 1.1 Definitions: Space, Time, Motion, . . . , Tangent space, Velocity, Motion equation

### Space

$\boxed{\textbf{\textit{Space}}}$ is taken to be a smooth manifold $Q$ with points $q \in Q$ (Positions, States, Configurations).

Let $Q$ be a $\boxed{\textbf{\textit{smooth manifold}}}$ $\dim Q = n$. That is, $Q$ is a smooth space that is locally $\mathbb{R}^n$. Operationally, a smooth manifold is a space on which the rules of calculus apply.

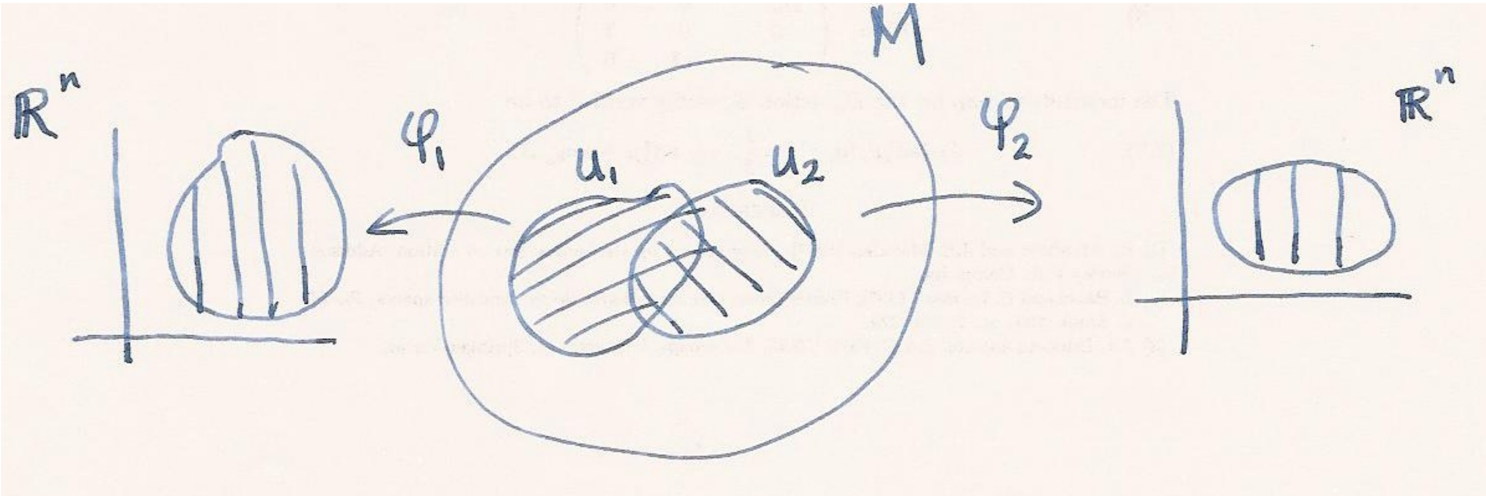


Figure 2: A manifold $Q$ is defined by the disjoint union (or, atlas) of local charts, each of which is isomorphic to $\mathbb{R}^{\dim Q}$.

**Examples of manifolds**

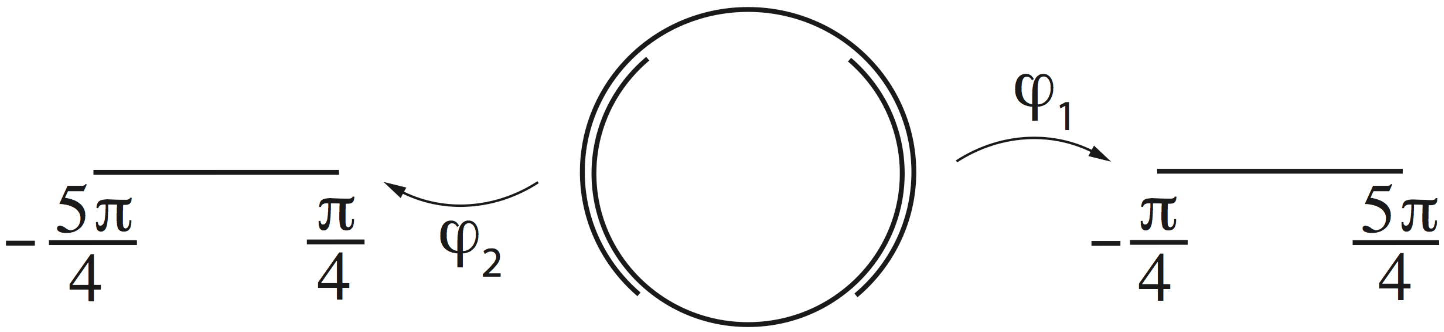


Figure 3: The circle $S^1$ is an example of a manifold that can be covered with two charts that are each locally $\mathbb{R}^1$.

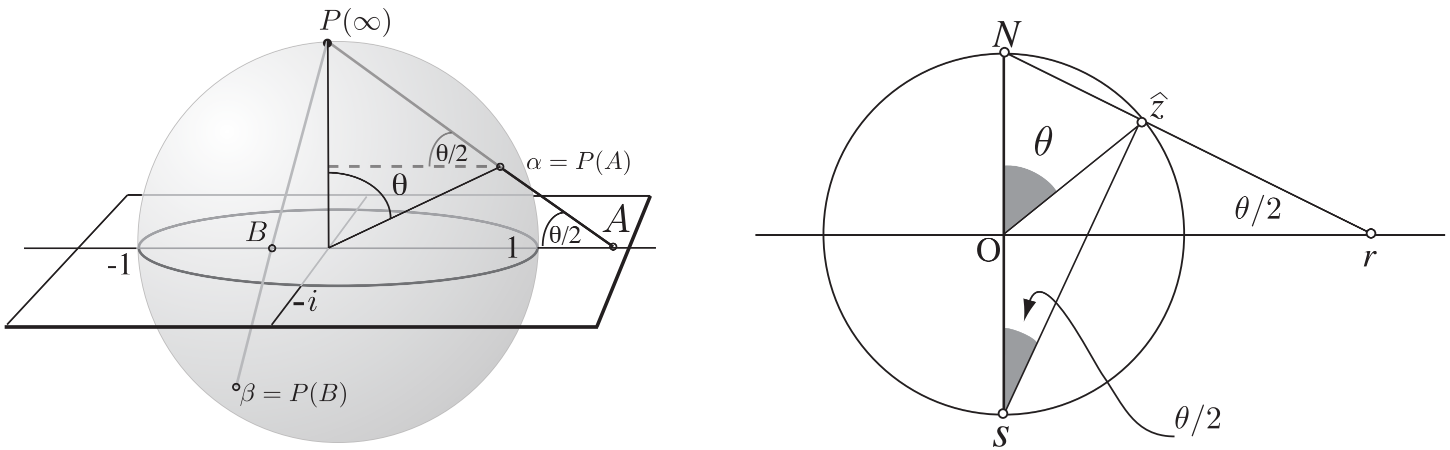


Figure 4: The Riemann map shows that the unit sphere $S^2$ is a manifold that can be covered with two charts that are each locally $\mathbb{R}^2$.

**Exercise.** Figure 4 illustrates Riemann's stereoscopic projection, used in class to show that the circle $S^1$ is a manifold which may be covered by two charts. Derive the values of the stereoscopic projections $x_N$ and $x_S$ from the North and South poles onto the $x$-axis, respectively, of a point on the circle at polar angle $\theta$. Explain the angle $\theta/2$. How are $x_N$ and $x_S$ related to each other? Hint: you may use trigonometry. ★

**Answer.** A point on the circle at polar angle $\theta$ from the North pole has height $z = \cos\theta$. The intersection of its stereographic projection with the $x$-axis is found from the proportion $r = \frac{x_N}{1} = \frac{\sin\theta}{1-\cos\theta} = \cot(\theta/2)$, provided $\cos\theta \neq 1$. The corresponding stereographic projection from the South pole in Figure 4 satisfies the proportion $\frac{x_S}{1} = \frac{\sin\theta}{1+\cos\theta}$, provided $\cos\theta \neq -1$. Consequently, $x_S x_N = 1$, so that $x_S = 1/x_N = \tan(\theta/2)$ for $\theta \neq 0, \pi$. ▲

***Remark.*** The manifold $Q$ may sometimes be identified with a Lie group $G$. We will do this when we consider rotation and translation, for example. In this case, the configurations are obtained from the group action $G \times Q_0 \to Q$ where $Q_0$ is a reference configuration and the group is $G = SE(3)$ the special Euclidean Lie group of motions in three dimensions.

## Time

Time is taken to be a manifold $T$ with points $t \in T$. Usually $T = \mathbb{R}$ (for real 1D time), but we will also consider $T = \mathbb{R}^2$, and the option to let $T$ and $Q$ both be complex manifolds is not out of the question.

## Motion

Motion is a map $\phi_t : T \to Q$, where subscript $t$ denotes dependence on time $t$. For example, when $T = \mathbb{R}$, the motion is a curve $q_t = \phi_t \circ q_0$ obtained by composition of functions. The motion is called a *flow* if $\phi_{t+s} = \phi_t \circ \phi_s$, for $s, t \in \mathbb{R}$, and $\phi_0 = \mathrm{Id}$, so that $\phi_t^{-1} = \phi_{-t}$. Note that the composition of functions is associative, $(\phi_t \circ \phi_s) \circ \phi_r = \phi_t \circ (\phi_s \circ \phi_r) = \phi_t \circ \phi_s \circ \phi_r = \phi_{t+s+r}$, but in general it is not commutative.

When the motion is obtained from a group action $G \times Q_0 \to Q$, then it may be identified with a map $\phi_t : T \to G$, which we may regard as a curve on the group $G$.

Thus, we should anticipate motion and mechanics on Lie group manifolds.

## 1.2 Curves on manifolds and their tangent spaces

The $\boxed{\textit{tangent space}}$ $T_qQ$ contains vectors $v_q = \dot{q}(t) \in T_qQ$, tangent to curve $q(t) \in Q$ at point $q$. The coordinates are $(q, v_q) \in TQ_q$. Note, $\dim T_qQ = 2n$ and subscript $q$ reminds us that $v_q$ is an element of the tangent space at the point $q$ and that on manifolds we must keep track of base points.

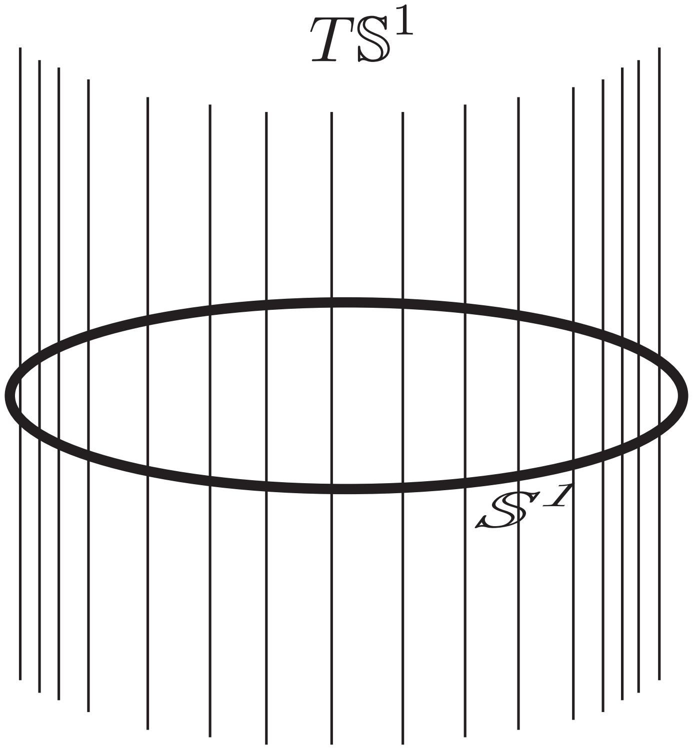


Figure 5: This is a sketch of the tangent bundle $TS^1$ of the circle $S^1$, $TS^1 = \{(\mathbf{x}, \mathbf{v}) \in T\mathbb{R}^2 : |\mathbf{x}|^2 = 1 \text{ and } \mathbf{x} \cdot \mathbf{v} = 0\}$.

The union of tangent spaces $TQ := \cup_{q\in Q} T_qQ$ is also called the $\boxed{\textit{tangent bundle}}$ of the manifold $Q$.
The curve $q(t)$ describes the $\boxed{\textit{motion}}$ on manifold $Q$. The curve $\dot{q}(t) \in T_qQ$ is called the $\boxed{\textit{tangent lift}}$ of the

curve $q(t) \in Q$.

## 1.3 Velocity and the Motion Equation

### Velocity

The tangent lift vector $v_q = \dot{q}(t) \in T_qQ$ is called the *velocity* along a flow $q(t)$ that describes a smooth curve in $Q$.

### Motion Equation

The *motion equation* that determines the flow $q_t \in Q$ takes the form

$$\dot{q}_t = f(q_t)$$

where the map $f : q \in M \to f(q) \in T_qM$ is a prescribed ***vector field*** over $Q$.

For example, if the curve $q_t = \phi_t \circ q_0$ is a flow, then

$$\dot{q}_t = \dot{\phi}_t \phi_t^{-1} \circ q_t = f(q_t)$$

so that

$$\dot{\phi}_t = f \circ \phi_t =: \phi_t^* f$$

which defines the *pullback* of $f$ by $\phi_t$.

### 1.4 Variational Principles for classical mechanics

- Define $\boxed{\textbf{\textit{kinetic energy}}}$ $KE : TM \to \mathbb{R}$, via a *Riemannian metric* $g_q(\,\cdot\,,\,\cdot\,) : TM \times TM \to \mathbb{R}$. Explicitly, $KE = \frac{1}{2} g_q(\dot{q}, \dot{q}) =: \frac{1}{2}\|\dot{q}\|^2$.

- Choose the $\boxed{\textbf{\textit{Lagrangian}}}$ $L : TM \to \mathbb{R}$. (For example, one could choose $L$ to be $KE$.)

- $\boxed{\textbf{\textit{Hamilton's principle}}}$ is $\delta S = 0$ with $S = \int_a^b L(q, \dot{q})dt$, for a family of curves $q(t, \epsilon)$ parameterised smoothly by $(t, \epsilon) \in \mathbb{R} \times \mathbb{R}$. The linearisation

$$\delta S := \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \int_a^b L(q(t,\epsilon), \dot{q}(t,\epsilon))dt \quad \text{with} \quad \delta q(t) := \frac{dq(t,\epsilon)}{d\epsilon}\bigg|_{\epsilon=0}$$

  defines the $\boxed{\textbf{\textit{variational derivative}}}$ $\delta S$ of $S$ near the identity $\epsilon = 0$. The variations in $q$ are assumed to vanish at endpoints in time, so that $q(a, \epsilon) = q(a)$ and $q(b, \epsilon) = q(b)$.

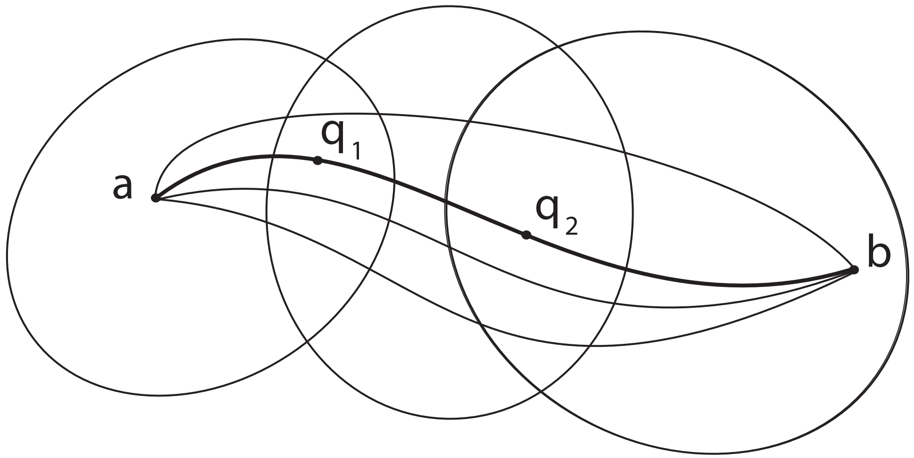


Figure 6: This is a sketch of variations of a family of curves on a manifold.

## 2 Classical mechanics: Euler–Lagrange equation

**Theorem 1** (Hamilton 1835, Euler 1750, Lagrange 1756). *Hamilton's principle* $\delta S = 0$ *with* $S = \int_a^b L(q,\dot{q})dt$ *implies the* $\boxed{\textbf{\textit{Euler–Lagrange (EL) equation:}}}$

$$\frac{d}{dt}\frac{\partial L(q,\dot{q})}{\partial \dot{q}} = \frac{\partial L(q,\dot{q})}{\partial q}\,, \quad \textit{for any } L(q,\dot{q})\,.$$

$\boxed{\textbf{Proof 1}}$ Vary the curve $q(t)$ in the family $q(t,\epsilon) \in \mathcal{C}(Q)$ using the linearisation

$$\delta S := \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \int_a^b L(q(t,\epsilon),\dot{q}(t,\epsilon))dt \quad \text{with} \quad \delta q(t) := \frac{dq(t,\epsilon)}{d\epsilon}\bigg|_{\epsilon=0}$$

and set $\delta\frac{dq}{dt} = \frac{d}{dt}\delta q$ in the variation of the action $S$ as

$$\delta S = \delta \int_a^b L(q,\dot{q})dt = \int_a^b \delta L(q,\dot{q})dt = \int_a^b \left\langle \frac{\partial L}{\partial \dot{q}}, \delta\dot{q} \right\rangle + \underbrace{\left\langle \frac{\partial L}{\partial q}, \delta q \right\rangle}_{\text{Pairing}} dt = \int_a^b \left\langle \frac{\partial L}{\partial \dot{q}}, \frac{d}{dt}\delta q \right\rangle + \left\langle \frac{\partial L}{\partial q}, \delta q \right\rangle dt$$

$$= \int_a^b \left\langle \underbrace{-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} + \frac{\partial L}{\partial q}}_{\text{EL equation}}\,, \delta q \right\rangle dt \;+\; \underbrace{\left\langle \frac{\partial L}{\partial \dot{q}}, \delta q \right\rangle\bigg|_a^b}_{\text{Endpoint term} \,=\, 0} \qquad \square$$

$\boxed{\textbf{Proof}}$ Vary coordinates $(q, v) \in TQ$, subject to the constraint $v = \frac{dq}{dt}$ (tangent lift)

$$\delta S = \delta \int_a^b L(q,v) + \left\langle p, \frac{dq}{dt} - v \right\rangle dt \ = \int_a^b \left\langle \frac{\partial L}{\partial v} - p, \delta v \right\rangle + \left\langle \frac{\partial L}{\partial q} - \frac{dp}{dt}, \delta q \right\rangle + \Big\langle \delta p, \dot{q} - v \Big\rangle dt + \Big\langle p, \delta q \Big\rangle \Big|_a^b$$

Then we assemble the EL equation from the various stationary conditions, and evaluate $\frac{\partial L}{\partial v}\big|_{v=\dot{q}}$.

## 2.1 What to do in solving a Lagrangian mechanics problem

1. Define the configuration manifold of a given mechanical system and find a suitable coordinate system on that manifold.
2. Find the Lagrangian and compute the Euler-Lagrange equations.
3. For simple (one-dimensional) systems, sketch phase portraits.
4. Compute energy and find how it evolves as a function of time.
5. Find the equilibria of the system and compute linear oscillations about that equilibrium by finding the normal frequencies and normal modes.
6. Find integrals of motion for a Lagrangian system using symmetries of the Lagrangian and Noether's theorem.
7. Compute the Legendre transformation from the Lagrangian to the Hamiltonian description and derive Hamilton's canonical equations.
8. Write the Poisson brackets and find the evolution equations for phase space functions of $(q, p)$ and possibly of $t$.
9. Determine the equilibria of the system from critical points of the sum of the Hamiltonian and constants of the motion.
10. Determine the stability of the equilibrium solutions by taking a second variation around the equilibrium and finding whether the corresponding Hamiltonian for the linearised problem is definite in sign.
11. Reduce by a symmetry of the Lagrangian
12. Repeat from 1. above.

## 2.2 Example - The isoperimetric problem (what Lagrange wrote to Euler about).

The problem is to find the curve between two points $(x_1, y_1)$ and $(x_2, y_2)$, of specified length, that maximises the area integral $\int_{x_1}^{x_2} y(x)dx$.

In this example the length of the curve is

$$L[y] = \int_{x_1}^{x_2} \sqrt{dx^2 + dy^2} = \int_{x_1}^{x_2} \sqrt{1 + y'^2}dx, \quad \text{with} \quad y'(x) = dy/dx\,,$$

which takes the specified value $l = const$. The area is

$$A[y] = \iint dx \wedge dy = \int_{x_1}^{x_2} y(x)dx.$$

We look for extrema of the modified functional

$$S[y] = \int_{x_1}^{x_2} y dx - \lambda \int_{x_1}^{x_2} (\sqrt{1 + y'^2}dx - l\,),$$

where $\lambda$ is a scalar constant (Lagrange multiplier), to be determined. The Euler-Lagrange equation is

$$\lambda \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y'}{\sqrt{1 + y'^2}}\right) + 1 = 0\,. \tag{1}$$

Hence, a first integration yields $\frac{y'}{\sqrt{1+y'^2}} = -(x - x_0)/\lambda$, giving the parametric solution, after solving for $y'^2$,

$$x = x_0 \pm \lambda \sin(\theta), \qquad y = y_0 \pm \lambda \cos(\theta), \tag{2}$$

so $(x - x_0)^2 + (y - y_0)^2 = \lambda^2$ and the extremum is the arc of a circle of radius $\lambda$.

The variational problem satisfied by a soap bubble is analogous to the isoperimetric problem. For the soap bubble, the surface area is extremised, holding the volume integral constant. The Lagrange multiplier is the pressure, $p$.

## 2.3 Example: Hamilton's Principle for geodesics (covariant derivatives)

- $\boxed{\textit{Geodesics:}}$ When $L = KE = \frac{1}{2}g_q(\dot{q},\dot{q}) =: \frac{1}{2}\|\dot{q}\|^2$, the solution $q(t)$ of the EL equations that passes from point $q(a)$ to $q(b)$ is called the *geodesic path* with respect to the metric $g_q : TM \times TM \to \mathbb{R}$. The geodesic represents the path of shortest distance $q(a) \to q(b)$ measured by

$$ds^2 := dq^a g_{ab}(q) dq^b = g_q(\dot{q},\dot{q})dt^2 = \|\dot{q}\|^2\, dt^2$$

- $\boxed{\textit{Exercise:}}$ Compute the EL equations for a geodesic with respect to the metric $g_q : TM \times TM \to \mathbb{R}$. That is, compute the EL equations for $L = KE = \frac{1}{2}g_q(\dot{q},\dot{q}) =: \frac{1}{2}\|\dot{q}\|^2$.

- $\boxed{\textit{Answer:}}$ The KE Lagrangian is

$$L(q,\,\dot{q}) = \frac{1}{2}\dot{q}^b g_{bc}(q)\dot{q}^c\,.$$

Its partial derivatives are given by

$$\frac{\partial L}{\partial \dot{q}^a} = g_{ac}(q)\dot{q}^c \quad \text{and} \quad \frac{\partial L}{\partial q^a} = \frac{1}{2}\frac{\partial g_{bc}(q)}{\partial q^a}\dot{q}^b\dot{q}^c\,.$$

Consequently, its Euler–Lagrange equations are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^a} - \frac{\partial L}{\partial q^a} \;=\; g_{ae}(q)\ddot{q}^{\,e} + \frac{\partial g_{ae}(q)}{\partial q^b}\dot{q}^b\dot{q}^e - \frac{1}{2}\frac{\partial g_{be}(q)}{\partial q^a}\dot{q}^b\dot{q}^e = 0\,.$$

Symmetrising the coefficient of the middle term and contracting with co-metric $g^{ca}$ satisfying $g^{ca}g_{ae} = \delta^c_e$ yields

$$\boxed{\ddot{q}^{\,c} + \Gamma^c_{be}(q)\dot{q}^b\dot{q}^e = 0} \quad \text{with} \quad \Gamma^c_{be}(q) = \frac{1}{2}g^{ca}\left[\frac{\partial g_{ae}(q)}{\partial q^b} + \frac{\partial g_{ab}(q)}{\partial q^e} - \frac{\partial g_{be}(q)}{\partial q^a}\right], \tag{3}$$

in which the $\Gamma^c_{be}$ are called the ***Christoffel symbols*** for the Riemannian metric $g_{ab}$.

These Euler–Lagrange equations are the ***geodesic equations*** of a free particle moving in a Riemannian space. They are often written as

$$\ddot{q} + \nabla_{\dot{q}}\dot{q} = 0,$$

in terms of the $\boxed{\textit{covariant derivative}}$ $\nabla_{\dot{q}}$.

## 2.4 Ten examples of Hamilton's principle for Simple Mechanical Systems

For simple mechanical systems, $L(q, \dot{q}) = T(\dot{q}) - V(q) = KE - PE$. For example,

1. Planar isotropic oscillator, $(\mathbf{x}, \dot{\mathbf{x}}) \in T\mathbb{R}^2$: $\quad L = \frac{m}{2}|\dot{\mathbf{x}}|^2 - \frac{k}{2}|\mathbf{x}|^2 \quad \Longrightarrow \quad \ddot{\mathbf{x}} = -\,\omega^2\mathbf{x} \quad \text{with} \quad \omega^2 = k/m$

2. Planar anisotropic oscillator, $(\mathbf{x}, \dot{\mathbf{x}}) \in T\mathbb{R}^2$: $\quad L = \frac{m}{2}|\dot{\mathbf{x}}|^2 - \frac{k_1}{2}x_1^2 - \frac{k_2}{2}x_2^2 \quad \Longrightarrow \quad \ddot{x}_i = -\,\omega_i^2 x_i \quad \text{with} \quad \omega_i^2 = k_i/m \quad i = 1, 2$

3. Planar pendulum, $(\mathbf{x}, \dot{\mathbf{x}}) \in T\mathbb{R}^2$, constrained to $TS^1 = \{\mathbf{x}, \dot{\mathbf{x}} \in T\mathbb{R}^2 |\, 1 - |\mathbf{x}|^2 = 0 \;\&\; \mathbf{x}\cdot\dot{\mathbf{x}} = 0\}$: $\quad L = \frac{m}{2}|\dot{\mathbf{x}}|^2 - mg\,\hat{\mathbf{e}}_2\cdot\mathbf{x} - \frac{\mu}{2}(1 - |\mathbf{x}|^2) \quad \Longrightarrow \quad m\ddot{\mathbf{x}} = -\,mg\hat{\mathbf{e}}_2\cdot(\mathrm{Id} - \mathbf{x}\otimes\mathbf{x}) - \,\mathrm{m}|\dot{\mathbf{x}}|^2\mathbf{x}$, (gravity & centripetal force)

4. Planar pendulum motion lifted to a curve in $SO(2)$: $\mathbf{x}(t) = O(\theta(t))\mathbf{x}_0 \in \mathbb{R}^2\,, \quad O(\theta(t)) \in SO(2)\,, \quad |\mathbf{x_0}|^2 = R^2$, where $\mathbf{x}(0) = \mathbf{x}_0$.

   $\dot{\mathbf{x}}(t) = \dot{O}O^{-1}(t)\mathbf{x} = \dot{\theta}(t)\,\hat{\mathbf{e}}_3\times\mathbf{x}$ for $(\theta, \dot{\theta}) \in TSO(2)\,,$

   $L = \frac{m}{2}R^2\dot{\theta}^2 - mgR(1 - \cos\theta) \quad \Longrightarrow \quad \ddot{\theta} = -\omega^2\sin\theta \quad \text{with} \quad \omega^2 = g/R$

5. Charged particle in a magnetic field, $(\mathbf{x}, \dot{\mathbf{x}}) \in T\mathbb{R}^2$: $\quad L = \frac{m}{2}|\dot{\mathbf{x}}|^2 + \frac{e}{c}\dot{\mathbf{x}} \cdot \mathbf{A}(\mathbf{x}) \quad \Longrightarrow \quad \ddot{\mathbf{x}} = \frac{e}{mc}\dot{\mathbf{x}} \times \mathbf{B}$ with $\mathbf{B} = \operatorname{curl} \mathbf{A}$

6. Kepler problem, $(r, \dot{r}, \theta, \dot{\theta}) \in T\mathbb{R}_+ \times TS^1$: $\quad L = \frac{m}{2}\Big(\dot{r}^2 + r^2\dot{\theta}^2\Big) + \frac{GMm}{r} \quad \Longrightarrow \quad \ddot{r} = -\frac{GM}{r^2} + \frac{J^2}{r^3}$ with $J = r^2\dot{\theta} = const$

7. Free motion on a sphere, $(\mathbf{x}, \dot{\mathbf{x}}) \in T\mathbb{R}^3$, constrained to $TS^2 = \{(\mathbf{x}, \dot{\mathbf{x}}) \in T\mathbb{R}^3 : |\mathbf{x}| = 1 \;\&\; \mathbf{x} \cdot \dot{\mathbf{x}} = 0\}$: $L = \frac{1}{2}|\dot{\mathbf{x}}|^2 + \frac{1}{2}\mu(1 - |\mathbf{x}|^2)$

8. Spherical pendulum (a), $(\mathbf{x}, \dot{\mathbf{x}}) \in T\mathbb{R}^3$, on $TS^2 = \{(\mathbf{x}, \dot{\mathbf{x}}) \in T\mathbb{R}^3 : |\mathbf{x}| = 1 \;\&\; \mathbf{x} \cdot \dot{\mathbf{x}} = 0\}$: $L = \frac{m}{2}|\dot{\mathbf{x}}|^2 - mg\,\hat{\mathbf{e}}_3 \cdot \mathbf{x} + \frac{1}{2}\mu(1 - |\mathbf{x}|^2)$

9. Spherical pendulum (b), setting $\mathbf{x}(t) = O(t)\mathbf{x}_0\,, \quad \dot{\mathbf{x}}(t) = \dot{O}(t)\mathbf{x}_0$ for $(O,\, \dot{O}) \in TSO(3)\,$, where $\mathbf{x}_0 = \mathbf{x}(0)$ initially and

$$
\begin{aligned}
L &= \frac{m}{2}|\dot{\mathbf{x}}|^2 - mg\,\hat{\mathbf{e}}_3 \cdot \mathbf{x} \\
&= \frac{m}{2}|\dot{O}(t)\mathbf{x}_0|^2 - mg\,O^T(t)\hat{\mathbf{e}}_3 \cdot \mathbf{x}_0 \\
&= \frac{m}{2}|O^{-1}\dot{O}(t)\mathbf{x}_0|^2 - mg\,O^{-1}(t)\hat{\mathbf{e}}_3 \cdot \mathbf{x}_0 \\
&= \frac{m}{2}|\boldsymbol{\Omega} \times \mathbf{x}_0|^2 - mg\,\boldsymbol{\Gamma} \cdot \mathbf{x}_0
\end{aligned}
$$

where $(O^{-1}\dot{O})_{ij} = -\epsilon_{ijk}\Omega^k$ and $\boldsymbol{\Gamma} = O^{-1}(t)\hat{\mathbf{e}}_3$. Rotations preserve length, so $|\mathbf{x}|^2 = \mathbf{x}_0|^2 = 1$. Set $g = 0$ for free motion on $TS^2$.

10. Rotating rigid body, $\widehat{\Omega} = O^{-1}\dot{O} \in T(SO(3) \simeq \mathfrak{so}(3) \quad \ell = \frac{1}{2}\boldsymbol{\Omega} \cdot I\boldsymbol{\Omega}$ with $\boldsymbol{\Omega}\times = \widehat{\Omega}\,,$ that is, $-\epsilon_{ijk}\Omega^k = \widehat{\Omega}_{ij}$

# 3 Classical mechanics: Hamilton's equations

## 3.1 Legendre transform

- $LT : (q, \dot{q}) \in TM \to (q, p) \in T^*M$ defines *momentum* $p$ as the *fibre derivative* of $L$, namely

$$p := \frac{\partial L(q, \dot{q})}{\partial \dot{q}} \in T^*M \qquad \textit{(fibre derivative)}\,.$$

The LT is *invertible* for $\dot{q} = f(q, p)$, provided the *Hessian* $\partial^2 L(q, \dot{q})/\partial\dot{q}\partial\dot{q}$ has nonzero determinant. Note, $\dim T^*M = 2n$.

- In terms of the LT, the ***Hamiltonian*** $H : T^*M \to \mathbb{R}$ is defined by

$$H(q, p) = \Big\langle p, \dot{q} \Big\rangle - L(q, \dot{q})$$

in which the expression $\langle p, \dot{q}\rangle$ in this calculation identifies a *pairing* $\langle \,\cdot\,,\,\cdot\,\rangle : T^*M \times TM \to \mathbb{R}$.

Taking the differential of this definition yields

$$\begin{aligned} dH &= \Big\langle H_p, dp \Big\rangle + \Big\langle H_q, dq \Big\rangle \\ &= \Big\langle dp, \dot{q} \Big\rangle + \Big\langle p - L_{\dot{q}}, d\dot{q} \Big\rangle - \Big\langle L_q, dq \Big\rangle \end{aligned}$$

from which Hamilton's principle $\delta S = 0$ for $S = \int_a^b \langle p, \dot{q}\rangle - H(q, p)\, dt$ produces *Hamilton's canonical equations* on phase space $T^*M$,

$$H_p = \dot{q} \quad \text{and} \quad H_q = -\,L_q = -\,\dot{p}\,.$$

- Hamilton's principle $\delta S = 0$ for $S = \int_a^b \langle p, \dot{q}\rangle - H(q,p)\,dt$ produces *Hamilton's canonical equations* on phase space $T^*M$,

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**Exercise.** Verify the previous statement. That is, compute the results of the following Phase-space form of Hamilton's principle on $T^*M$, given by $\delta S = 0$ with $S = \int_a^b \langle p, \dot{q}\rangle - H(q,p)\,dt$. ★

- **Answer.** One computes

$$\begin{aligned}
\delta S &= \delta \int_a^b \langle p, \dot{q}\rangle - H(q,p)\,dt = \int_a^b \delta\langle p, \dot{q}\rangle - \delta H(q,p)\,dt \\
&= \int_a^b \Big\langle \delta p\,, \dot{q} - H_p \Big\rangle - \Big\langle \dot{p} + H_q, \delta q \Big\rangle dt + \underbrace{\Big\langle p, \delta q \Big\rangle\Big|_a^b}_{\text{Endpoint term}}
\end{aligned}$$

▲

**Remark 2.** *We will return to the endpoint term in formulating Noether's theorem on phase space, that is, on* $T^*M$.

### 3.2 Legendre transform in simple mechanical systems – Exercise sheet

- Legendre transform: $H(q, p) = \langle p,\, \dot{q}\rangle - L(q, \dot{q}) = T(p) + V(q) = KE + PE$.

  For example,

  1. Planar isotropic oscillator, $(\mathbf{x}, \mathbf{p}) \in T^*\mathbb{R}^2$: $\quad H = \frac{1}{2m}|\mathbf{p}|^2 + \frac{k}{2}|\mathbf{x}|^2$
  2. Planar anisotropic oscillator, $(\mathbf{x}, \mathbf{p}) \in T^*\mathbb{R}^2$: $\quad H = \frac{1}{2m}|\mathbf{p}|^2 + \frac{k_1}{2}x_1^2 + \frac{k_2}{2}x_2^2$
  3. Planar pendulum in polar coordinates, $(\theta, p_\theta) \in T^*S^1$: $\quad H = \frac{1}{2mR^2}p_\theta^2 + mgR(1 - \cos\theta)$
  4. Planar pendulum, $(\mathbf{x}, \mathbf{p}) \in T^*\mathbb{R}^2$, constrained to $S^1 = \{\mathbf{x} \in \mathbb{R}^2 : 1 - |\mathbf{x}|^2 = 0\}$: $\quad H = \frac{1}{2m}|\mathbf{p}|^2 + mg\,\hat{\mathbf{e}}_2 \cdot \mathbf{x} - \mu(1 - |\mathbf{x}|^2)$
  5. Charged particle in a magnetic field, $(\mathbf{x}, \mathbf{p}) \in T^*\mathbb{R}^2$: $\quad H = \frac{1}{2m}|\mathbf{p} - \frac{e}{c}\mathbf{A}(\mathbf{x})|^2 \quad \mathbf{p} := \partial L/\partial \dot{\mathbf{q}} = m\dot{\mathbf{x}} + \frac{e}{c}\mathbf{A}(\mathbf{x}) \in T^*M$
  6. Kepler problem, $(r, p_r, \theta, p_\theta) \in T^*\mathbb{R}_+ \times T^*S^1$: $\quad H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} - \frac{GMm}{r}$ with $p_\theta = r^2\dot{\theta} = const$
  7. Free motion on a sphere, $(\mathbf{x}, \mathbf{p}) \in T^*\mathbb{R}^3$, constrained to $S^2 = \{\mathbf{x} \in \mathbb{R}^3 : 1 - |\mathbf{x}|^2 = 0\}$: $\quad H = \frac{1}{2m}|\mathbf{p}|^2 - \mu(1 - |\mathbf{x}|^2)$
  8. Spherical pendulum (a), $(\mathbf{x}, \mathbf{p}) \in T^*\mathbb{R}^3$, constrained to $S^2 = \{\mathbf{x} \in \mathbb{R}^3 : 1 - |\mathbf{x}|^2 = 0\}$: $\quad H = \frac{1}{2m}|\mathbf{p}|^2 + mg\,\hat{\mathbf{e}}_3 \cdot \mathbf{x} - \mu(1 - |\mathbf{x}|^2)$

9. Spherical pendulum (b), $(O,\,\dot{O}) \in TSO(3)$, $\widehat{\xi} = O^{-1}\dot{O} \in T(SO(3) \simeq \mathfrak{so}(3)$, $\mathbf{\Pi} = \partial\ell/\partial\mathbf{\Omega} \in T^*(SO(3) \simeq \mathfrak{so}(3)^* \simeq \mathbb{R}^3 \quad H = \frac{1}{2}\mathbf{\Pi}\cdot I^{-1}\mathbf{\Pi} + g\,\mathbf{\Gamma}\cdot\mathbf{x}_0$ with $\mathbf{\Pi} = \frac{\partial\ell}{\partial\mathbf{\Omega}} = I\mathbf{\Omega}$. Set $g=0$ to get freely rotating rigid body motion.
10. Rotating rigid body, $\mathbf{\Pi} \in T^*(SO(3) \simeq \mathfrak{so}(3)^* \simeq \mathbb{R}^3 \quad H = \frac{1}{2}\mathbf{\Pi}\cdot I^{-1}\mathbf{\Pi}$ with $\mathbf{\Pi} = \frac{\partial\ell}{\partial\mathbf{\Omega}} = I\mathbf{\Omega}$.

### 3.3 Canonical Poisson bracket

- The Hamiltonian dynamics of a phase-space function is given by

$$\frac{d}{dt}F(q,p) = \frac{\partial F}{\partial q}\dot{q} + \frac{\partial F}{\partial p}\dot{p} = \frac{\partial F}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial F}{\partial p}\frac{\partial H}{\partial q} := \{F,\,H\}$$

The operation $\{F,\,H\}$ is called the *canonical Poisson bracket* of $F$ with $H$ on the phase space $T^*M$.

The canonical Poisson bracket operation $\{\cdot\,,\,\cdot\}$ is a map among smooth real functions $\mathcal{F}(T^*M) : T^*M \to \mathbb{R}$

$$\{\cdot\,,\,\cdot\} : \mathcal{F}(T^*M) \times \mathcal{F}(T^*M) \to \mathcal{F}(T^*M)\,, \tag{4}$$

so that Hamiltonian dynamics *on phase space* $T^*M$ is given by $\dot{F} = \{F\,,\,H\}$ for any $F \in \mathcal{F}(T^*M)$.

**Definition 3** (Poisson bracket)**.** *A* ***Poisson bracket operation*** $\{\cdot\,,\,\cdot\}$ *is defined by its properties listed below:*

- *It is* ***bilinear****.*
- *It is* ***skew-symmetric****,* $\{F\,,\,H\} = -\{H\,,\,F\}$.
- *It satisfies the* ***Leibniz rule*** *(product rule),*

$$\{FG\,,\,H\} = \{F\,,\,H\}G + F\{G\,,\,H\}\,,$$

*for the product of any two functions $F$ and $G$ on $M$.*

– *It satisfies the* ***Jacobi identity****,*

$$\{F\,,\,\{G\,,\,H\}\}+\{G\,,\,\{H\,,\,F\}\}+\{H\,,\,\{F\,,\,G\}\}=0\,, \tag{5}$$

*for any three functions $F$, $G$ and $H$ on $M$.*

***Remark.*** The Leibniz rule associates Poisson brackets with differential operators on smooth functions $F \in \mathcal{F}(T^*M)$.

The differential operator or $\boxed{\textbf{\textit{Hamiltonian vector field}}}$ generated by the canonical Poisson bracket with $F$ is

$$X_F := \{\,\cdot\,,\,F\,\} = \frac{\partial F}{\partial p}\frac{\partial}{\partial q} - \frac{\partial F}{\partial q}\frac{\partial}{\partial p}\,.$$

- $\boxed{\textbf{\textit{Exercise:}}}$ What is $\boxed{\textbf{\textit{Noether's theorem for Hamilton's principle in phase-space, on }} T^*M}$?

- $\boxed{\textbf{\textit{Answer:}}}$ For an infinitesimal transformation $(\delta q\,,\delta p)$ that induces $\delta L = \delta\big(\langle p,\dot q\rangle - H(q,p)\big)$ we have

$$\delta S = \delta\int_a^b \Big\langle p,\dot q\Big\rangle - H(q,p)\,dt = \int_a^b \delta\Big\langle p,\dot q\Big\rangle - \delta H(q,p) = \int_a^b \Big\langle \delta p\,,\dot q - H_p\Big\rangle - \Big\langle \dot p + H_q,\delta q\Big\rangle dt + \underbrace{\Big\langle p,\delta q\Big\rangle\Big|_a^b}_{\text{Endpoint}}$$

### 3.4 Cotangent lift and Noether's theorem on the Hamiltonian side

Suppose the variations due to the infinitesimal transformations on $M$ take the form $\delta q = \xi_M(q)$. Then the corresponding Hamiltonian for these infinitesimal transformations is

$$J^\xi := \Big\langle p, \xi_M(q) \Big\rangle \quad \text{so that} \quad \delta q = \frac{\partial J^\xi}{\partial p} = \xi_M(q) \quad \text{and} \quad \delta p = -\,\frac{\partial J^\xi}{\partial q} = -\,\xi'_M(q)^T p$$

The last expression is called the $\boxed{\textbf{\textit{cotangent lift}}}$ to $T_q^*M$ of the infinitesimal transformation $q \to q_\epsilon = q + \epsilon\xi_M(q)$ on $M$.

The cotangent lift specifies the infinitesimal transformation of $p \in T_q^*M$, given the infinitesimal transformation of $q \in M$.

$$q \to q_\epsilon = q + \epsilon\xi_M(q) \text{ on } M \Longrightarrow (q,p) \to (q_\epsilon, p_\epsilon) = (q + \epsilon\xi_M(q),\, p -\, \epsilon\xi'_M(q)^T p) \text{ on } T_q^*M\,.$$

The time derivative of $J^\xi(q,p)$ is given by

$$\frac{d}{dt}J^\xi(q,p) = \frac{\partial J^\xi}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial J^\xi}{\partial p}\frac{\partial H}{\partial q} = -\,\frac{\partial H}{\partial p}\delta p - \frac{\partial H}{\partial q}\delta q = -\,\delta H = \{J^\xi, H\} = -\,\{H, J^\xi\} = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} H(q,p).$$

In the last step we defined the infinitesimal transformation of $H$ under canonical transformations generated by $J^\xi := \langle p, \xi_M(q)\rangle := \langle p, \delta q\rangle$, the conserved endpoint term in Noether's theorem. This calculation proves the following.

**Corollary 4.** *On the Hamiltonian side, Noether's theorem for conservation of the endpoint term* $J^\xi := \langle p, \xi_M(q)\rangle := \langle p, \delta q\rangle$ *follows from Lie symmetry of the Hamiltonian function under* $\delta H = \{H, J^\xi\} = 0$.

**Remark 5.** *The differential operator or* $\boxed{\textbf{\textit{Hamiltonian vector field}}}$ *generated by the canonical Poisson bracket with* $J^\xi$ *is defined by*

$$\frac{d}{d\epsilon} = X_{J^\xi} := \{\,\cdot\,,\,J^\xi\,\} = \frac{\partial J^\xi}{\partial p}\frac{\partial}{\partial q} - \frac{\partial J^\xi}{\partial q}\frac{\partial}{\partial p} = \xi_M(q)\frac{\partial}{\partial q} - \xi\,'(q)^T p\frac{\partial}{\partial p} = \delta q\frac{\partial}{\partial q} + \delta p\frac{\partial}{\partial p}\,.$$

### 3.5 Quick review: Angular velocity and angular momentum

Let $G \times M \to M$ with $G = SO(3)$ and $M = \mathbb{R}^3$. That is, $SO(3) \times \mathbb{R}^3 \to \mathbb{R}^3$.

Let $q(\epsilon) = O(\epsilon)q(0)$ with $O \in SO(3)$, so that $O^TO = Id$ and $q \in \mathbb{R}^3$. Then the infinitesimal transformation is[1]

$$\delta q := q'(\epsilon)\big|_{\epsilon=0} = \big[O'(\epsilon)q(0)\big]_{\epsilon=0} = \big[O'(\epsilon)O^{-1}(\epsilon)q(\epsilon)\big]_{\epsilon=0} := \widehat{\xi}q = \xi \times q \quad \text{with} \quad \widehat{\xi}_{ab} = -\epsilon_{abc}\xi^c.$$

**Remark 6** ($\boxed{\textbf{\textit{Hat map}}}$)**.** The components of any $3 \times 3$ skew matrix $\widehat{\xi}$ may be identified with the corresponding components of a vector $\boldsymbol{\xi} \in \mathbb{R}^3$, by the linear invertible relation,

$$\widehat{\xi} = \begin{pmatrix} 0 & -\xi^3 & \xi^2 \\ \xi^3 & 0 & -\xi^1 \\ -\xi^2 & \xi^1 & 0 \end{pmatrix} \quad \text{with} \quad \widehat{\xi}_{ab} = -\epsilon_{abc}\xi^c\,, \tag{6}$$

for $a, b, c = 1, 2, 3$. This is an ***isomorphism*** (one-to-one invertible map) between $3 \times 3$ skew-symmetric matrices and vectors in $\mathbb{R}^3$.

**Remark 7** ($\boxed{\textbf{\textit{Hat map}}}$)**.** The overscript hat ($\widehat{\ }$) applied to a vector identifies that vector in $\mathbb{R}^3$ with a $3 \times 3$ skew-symmetric matrix. For example, the unit vectors in the Cartesian basis set, $\{\mathbf{e}_1,\,\mathbf{e}_2,\,\mathbf{e}_3\}$, are associated with the basis elements of $\mathfrak{so}(3)$, by $\widehat{\mathbf{e}}_a$, or in matrix components,

$$(\widehat{\mathbf{e}}_a)_{bc} = -\delta^d_a\epsilon_{dbc} = -\epsilon_{abc} = (\mathbf{e}_a\times)_{bc}\,.$$

[1] The matrix $\widehat{\xi} = \dot{O}O^{-1} = \dot{O}O^T$ is skew, since $\frac{d(OO^T)}{dt} = \frac{d(Id)}{dt} = \dot{O}O^T + O\dot{O}^T = \dot{O}O^T + (\dot{O}O^T)^T = \widehat{\xi} + \widehat{\xi}^T = 0$.

This equation introduces the convenient notation $\widehat{\boldsymbol{e}}$ that denotes the basis for the $3 \times 3$ skew-symmetric matrices $\widehat{e}_a$, with $a = 1, 2, 3$ as a vector of matrices. One may check the commutator $[\widehat{e}_a, \widehat{e}_b] = \epsilon_{abc}\widehat{e}_c$; so that

$$[\,\widehat{\xi},\,\widehat{\eta}\,] = \boldsymbol{\xi} \times \boldsymbol{\eta} \cdot \widehat{\boldsymbol{e}} =: (\boldsymbol{\xi} \times \boldsymbol{\eta})\widehat{\ }$$

for $\widehat{\xi} = \xi^a \widehat{e}_a$ and $\widehat{\eta} = \eta^b \widehat{e}_b$.

### 3.5.1 Quick review of the angular momentum map

The Hamiltonian

$$J^\xi(q,p) = q \times p \cdot \xi = p \cdot \xi_M(q) = p \cdot \xi \times q$$

generates infinitesimal $SO(3)$ rotations around the vector $\xi \in \mathfrak{so}(3) \simeq \mathbb{R}^3$, as we compute

$$\delta q = \big\{q, J^\xi(q,p)\big\} = \xi \times q(t), \quad \delta p = \big\{p, J^\xi(q,p)\big\} = \xi \times p(t),$$

using the canonical Poisson bracket $\big\{\,\boldsymbol{\cdot}\,,\,\boldsymbol{\cdot}\,\big\}$. Thus, the $\boxed{\textit{\textbf{cotangent lift}}}$ of an infinitesimal rotation of $q$ given by $\xi_M(q) = \xi \times q$ is an infinitesimal rotation of $p$ given by $-\,\xi_M'(q)^T p = \xi \times p$. These equations imply the following variation for $J(q,p) = q \times p \in \mathfrak{so}(3)^* \simeq \mathbb{R}^3$

$$\delta J = \xi \times J(t) \quad \text{for } \xi \in \mathfrak{so}(3) \simeq \mathbb{R}^3 \text{ and } J \in \mathfrak{so}(3)^* \simeq \mathbb{R}^3\,,$$

as obtained by using the product rule for the Poisson bracket and the Jacobi identity for the cross product of vectors in $\mathbb{R}^3$. [2]

The quantity $J(q,p) = q \times p \in \mathfrak{so}(3)^* \simeq \mathbb{R}^3$ is called the $\boxed{\textit{\textbf{angular momentum}}}$.

The map $J(q,p) = q \times p : T_q^*M \to \mathfrak{so}(3)^* \simeq \mathbb{R}^3$ is the $\boxed{\textit{\textbf{cotangent lift momentum map}}}$ for the action of the Lie group of spatial rotations $G = SO(3)$ on the manifold $M = \mathbb{R}^3$.

[2] Thus, $\delta J = \{q \times p, J^\xi(q,p)\} = \delta(q \times p) = \delta q \times p + q \times \delta p = (\xi \times q) \times p + q \times (\xi \times p) = p \times (q \times \xi) + q \times (\xi \times p) = -\,\xi \times (p \times q) = \xi \times (q \times p)$.

### 3.6 The angular momentum map: Lie-Poisson brackets for $SO(3)$.

$\boxed{\textbf{\textit{Exercise:}}}$ Show for vectors $\xi, \eta \in \mathbb{R}^3$ that for angular momentum $J(q,p) = q \times p \in \mathfrak{so}(3)^* \simeq \mathbb{R}^3$ and $J^\xi = \xi \cdot J(q,p)$ that

$$\Big\{ J^\xi ,\, J^\eta \Big\} = J^{\xi \times \eta}.$$

$\boxed{\textbf{\textit{Answer:}}}$ The proof follows by a direct calculation using Jacobi's identity for vector cross products:[3]

$$\Big\{ J^\xi ,\, J^\eta \Big\} = \Big\{ J \cdot \xi ,\, J \cdot \eta \Big\} = \Big\{ q \times p \cdot \xi ,\, q \times p \cdot \eta \Big\}_{can} = (q \times p) \cdot (\xi \times \eta) = J \cdot (\xi \times \eta) = J^{\xi \times \eta}.$$

Hence, for functions $F(J(q,p)) = F \circ J$ and $H(J(q,p)) = H \circ J$ of the angular momentum map $J$ we have

$$\Big\{ J_k ,\, J_l \Big\} = \epsilon_{kl}{}^m J_m \quad \text{and} \quad \Big\{ F(J) ,\, H(J) \Big\} = J \cdot \frac{\partial F}{\partial J} \times \frac{\partial H}{\partial J} \quad \text{so that} \quad \frac{dJ}{dt} = \Big\{ J ,\, H(J) \Big\} = -J \times \frac{\partial H}{\partial J}\,.$$

Thus, the angular momentum map $J(q,p) : T^*\mathbb{R}^3 \to \mathbb{R}^3$ is $\boxed{\textbf{\textit{Poisson}}}$, which means that

$$\{ F \circ J \,, H \circ J \,\} = \{ F ,\, H \} \circ J \,.$$

[3]Using the calculation in the previous footnote, $\Big\{ q \times p \cdot \xi ,\, q \times p \cdot \eta \Big\}_{can} = -\eta \cdot \{ q \times p, J^\xi(q,p) \}_{can} = -\eta \cdot \xi \times (q \times p) = -J \cdot \eta \times \xi = J \cdot \xi \times \eta.$

### 3.7 An angular momentum map that generalises from $SO(3)$ to other Lie groups.

For any Lie algebra $\mathfrak{g}$, the cotangent lift momentum map satisfies (recall the commutator for the hat map!)

$$\Big\{J^\xi,\, J^\eta\Big\} = \pm J^{[\xi,\,\eta]},$$

where $[\xi,\,\eta] = -[\eta,\,\xi]$ is the Lie bracket between $\xi,\,\eta \in \mathfrak{g}$, which we also denote as $[\xi,\,\eta] =: \mathrm{ad}_\xi\eta$. The $\pm$ sign convention is $(-)$ for left-invariant $(g^{-1}\dot g)$ and $(+)$ for right-invariant $(\dot g g^{-1})$ Lie algebra actions, respectively.

The corresponding $\boxed{\textbf{\textit{Lie–Poisson bracket}}}$ is

$$\Big\{F(J),\, H(J)\Big\} := \pm\left\langle J\,,\left[\frac{\partial F}{\partial J},\,\frac{\partial H}{\partial J}\right]\right\rangle_{\mathfrak{g}^*\times\mathfrak{g}} =: \mp\left\langle J\,,\mathrm{ad}_{\partial H/\partial J}\frac{\partial F}{\partial J}\right\rangle_{\mathfrak{g}^*\times\mathfrak{g}} =: \mp\left\langle \mathrm{ad}^*_{\partial H/\partial J}J\,,\frac{\partial F}{\partial J}\right\rangle_{\mathfrak{g}^*\times\mathfrak{g}}.$$

Consequently, for Lie-Poisson systems, the dynamics of the cotangent lift momentum map is governed by

$$\frac{dJ}{dt} = \Big\{J,\, H(J)\Big\} = \mp\,\mathrm{ad}^*_{\partial H/\partial J}J\,.$$

$\boxed{\text{This generalises the angular momentum map Exercise for } SO(3) \text{ to arbitrary Lie groups and their Lie algebras.}}$

The proof follows by a direct calculation using the Lie-Poisson bracket:

$$\Big\{J^\xi,\, J^\eta\Big\} = \Big\{\big\langle J\,,\,\xi\big\rangle,\,\big\langle J\,,\,\eta\big\rangle\Big\} = \pm\big\langle J\,,\,[\xi\,,\,\eta]\big\rangle = \pm J^{[\xi\,,\,\eta]}$$

where we have used $\xi = \xi^j e_j$, $\eta = \eta^k e_k$ and $[e_j,\, e_k] = c_{jk}{}^i e_i$ to compute

$$[\xi,\eta] = [\xi^j e_j,\,\eta^k e_k] = \xi^j[e_j,\,e_k]\eta^k = \xi^j\eta^k c_{jk}{}^i e_i = [\xi,\eta]^i e_i\,.$$

Hence, for functions of the $\boxed{\textbf{\textit{momentum map}}}$ $J$ we now have the result that

$$\Big\{J_k,\, J_l\Big\} = \pm c_{kl}{}^m J_m \quad \text{and} \quad \Big\{F(J),\, H(J)\Big\} = \mp \left\langle J\,, \mathrm{ad}_{\partial H/\partial J}\frac{\partial F}{\partial J}\right\rangle_{\mathfrak{g}^*\times\mathfrak{g}} \quad \text{so} \quad \frac{dJ}{dt} = \Big\{J,\, H(J)\Big\} = \mp\,\mathrm{ad}^*_{\partial H/\partial J}\, J\,.$$

Thus, the momentum map $J(q,p) : T^*M \to \mathfrak{g}^*$ is Poisson, which means that $\{F\circ J\,, H\circ J\,\} = \{F,\, H\}\circ J$.

The Lagrangian counterpart of Lie–Poisson theory is $\boxed{\textbf{\textit{Euler–Poincaré}}}$ theory, from Poincaré [1901] that we will study next.

**Lecture 1, Friday 10 Jan 2020: Introduction to the Course and to Smooth Manifolds**

# 4 What is Geometric Mechanics?

GM lifts mechanics on a *manifold* $M$ to mechanics on a *Lie group* $G$ which *acts* (transitively) on $M$.

This sentence defining GM introduces three main concepts into classical Lagrangian and Hamiltonian mechanics:

- The configuration spaces are Manifolds. A manifold $M$ is a space which admits differentiable transformations along curves (motions).
  In particular, manifolds admit the rules of calculus.
- Lie groups are groups of *transformations* which depend smoothly on a set of parameters, e.g. rotations or translations.
  In particular, Lie groups are groups which are also manifolds.
- A Group action is a transformation of a Lie group $G$ which takes an initial point $q_0 \in M$ in a manifold $M$ to another one along a smooth curve $q_t \in M$, denoted $q_t = g_t q_0$, for $g_t$ a curve in Lie group $G$ parameterised by $t$.

This class provides examples of how these concepts are used!

Lie groups describe the symmetries of Hamilton's principle $0 = \delta S$, with $S = \int_a^b L(q.\dot{q})\, dt$ for a Lagrangian $L : (q, \dot{q}) \in TM \to \mathbb{R}$, where $TM$ (tangent bundle of $M$) is the union of the set of tangent vectors to $M$ at all points $q \in M$.

## 4.1 The Geometric Mechanics Framework (GMF) of relationships for understanding dynamics

**Classical mechanics may be visualised as the top face of the GMF cube.**

$$\begin{tikzcd}
 & L : TM \to \mathbb{R} \arrow[rr, leftrightarrow, "\text{Legendre Transform}"] \arrow[dl, "\text{Lagrange variational principle}"] & & H : T^*M \to \mathbb{R} \arrow[dl, "\text{Hamilton variational principle}"] \\
\text{Euler-Lagrange eqns} \arrow[rr, Leftrightarrow] & & \text{Hamilton's eqns} &
\end{tikzcd}$$

Figure 7: Framework for Classical Mechanics

**Geometric mechanics will trace through all of the corners and edges of the GMF cube.**

$$\begin{tikzcd}
 & & L : TM \to \mathbb{R} \arrow[rr, leftrightarrow, "\text{Legendre Transform}"] \arrow[dl, "\text{Lag vp}"] \arrow[dd, "TM/G"] & & H : T^*M \to \mathbb{R} \arrow[dl, "\text{Ham vp}"] \arrow[dd, "\textbf{Reduction via Momentum Map } J(q,p) \in \mathfrak{g}^* \simeq T^*M/G"] & \\
 & \text{EL eqns} \arrow[rr, Leftrightarrow] \arrow[dd, "TM/G"] & & \text{Ham eqns} \arrow[dd, "J(q,p)"] & & \\
\text{Noether's Theorem} \arrow[r, dash] \arrow[uurr, dash, bend left] & {} \arrow[r, dash] & \ell : \mathfrak{g} \to \mathbb{R} \arrow[rr, leftrightarrow, "\textbf{LT}"] \arrow[dl, "\text{EP vp}"] & & h : \mathfrak{g}^* \to \mathbb{R} \arrow[r, dash] \arrow[dl, "\text{LP vp}"] & \text{Noether's Theorem} \arrow[uul, dash, bend right] \\
 & \text{Euler-Poincaré eqns} \arrow[rr, Leftrightarrow] & & \text{Lie-Poisson eqns} & &
\end{tikzcd}$$

Figure 8: Framework for Geometric Mechanics

1. **Space** is taken to be a smooth manifold $M$ with points $q \in M$ (Positions, States, Configurations).

   **Time** is taken to be a manifold $T$ with points $t \in T$. Usually $T = \mathbb{R}$

   Let $M$ be a **smooth manifold** $\dim M = n$. That is, $M$ is a smooth space that is locally $\mathbb{R}^n$.

   Operationally, a smooth manifold $M$ is a space on which the rules of calculus apply.

   We will consider curves $q(t) \in M$.

2. Define the **Lagrangian** function as a smooth real-valued mapping defined on $TM$, the tangent space of the configuration manifold $M$; namely,

$$
\begin{aligned}
&\text{Lagrangian } L(q(t), v(t)) : TM \to \mathbb{R} \\
&\text{Action integral } S := \int_a^b L(q, v) + \left\langle p, \frac{dq}{dt} - v \right\rangle dt\,.
\end{aligned}
$$

3. Embed the curve $q(t) \in M$ in a *family of curves* given by the one-parameter smooth map

$$
q(t) \to q(t, \epsilon) \in M \quad \text{with} \quad q(t, 0) = q(t),
$$

   so that $\epsilon = 0$ is the identity map.

   Define the *variation operation* $\delta q := dq/d\epsilon|_{\epsilon=0}$ and invoke **Hamilton's Principle**:

$$
0 = \delta S = \int_a^b \left\langle \delta p, \frac{dq}{dt} - v \right\rangle + \left\langle \frac{\partial L}{\partial v} - p\,,\, \delta v \right\rangle + \left\langle \frac{\partial L}{\partial q} - \frac{dp}{dt}\,,\, \delta q \right\rangle dt + \langle p, \delta q \rangle \Big|_a^b
$$

**Remark 8.** *The variations define the* **pairing** $\langle\,\cdot\,,\,\cdot\,\rangle : TM \times T^*M \to \mathbb{R}$*, where* $(q, v) \in TM$ *(tangent space) and* $(q, p) \in T^*M$ *(cotangent space); and* $p := \partial L/\partial v$ *is the fiber derivative of the Lagrangian* $L(q, v)$*.*

Consequently, for variations that vanish at the endpoints in time, so that $\delta q(a) = 0 = \delta q(b)$, and are otherwise arbitrary, we have the *stationarity conditions*

$$\delta p : \frac{dq}{dt} - v = 0\,, \quad \delta v : \frac{\partial L}{\partial v} - p = 0\,, \quad \delta q : \frac{dp}{dt} - \frac{\partial L}{\partial q} = 0$$

**Remark 9.** *The variable* $p$ *is a* **Lagrange multiplier**. *Its variation enforces the constraint* $v = \frac{dq}{dt}$ *known as the* **tangent lift** *of the curve* $q(t)$*.*

4. The three stationarity conditions obtained from Hamilton's Principle imply the **Euler-Lagrange equation**.

$$\frac{d}{dt}\frac{\partial L}{\partial (dq/dt)} - \frac{\partial L}{\partial q} = 0\,.$$

5. Define the **Hamiltonian** $H(p, q) : T^*M \to \mathbb{R}$ via the **Legendre transformation** $LT : TM \to T^*M$,

$$H(q, p) = \langle p, v\rangle - L(q, v)\,,$$

with differential

$$\begin{aligned} dH(q, p) &= \left\langle \frac{\partial H}{\partial p}, dp \right\rangle + \left\langle \frac{\partial H}{\partial q}, dq \right\rangle \\ &= \langle v, dp\rangle - \left\langle \frac{\partial L}{\partial q}, dq \right\rangle + \left\langle p - \frac{\partial L}{\partial v}, dv \right\rangle . \end{aligned}$$

6. Introduce **Hamilton's principle on Phase Space** $(p,q) \in T^*M$

$$\text{Hamiltonian } H(p,q) : T^*M \to \mathbb{R}$$

$$\text{Phase Space Action } S := \int_a^b \left\langle p, \frac{dq}{dt} \right\rangle - H(q,p)dt$$

$$\text{Hamilton's Principle: } 0 = \delta S = \int_a^b \left\langle \delta p, \frac{dq}{dt} - \frac{\partial H}{\partial p} \right\rangle - \left\langle \frac{\partial H}{\partial q} + \frac{dp}{dt}\,,\, \delta q \right\rangle dt + \langle p, \delta q \rangle \Big|_a^b$$

$$\text{Hamilton's Canonical Equations: } \delta p : \frac{dq}{dt} - \frac{\partial H}{\partial p} = 0\,, \quad \delta q : \frac{dp}{dt} + \frac{\partial H}{\partial q} = 0$$

7. Prove **Noether's theorem** on both the Lagrangian and Hamiltonian sides.

**Theorem 10** (Noether's theorem)**.** *If the action integral $S$ is invariant under a smooth infinitesimal transformation $\delta q = \Phi_\xi(q)$, then the quantity*

$$J_\xi(q,p) := \langle p, \Phi_\xi(q) \rangle$$

*is a constant of the motion. That is, $J_\xi(q,p)$ is conserved when the equations of motion hold.*

*Proof.* On the Lagrangian side, suppose that $L(q, \frac{dq}{dt})$ is invariant under $q \to q + \epsilon\Phi_\xi(q)$. Then, if the Euler-Lagrange equation holds, we have $\langle p, \Phi_\xi(q) \rangle \Big|_a^b = 0$, which means that $J_\xi(q,p)$ must be constant.

Likewise, the same conclusion $\langle p, \Phi_\xi(q) \rangle \Big|_a^b = 0$ follows on the Hamiltonian side, when Hamilton's canonical equations hold, and the phase space action is invariant. □

8.

**Definition 11** (**Canonical Poisson bracket).** *By direct computation, one finds*

$$\frac{d}{dt}J_\xi(q,p) = \frac{\partial J_\xi}{\partial q}\frac{dq}{dt} + \frac{\partial J_\xi}{\partial p}\frac{dp}{dt} = \frac{\partial J_\xi}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial H}{\partial q}\frac{\partial J_\xi}{\partial p} =: \Big\{ J_\xi \,,\, H \Big\}$$

Consequently, Hamilton's Canonical equations may be rewritten as

$$\frac{dq}{dt} = \Big\{ q \,,\, H \Big\} \quad \text{and} \quad \frac{dp}{dt} = \Big\{ p \,,\, H \Big\}$$

**Exercise.** Show that the canonical Poisson brackets defined by $\{F, H\} := F_q H_p - H_q F_p$ have the following properties

(a) Binary map on smooth phase space functions $\{\,\cdot\,,\,\cdot\,\} : \mathcal{C}^\infty(T^*M,\mathbb{R}) \times \mathcal{C}^\infty(T^*M,\mathbb{R}) \to \mathcal{C}^\infty(T^*M,\mathbb{R})$

(b) Skew symmetric: $\{F, H\} = -\{H, F\}$, for $F, H \in \mathcal{C}^\infty(T^*M,\mathbb{R})$

(c) Bilinear: $\{F, aH + bJ\} = a\{F, H\} + b\{F, J\}$, for $a, b \in \mathbb{R}$

(d) Leibnitz: $\{F, HJ\} = \{F, H\}J + H\{F, J\}$

(e) Jacobi: $\{F, \{H, J\}\} + \{H, \{J, F\}\} + \{J, \{F, H\}\} = 0$
The Jacobi identity may be verified formally by writing $\{G,\, H\} = G(H) - H(G)$ symbolically. Then write

$$\{F \,,\, \{G \,,\, H\}\,\} = F(G(H)) - F(H(G)) - G(H(F)) + H(G(F))\,.$$

Summation over cyclic permutations then yields the result.

★

**Lie groups.** Consider a group of transformations which depend smoothly on a set of parameters, labelled $\epsilon$,

$$q(t) \to q(t,\epsilon) \in M \quad \text{with} \quad q(t,0) = q(t),$$

so that $\epsilon = 0$ is the identity map. Such a group is called a **Lie group**.

The **infinitesimal transformation** of $q$ under this group is given by the tangent at the identity, denoted as

$$\delta q := dq/d\epsilon|_{\epsilon=0} \,.$$

**Cotangent lift.** Suppose the infinitesimal transformation of $q$ is given by $\delta q = \Phi_\xi(q)$.

This $\delta q = \Phi_\xi(q)$ lifts to the infinitesimal transformation of $p$ by analogy with Hamilton's equations, as

$$\delta q := \frac{dq}{d\epsilon}\Big|_{\epsilon=0} = \Big\{ q \,,\, J_\xi \Big\} = \Phi_\xi(q) \qquad \text{and} \qquad \delta p := \frac{dp}{d\epsilon}\Big|_{\epsilon=0} = \Big\{ p \,,\, J_\xi \Big\} = -p\frac{\partial \Phi_\xi(q)}{\partial q}$$

Consequently, (Conservation of $J_\xi$) implies (Invariance of $H$) under the infinitesimal transformations associated with $J_\xi$, and *vice versa* since

$$\delta H = \frac{dH}{d\epsilon}\Big|_{\epsilon=0} = \Big\{ H \,,\, J_\xi \Big\} = -\Big\{ J_\xi \,,\, H \Big\} = -\frac{d}{dt} J_\xi(q,p) = 0$$

On the other hand, if the Hamiltonian $H$ depends on the variables $J_\xi(q,p)$, but $H(J)$ is *not* invariant under the transformations generated by $\{\,\cdot\,,\, J_\xi\}$ then we may still write

$$\frac{d}{dt} J_\xi = \Big\{ J_\xi \,,\, H \Big\} = \Big\{ J_\xi \,,\, J_\eta \Big\} \frac{\partial H}{\partial J_\eta}$$

which yields the **transformation law for Poisson brackets**,

$$\frac{d}{dt} F(J) = \Big\{ F \,,\, H \Big\}(J(q,p)) = \frac{\partial F}{\partial J_\xi} \Big\{ J_\xi \,,\, J_\eta \Big\} \frac{\partial H}{\partial J_\eta}$$

**Closure.** If the Poisson brackets of the components of $J$ *close* among themselves, so that $\Big\{ J_\xi \,,\, J_\eta \Big\} = J_\gamma C^\gamma_{\xi\eta}$ where $C^\gamma_{\xi\eta}$ comprise a set of constants, then the dynamics on the $J$ space reduces to

$$\frac{d}{dt} J_\xi = J_\gamma C^\gamma_{\xi\eta} \frac{\partial H}{\partial J_\eta} \qquad \text{and} \qquad \frac{d}{dt} F(J) = J_\gamma C^\gamma_{\xi\eta} \frac{\partial F}{\partial J_\xi} \frac{\partial H}{\partial J_\eta} =: \Big\{ F \,,\, H \Big\}_{LP} (J)\,,$$

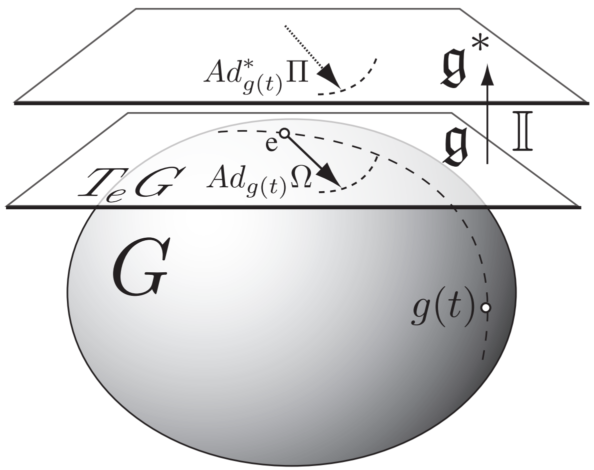
in which $\{\,\cdot\,,\,\cdot\,\}_{LP}$ preserves the properties of the canonical Poisson bracket provided the constants $C^\gamma{}_{\xi\eta}$ for $\xi,\eta,\gamma = 1,2,\dots,r$ are structure constants for a Lie algebra, where $[e_\xi, e_\eta] = e_\gamma C^\gamma{}_{\xi\eta}$ for a Lie algebra whose structure constants are $C^\gamma_{\xi\eta}$ in the basis $e_\xi$ with $\xi = 1,2,\dots,r$.

# 5 Quick definitions of what we need here about Lie groups and Lie algebras

**Definition 12** (Group)**.** *A **group** $G$ is a set of elements that possesses a binary product (multiplication), $G\times G \to G$, such that the following properties hold:*

- *The product $gh$ of $g$ and $h$ is associative, that is, $(gh)k = g(hk)$.*
- *An identity element exists, $e:\ eg = g$ and $ge = g$, for all $g \in G$.*
- *The inverse operation exists, $G \to G$, so that $gg^{-1} = g^{-1}g = e$.*

**Definition 13** (Lie group)**.** *A **Lie group** is a group that depends smoothly on a set of parameters. That is, a Lie group is both a group and a smooth manifold, for which the group operation is by composition of smooth invertible functions.*



**Definition 14.** *A* ***Lie algebra*** *is a vector space* $\mathfrak{g}$ *together with a bilinear operation*

$$[\,\cdot\,,\,\cdot\,]:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}\,,$$

*called the* ***Lie bracket*** *for* $\mathfrak{g}$*, that satisfies the defining properties:*

- *bilinearity, e.g.,*
$$[a\mathbf{u}+b\mathbf{v}\,,\,\mathbf{w}]=a[\mathbf{u}\,,\,\mathbf{w}]+b[\mathbf{v}\,,\,\mathbf{w}]\,,$$
*for constants* $(a,b)\in\mathbb{R}$ *and any vectors* $(\mathbf{u},\mathbf{v},\mathbf{w})\in\mathfrak{g}$*;*
- *skew-symmetry,*
$$[\mathbf{u}\,,\,\mathbf{w}]=-[\mathbf{w}\,,\,\mathbf{u}]\,;$$
- *Jacobi identity,*
$$[\mathbf{u}\,,\,[\mathbf{v}\,,\,\mathbf{w}]\,]+[\mathbf{v}\,,\,[\mathbf{w}\,,\,\mathbf{u}]\,]+[\mathbf{w}\,,\,[\mathbf{u}\,,\,\mathbf{v}]\,]=0\,,$$
*for all* $\mathbf{u},\mathbf{v},\mathbf{w}$ *in* $\mathfrak{g}$.

## 5.1 Structure constants

Suppose $\mathfrak{g}$ is any finite-dimensional Lie algebra. The Lie bracket for any choice of basis vectors $\{\mathbf{e}_1,\,\dots,\,\mathbf{e}_r\}$ of $\mathfrak{g}$ must again lie in $\mathfrak{g}$. Thus, constants $c_{ij}^k$ exist, where $i,j,k=1,2,\dots,r$, called the ***structure constants*** of the Lie algebra $\mathfrak{g}$, such that

$$[\,\mathbf{e}_i\,,\,\mathbf{e}_j\,]=c_{ij}^k\mathbf{e}_k\,. \tag{7}$$

Since $\{\mathbf{e}_1,\,\dots,\,\mathbf{e}_r\}$ form a vector basis, the structure constants in (7) determine the Lie algebra $\mathfrak{g}$ from the bilinearity of the Lie bracket. The conditions of skew-symmetry and the Jacobi identity place further constraints on the structure constants. These constraints are

- skew-symmetry

$$c^k_{ji} = -\, c^k_{ij}\,, \tag{8}$$

- Jacobi identity

$$c^k_{ij}c^m_{lk} + c^k_{li}c^m_{jk} + c^k_{jl}c^m_{ik} = 0\,. \tag{9}$$

Conversely, any set of constants $c^k_{ij}$ that satisfy relations (8) and (9) defines a Lie algebra $\mathfrak{g}$.

**Exercise.** Prove that the Jacobi identity requires the relation (9). ★

**Answer.** The Jacobi identity involves summing three terms of the form

$$[\,\mathbf{e}_l\,,\,[\,\mathbf{e}_i\,,\,\mathbf{e}_j\,]\,] = c^k_{ij}[\,\mathbf{e}_l\,,\,\mathbf{e}_k] = c^k_{ij}c^m_{lk}\mathbf{e}_m\,.$$

Summing over the three cyclic permutations of $(l, i, j)$ of this expression yields the required relation (9) among the structure constants for the Jacobi identity to hold. ▲

## 5.2 Lie group symmetries and Noether's theorem

***Introduction of Lie group symmetries:***

- A ***group*** is a set of elements with an associative binary product that has a unique inverse and identity element.
- A ***Lie group*** $G$ is a group whose transformations depends smoothly on a set of parameters in $\mathbb{R}^{\dim(G)}$. (A Lie group is also a smooth manifold, so it is an ideal arena for geometric mechanics, e.g., rigid body motion on $SO(3)$.)

***Noether's theorem:*** Suppose $q(t,\epsilon) = q_\epsilon(t) = \phi_\epsilon \circ q(t)$ represents a Lie group, i.e., group of transformations of $q(t)$ that depends smoothly on a set of parameters $\epsilon$. Its linearisation is computed from a Taylor series as

$$q(t) \to q_\epsilon(t) = q(t) + \epsilon \frac{dq(t,\epsilon)}{d\epsilon}\bigg|_{\epsilon=0} + O(\epsilon^2) = q(t) + \epsilon\delta q(t) + O(\epsilon^2),$$

where the linear term is a vector field on $Q$

$$\delta q(t) := \frac{dq(t,\epsilon)}{d\epsilon}\bigg|_{\epsilon=0} = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} (\phi_\epsilon \circ q_0) =: \Phi(q)\,, \quad \text{called the } \boxed{\textit{infinitesimal transformation}}$$

That is, $\Phi(q)$ is the ***linearisation*** of the flow map $\phi_\epsilon$ at the point $q \in Q$.

Suppose also that the Lagrangian $L(q,\dot q)$ in Hamilton's principle $\delta S = 0$ with $S = \int_a^b L(q,\dot q)dt$ is *invariant* under these infinitesimal transformations, so that $\delta S = 0$ as a consequence of this invariance. Then the endpoint term above $\langle \frac{\partial L}{\partial \dot q}, \delta q\rangle = \langle p, \delta q\rangle$ is a *constant of the motion*. That is, the quantity $\langle \frac{\partial L}{\partial \dot q}, \delta q\rangle = \langle p, \delta q\rangle$ is a constant, whenever $q(t)$ is a solution of the EL equations for this invariant Lagrangian. This argument proves the following.

**Theorem 15** (Noether, 1918)**.**
*To each Lie symmetry of the Lagrangian,* $(\frac{d}{d\epsilon}\big|_{\epsilon=0}L(q,\dot{q})) = 0$*, there corresponds a conservation law,* $\langle\frac{\partial L}{\partial \dot{q}}, \Phi(q)\rangle = \langle p, \Phi(q)\rangle$.

**Example:** *Ignorable coordinates*: For $L(q,\dot{q},\dot{\theta})$ invariant under $\theta \to \theta + \epsilon$, $\delta\theta = \epsilon$, we have $\frac{d}{dt}\left\langle \frac{\partial L}{\partial \dot{\theta}},\, \epsilon \right\rangle = \left\langle \frac{\partial L}{\partial \theta},\, \epsilon \right\rangle = 0$.

### 5.3 Infinitesimal vs Finite Transformation of a Lie Group

The infinitesimal transformation of a Lie group $G$ acting on a manifold $Q$ as $G \times Q \to Q$ is given by the linear term in the Taylor series of the finite transformation

$$
\begin{aligned}
q_\epsilon = \phi_\epsilon(q_0) &= q_0 + \epsilon \left[ \frac{d}{d\epsilon} \phi_\epsilon(q_0) \right]_{\epsilon=0} + O(\epsilon^2) \\
&= q_0 + \epsilon \Phi(q_0) + O(\epsilon^2)
\end{aligned}
$$

and denoted as

$$
\delta q = \frac{dq_\epsilon}{d\epsilon}\bigg|_{\epsilon=0} = \Phi(q)
$$

In more generality, for smooth functions $f \in C^\infty(Q)$ we have the **pull-back relation**

$$
\frac{d}{d\epsilon}({\phi_\epsilon}^* f) = {\phi_\epsilon}^* \mathcal{L}_{v_\Phi} f
$$

where the vector field $v_\Phi$ generates the smooth flow $\phi_\epsilon$.

Then, evaluating at $\epsilon = 0$ gives

$$
\frac{d}{d\epsilon}({\phi_\epsilon}^* f)\bigg|_{\epsilon=0} = \mathcal{L}_{v_\Phi} f
$$

This is the dynamical definition of the Lie derivative of the function $f$ by the vector field

$$
v_\Phi := \Phi(q) \cdot \nabla_q = \Phi^j(q) \frac{\partial}{\partial q^j}
$$

This definition identifies infinitesimal transformations of Lie groups with vector fields, as

$$\frac{d}{d\epsilon} = \Phi^j(q_\epsilon)\frac{\partial}{\partial {q_\epsilon}^j}$$

Consequently, the finite transformation of the Lie group can be determined from the characteristic equations of the vector field $v_\Phi$ as

$$dv_\epsilon = \frac{dq^1}{\Phi^1(q)} = \cdots = \frac{dq^j}{\Phi^j(q)} = \cdots = \frac{dq^n}{\Phi^n(q)} \quad \text{with} \quad n = \dim Q$$

### 5.4 Class Review: Geometric Mechanics deals with group invariant variational principles (Noether)

$$
\begin{tikzcd}[column sep=large, row sep=huge]
\left.\begin{array}{c} \frac{d}{dt}\frac{\partial L}{\partial \dot q} - \frac{\partial L}{\partial q} = 0, \\ \Big\langle \frac{\partial L}{\partial \dot q}\,,\, \delta q\Big\rangle_{TM}\Big|_{t=a}^{t=b} = 0 \end{array}\right\} \arrow[d, dotted, leftrightarrow]
& L(q,v) : TM \to \mathbb{R} \arrow[l, "\delta S = 0"'] \arrow[r, "\mathcal{L}T"] \arrow[d, "\text{Reduction by Lie symmetry}"]
& H(q,p) : T^*M \to \mathbb{R} \arrow[r, "\delta S = 0"] \arrow[d, "{\langle p, \delta q\rangle_{TM} = \langle J(q,p), \xi\rangle_{\mathfrak{g}}\ \text{(momentum map)}}"]
& \left\{\begin{array}{c} \frac{\partial H}{\partial p} = \dot q, \qquad \frac{\partial H}{\partial q} = -\dot p, \\ \Big( X_H \,\lrcorner\, (dq \wedge dp) = dH \Big) \end{array}\right. \arrow[d, dotted, leftrightarrow] \\
\begin{array}{c} \frac{d}{dt}\frac{\partial l}{\partial \xi^\alpha} = \frac{\partial l}{\partial \xi^\gamma} c^\gamma_{\alpha\beta}\xi^\beta \\ \Big( \frac{d}{dt}\frac{\partial l}{\partial \xi} = \mathrm{ad}^*_\xi \frac{\partial l}{\partial \xi} \Big) \end{array}
& l(\xi) : \mathfrak{g} \to \mathbb{R} \arrow[l, "\delta S_{red} = 0"'] \arrow[r, leftrightarrow, "\text{Reduced } \mathcal{L}T"]
& h(J) : \mathfrak{g}^* \to \mathbb{R} \arrow[r, "\delta S_{red} = 0"]
& \begin{array}{c} \frac{dJ_\alpha}{dt} = \{J_\alpha, J_\beta\}\frac{\partial h}{\partial J_\beta} = J_\gamma c^\gamma_{\alpha\beta}\frac{\partial h}{\partial J_\beta} \\ \Big( \frac{dJ}{dt} = \mathrm{ad}^*_{\partial h/\partial J} J \Big) \end{array}
\end{tikzcd}
$$

Hamilton's principle (HP): $\delta S = 0$ with $S = \int_a^b L(q,v) + \langle p, \dot q - v\rangle_{TM}\, dt$ and $S = \int_a^b \langle p, \dot q\rangle_{TM} - H(q,p) dt$
Reduction by Lie symmetry $TM$: $q_t = g_t q_0$, $\dot q_t = \dot g_t q_0$, and $L(g,v) = L(kg, kv)$, $k \in G$, set $L(e, g^{-1}v) =: l(\xi)$
Noether's theorem: Lie symmetry of HP implies conservation of $\langle \frac{\partial L}{\partial \dot q}\,,\, \delta q\rangle_{TM} = \langle p\,,\, \delta q\rangle_{TM} = \langle J(q,p), \xi\rangle_{\mathfrak{g}}$
Legendre transformation ($\mathcal{L}T$): $p := \frac{\partial L}{\partial \dot q}$, $H(q,p) := \langle p, v\rangle_{TM} - L(q,v)$, and $J := \frac{\partial l}{\partial \xi}$, $h(J) := \langle J, \xi\rangle_{\mathfrak{g}} - l(\xi)$
Reduced Hamilton's principle: $S_{red} = \int_a^b l(\xi) + \langle J, g^{-1}\dot g - \xi\rangle_{\mathfrak{g}}\, dt$ and $S_{red} = \int_a^b \langle J, g^{-1}\dot g\rangle_{\mathfrak{g}} - h(J) dt$
Adjoint and co-adjoint actions: $\mathrm{Ad} : G \times \mathfrak{g} \to \mathfrak{g}$, $\mathrm{ad} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, $\mathrm{Ad}^* : G \times \mathfrak{g}^* \to \mathfrak{g}^*$, $\mathrm{ad}^* : \mathfrak{g} \times \mathfrak{g}^* \to \mathfrak{g}^*$

## 5.5 Worked example for angular momentum

**Exercise.** (a) What are the conditions on $\delta q^i = A^i_\xi(q)$ so that $\Big\{J_\xi\,,\,J_\eta\Big\} = J_\gamma C^\gamma_{\xi\eta}$ for a set of constants $C^\gamma_{\xi\eta}$?

(b) What are the conditions in Part (a) in the case that $A^i_\xi(q) = \widehat{\xi}\,^i_j q^j$ with $i,j = 1,2,3$, when $\widehat{\xi}\,^i_j$ is a constant matrix?

(c) Show that infinitesimal rotations of $\mathbb{R}^3$ are involved when $\widehat{\xi}\,^i_j$ is a constant $3\times 3$ skew-symmetric matrix.

(d) Compute the cotangent lift formulas in the ladder of commuting diagrams above for reduction in the case that $\mathbf{q}\in\mathbb{R}^3$ and $\delta\mathbf{q} = \boldsymbol{\xi}\times\mathbf{q}$; that is, $\delta q^i = \widehat{\xi}\,^i_j q^j$ for $i,j = 1,2,3$. ★

**Answer.** (a) Upon defining $J_\xi := p_i A^i_\xi(q)$, one finds

$$
\begin{aligned}
\Big\{J_\xi\,,\,J_\eta\Big\} = J_{[\xi,\eta]} &:= p_i\big[A_\xi\,,\,A_\eta\big]^i := p_i\Big[A^j_\xi(q)\frac{\partial}{\partial q^j}A^i_\eta(q) - A^j_\eta(q)\frac{\partial}{\partial q^j}A^i_\xi(q)\Big] \\
&=: p_i\Big[C_{\xi\eta}{}^\gamma(q)A^i_\gamma(q)\Big] =: J_\gamma C_{\xi\eta}{}^\gamma
\end{aligned}
$$

For $C_{\xi\eta}{}^\gamma$ to be a set of constants we must have linearity $A^i_\xi(q) = (A_\xi)^i_j q^j$, so that $J_\xi := p_i(A_\xi)^i_j q^j$. In that case, we have a matrix Lie algebra, with elements $(A_\xi)^i_j$, whose structure constants are determined from matrix commutators,

$$
\begin{aligned}
\Big\{J_\xi\,,\,J_\eta\Big\} = J_{[\xi,\eta]} &:= p_i\big[A_\xi\,,\,A_\eta\big]^i_k q^k := p_i\Big[(A_\xi)^i_j(A_\eta)^j_k - (A_\eta)^i_j(A_\xi)^j_k\Big]q^k \\
&:= p_i C_{\xi\eta}{}^\gamma (A_\gamma)^i_k q^k := J_\gamma C_{\xi\eta}{}^\gamma
\end{aligned}
$$

(b) In the case that $A^i_\xi(q) = \widehat{\xi}\,^i_j q^j$ with $i, j = 1, 2, 3$, when $\widehat{\xi}\,^i_j$ is a constant matrix, we have the bilinear form,

$$p_i \delta q^i = p_i A^i_\xi(q) = p_i(\epsilon^i_{jk}\xi^j)q^k = p_i \widehat{\xi}\,^i_k q^k \,.$$

(c) In the case that $\widehat{\xi}$ is a $3 \times 3$ skew-symmetric matrix, we discover the **hat map** isomorphism$\,\widehat{\ }$: $so(3) \simeq \mathbb{R}^3$, by which the Lie algebra $so(3)$ of infinitesimal rotations in $\mathbb{R}^3$ may be represented by $3 \times 3$ skew-symmetric matrices,

$$\widehat{\xi}\,^i_k = -\,\epsilon^i_{kj}\xi^j = -\,\widehat{\xi}\,^k_i \,, \quad \text{or} \quad \begin{pmatrix} 0 & -\xi^3 & \xi^2 \\ \xi^3 & 0 & -\xi^1 \\ -\xi^2 & \xi^1 & 0 \end{pmatrix} .$$

**Remark 16** (Properties of the hat map for $so(3)$)**.** *The hat map arises in the infinitesimal rotations*

$$\widehat{\xi}\,^i_k = (O^{-1}dO/ds)\,^i_k\big|_{s=0} = \epsilon^i_{kj}\xi^j \,,$$

*for $O \in SO(3)$ with $\det O = 1$ and $OO^T = Id$.*
*The matrix $\widehat{\xi} = O^{-1}\dot{O} = O^T\dot{O}$ is skew, since $\frac{d(O^TO)}{dt} = \frac{d(Id)}{dt} = \dot{O}^TO + O^T\dot{O} = (O^{-1}\dot{O})^T + O^{-1}\dot{O} = \widehat{\xi}^T + \widehat{\xi} = 0$.*
*The hat map is an isomorphism:*

$$(\mathbb{R}^3, \times) \mapsto (\mathfrak{so}(3), [\,\cdot\,,\,\cdot\,]\,) \,.$$

*That is, the hat map identifies the composition of two vectors in $\mathbb{R}^3$ using the cross product with the commutator of two skew-symmetric $3 \times 3$ matrices. Specifically, we write for any two vectors $\mathbf{q}, \boldsymbol{\xi} \in \mathbb{R}^3$,*

$$(\boldsymbol{\xi} \times \mathbf{q})^k = \epsilon^k{}_{jm}\xi^j q^m = \widehat{\xi}^k_m\, q^m \,.$$

*In matrix form, we may write*

$$\boldsymbol{\xi} \times \mathbf{q} = \widehat{\xi}\,\mathbf{q} \quad \textit{for all} \quad \boldsymbol{\xi},\, \mathbf{q} \in \mathbb{R}^3 \,.$$

***Exercise.*** *Verify the following formulas for* $\mathbf{p}, \mathbf{q}, \boldsymbol{\xi} \in \mathbb{R}^3$*:*

$$
\begin{aligned}
(\mathbf{p} \times \mathbf{q})\widehat{\ } &= \Big[\widehat{p}, \widehat{q}\Big] \,, \\
\Big[\widehat{p}, \widehat{q}\Big]\boldsymbol{\xi} &= (\mathbf{p} \times \mathbf{q}) \times \boldsymbol{\xi} \,, \\
\mathbf{p} \cdot \mathbf{q} &= \frac{1}{2}\,\mathrm{trace}\,\Big(\widehat{p}^T\,\widehat{q}\Big) = -\,\frac{1}{2}\,\mathrm{trace}\,\Big(\widehat{p}\,\widehat{q}\Big) \,.
\end{aligned}
$$

★

(d) In the case that $\mathbf{q} \in \mathbb{R}^3$ and $\delta\mathbf{q} = \boldsymbol{\xi} \times \mathbf{q} = \{\mathbf{q}, J_\xi\}$, the canonical Poisson bracket generates the cotangent lift infinitesimal transformation of the canonical momentum, $\delta\mathbf{p} = \boldsymbol{\xi} \times \mathbf{p} = \{\mathbf{p}, J_\xi\}$, and the momentum map turns out to be the familiar expression for the **angular momentum** of a particle in phase space $T^*\mathbb{R}^3$,

$$
J_\xi = \mathbf{J} \cdot \boldsymbol{\xi} = \mathbf{p} \cdot \boldsymbol{\xi} \times \mathbf{q} = \mathbf{q} \times \mathbf{p} \cdot \boldsymbol{\xi} \Longleftrightarrow \mathbf{J} = \mathbf{q} \times \mathbf{p}
$$

When we take $\boldsymbol{\xi} = \boldsymbol{e}_i$, with $i = 1, 2, 3$, as the basis of orthonormal unit vectors in $\mathbb{R}^3$, we find the Poisson bracket relations for the components with $J_i = \mathbf{J} \cdot \boldsymbol{e}_i$ to be

$$
\Big\{J_j \,,\, J_k\Big\} = \Big\{\mathbf{J} \cdot \boldsymbol{e}_j \,,\, \mathbf{J} \cdot \boldsymbol{e}_k\Big\} = \mathbf{J} \cdot \boldsymbol{e}_j \times \boldsymbol{e}_k = \mathbf{J} \cdot \epsilon^{\,i}_{jk}\boldsymbol{e}_i = J_i\epsilon^{\,i}_{jk} \quad \text{with} \quad i, j, k = 1, 2, 3 \,.
$$

Consequently, we may verify our previous calculation for arbitrary linear transformations in this case simply in terms of vector multiplication in $\mathbb{R}^3$, as

$$
\Big\{J_\xi ,\, J_\eta\Big\} = \Big\{J \cdot \xi ,\, J \cdot \eta\Big\} = \Big\{q \times p \cdot \xi ,\, q \times p \cdot \eta\Big\}_{can} = (q \times p) \cdot (\xi \times \eta) = J \cdot (\xi \times \eta) = J_{\xi \times \eta} = J_{[\widehat{\xi}, \widehat{\eta}]} ,
$$

where the middle part of the calculation follows by expanding as

$$
\Big\{q \times p \cdot \xi ,\, q \times p \cdot \eta\Big\}_{can} = -\eta \cdot \{q \times p ,\, J_\xi(q, p)\}_{can} = -\eta \cdot \xi \times (q \times p) = -J \cdot \eta \times \xi = J \cdot \xi \times \eta \,.
$$

▲

The corresponding Poisson bracket is given by

$$\{F,H\}(J) = \frac{\partial F}{\partial J_\alpha}\{J_\alpha, J_\beta\}\frac{\partial H}{\partial J_\beta} = J_\gamma \epsilon^\gamma_{\alpha\beta}\frac{\partial F}{\partial J_\alpha}\frac{\partial H}{\partial J_\beta} = \mathbf{J}\cdot\frac{\partial F}{\partial \mathbf{J}}\times\frac{\partial H}{\partial \mathbf{J}} \quad \text{with} \quad \alpha,\beta,\gamma = 1,2,3.$$

In particular, as one might have expected, since $\mathbf{J}\in\mathbb{R}^3$, the infinitesimal rotation of $\mathbf{J}$ generated by the Poisson bracket with $J_\xi = \mathbf{J}\cdot\boldsymbol{\xi}$ is given by

$$\delta\mathbf{J} = \{\mathbf{J}\,,\, J_\xi\} = \boldsymbol{\xi}\times\mathbf{J}$$

Upon denoting $\mathbf{J} = \mathbf{x}\in\mathbb{R}^3$ this Poisson bracket becomes

$$\{F,\,H\} = \nabla C\cdot\nabla F\times\nabla H$$

with motion equation

$$\dot{\mathbf{x}} = -\,\nabla C\times\nabla H \quad \text{where} \quad C(\mathbf{x}) = \frac{1}{2}|\mathbf{x}|^2\,.$$

This means the motion takes place on $\boxed{\textbf{\textit{spheres}}}$ along intersections of level sets of $C$ and $H$.

$$\begin{tikzcd}[column sep=large, row sep=huge]
\frac{dz}{dt} = \{z,\,H(z)\}_{can}\ \ \dim T^*\mathbb{R}^3 = 6 \\[-2em]
z=(q,p)\in T^*\mathbb{R}^3 \arrow[r] \arrow[d, "J(0)=q(0)\times p(0)"'] & T^*\mathbb{R}^3 \arrow[d, "J(t)=q(t)\times p(t)\ \text{(momap)}"] \arrow[dl, phantom, "\text{Equivariance}"] \\
J\in\mathbb{R}^3\simeq\mathfrak{so}(3)^* \arrow[r] & \mathbb{R}^3\simeq\mathfrak{so}(3)^* \\[-2em]
\frac{dJ}{dt} = \{J,\,H(J)\}_{LP} = -\,J\times\frac{\partial H}{\partial J}\ \ \dim(S^2)=2
\end{tikzcd}$$

This is **reduction by symmetry using the cotangent lift momentum map on the Hamiltonian side**.

### 5.6 Reduction by symmetry on the Lagrangian side

1. We may lift the dynamics for the curve $q(t) \in M$ on a manifold $M$ to a curve on a Lie group $G$ by setting $q(t) = g(t)q(0) \in M$ for a curve $g(t) \in G$. For geometric mechanics on a Lie group $G$, reduce $L : TG \to \mathbb{R}$ and $H : T^*G \to \mathbb{R}$ by the action of the Lie group $G$ on its tangent space $TG$ and its cotangent space, $T^*G$, via pullbacks $TG \to T_eG \simeq \mathfrak{g}$ and $T^*G \to T_e^*G \simeq \mathfrak{g}^*$.

2. Apply Hamilton's principle on the Lagrangian side after setting $q(t) = g(t)q(0) \in M$ for a curve $g(t) \in G$ with tangent $\dot{g}(t) = v_g \in T_gG$. Obtain
(i) the EL equation from the Lagrangian $L(g, v) : TG \to \mathbb{R}$ in the constrained Hamilton's principle

$$0 = \delta S(g, v) = \int_a^b L(g, v_g) + \langle p\,,\, \dot{g} - v_g \rangle \, dt$$

which implies as before, but now with $q(t) = g(t)q(0) \in M$

$$\delta p : \frac{dg}{dt} - v_g = 0\,, \quad \delta v_g : \frac{\partial L}{\partial v_g} - p = 0\,, \quad \delta g : \frac{dp}{dt} - \frac{\partial L}{\partial g} = 0$$

and
(ii) the reduced Euler–Poincaré (EP) dynamics on $\mathfrak{g}^*$, the dual of the Lie algebra, obtained when the Lagrangian is invariant under the action of any $k \in G$ so that

$$L(g, v_g) = L(kg, kv_g)$$

In this case, we choose $L(e, g^{-1}v_g) = \ell(\xi)$, where $\xi := g^{-1}v_g \in T_eG \simeq \mathfrak{g}$. Then we rewrite the previous constrained Hamilton's principle as the Hamilton–Pontryagin principle, given by

$$0 = \delta S(\xi, \mu, g) = \int_a^b \ell(\xi) + \langle \mu\,,\, g^{-1}\dot{g} - \xi \rangle \, dt$$

The stationarity conditions now are obtained from a side calculation which yields, with $\eta := g^{-1}\delta g$,

$$\delta(g^{-1}\dot{g}) = \frac{d\eta}{dt} + (\xi\eta - \eta\xi) := \frac{d\eta}{dt} + [\xi, \eta] =: \frac{d\eta}{dt} + \mathrm{ad}_\xi\, \eta$$

in which we use $\delta(g^{-1}) = -\, g^{-1}(\delta g)g^{-1}$ and $\delta(\dot{g}) = (\delta g)^{\dot{}}$.

By defining the *variation operation* $\delta q := dq/d\epsilon|_{\epsilon=0}$ and invoking the Hamilton–Pontryagin principle one finds:

$$0 = \delta S = \int_a^b \langle \delta\mu, g^{-1}\dot{g} - \xi\rangle + \left\langle \frac{\partial \ell}{\partial \xi} - \mu\,,\, \delta\xi \right\rangle + \left\langle \mu\,,\, \frac{d\eta}{dt} + \mathrm{ad}_\xi\, \eta \right\rangle\, dt + \langle \mu, \eta\rangle\Big|_a^b$$

$$\delta\mu : g^{-1}\dot{g} - \xi = 0\,, \qquad \delta\xi : \frac{\partial \ell}{\partial \xi} - \mu = 0\,, \qquad \delta g : \frac{d\mu}{dt} - \mathrm{ad}^*_\xi\, \mu = 0$$

where the operation $\mathrm{ad}^* : \mathfrak{g} \times \mathfrak{g}^* \to \mathfrak{g}^*$ is the dual of the operation $\mathrm{ad} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ with respect to the pairing $\langle\, \cdot\,,\, \cdot\,\rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$, according to

$$\langle \mathrm{ad}^*_\xi\, \mu\,,\, \eta\rangle = \langle \mu\,,\, \mathrm{ad}_\xi\, \eta\rangle\,.$$

3. The three stationarity conditions obtained from the Hamilton–Pontryagin principle imply the **Euler–Poincaré equation**.

$$\frac{d}{dt}\frac{\partial \ell}{\partial \xi} - \mathrm{ad}^*_\xi\, \frac{\partial \ell}{\partial \xi} = 0$$

4. Define the **Hamiltonian** $h(\mu) : \mathfrak{g}^* \to \mathbb{R}$ via the **reduced Legendre transformation** $LT : \mathfrak{g} \to \mathfrak{g}^*$.

$$h(\mu) = \langle \mu, \xi\rangle - \ell(\xi)$$

with differential

$$
\begin{aligned}
dh(\mu) &= \left\langle \frac{\partial h}{\partial \mu}, d\mu \right\rangle \\
&= \langle \xi, d\mu \rangle + \left\langle \mu - \frac{\partial \ell}{\partial \xi}, d\xi \right\rangle
\end{aligned}
$$

Thus, we find

$$
\frac{\partial h}{\partial \mu} = \xi \quad \text{and} \quad \frac{\partial \ell}{\partial \xi} = \mu \,.
$$

5. Introduce **Hamilton's principle on $\mathfrak{g}^*$** ($\mu \in \mathfrak{g}^*$)

   Hamiltonian $h(\mu) : \mathfrak{g}^* \to \mathbb{R}$

   Phase Space Action $S := \displaystyle\int_a^b \langle \mu, g^{-1}\dot{g} \rangle - h(\mu)\, dt$

   Hamilton's Principle: $0 = \delta S = \displaystyle\int_a^b \left\langle \delta\mu, g^{-1}\dot{g} - \frac{\partial h}{\partial \mu} \right\rangle - \left\langle \frac{d\mu}{dt} - \mathrm{ad}^*_{g^{-1}\dot{g}}\, \mu \,,\, g^{-1}\delta g \right\rangle dt + \langle \mu, g^{-1}\delta g \rangle \Big|_a^b$

   Lie–Poisson Equations: $\delta\mu : g^{-1}\dot{g} - \dfrac{\partial h}{\partial \mu} = 0 \,, \quad g^{-1}\delta g : \dfrac{d\mu}{dt} - \mathrm{ad}^*_{g^{-1}\dot{g}}\, \mu = 0$

## 5.7 Rigid body – Clebsch Hamilton's principle

**Review of the Clebsch Hamilton's principle for the Euler-Lagrange equations**

First, before deriving the Lagrangian and Hamiltonian formulations of rigid body dynamics, let's recall our earlier derivation of the Euler-Lagrange equations from the constrained Hamilton's principle, in which we varied coordinates $(q, v) \in TQ$, subject to the constraint $v = \frac{dq}{dt}$ (tangent lift). In this case, the constrained action integral was varied according to

$$\delta S = \delta \int_a^b L(q, v) + \Big\langle p, \frac{dq}{dt} - v \Big\rangle \, dt \; = \int_a^b \Big\langle \frac{\partial L}{\partial v} - p, \delta v \Big\rangle + \Big\langle \frac{\partial L}{\partial q} - \frac{dp}{dt}, \delta q \Big\rangle + \Big\langle \delta p, \dot{q} - v \Big\rangle dt + \Big\langle p, \delta q \Big\rangle \Big|_a^b .$$

Then we assembled the EL equation

$$\frac{d}{dt}\frac{\partial L(q, \dot{q})}{\partial \dot{q}} = \frac{\partial L(q, \dot{q})}{\partial q}$$

from the various stationary conditions, and evaluated $\frac{\partial L}{\partial v}\Big|_{v=\dot{q}} = \frac{\partial L(q,\dot{q})}{\partial \dot{q}}$.

In our next paragraph, we are going to do the same sort of variational calculation when $Q \in SO(3)$ and derive the equations for a rigidly rotating body described by a curve in $SO(3)$, for the case $G \times M \to M$ when both $G = SO(3)$ and $M = SO(3)$. That is, $SO(3) \times SO(3) \to SO(3)$, with flow $Q(t + s) = Q(t)Q(s)$ and $Q(t - t) = Q(0) = Id$, obtained from the rotation group $SO(3)$ acting on itself.

A sketch of the computation using the hat map isomorphism $\widehat{\xi} \in so(3) \to \xi \in \mathbb{R}^3$ follows, as preparation for the Lie algebra operations in our next paragraph. Here, the constrained action integral is varied according to

$$0 = \delta S = \delta \int_a^b \ell(\Omega) + \Big\langle p, \frac{dq}{dt} - \Omega \times q \Big\rangle \, dt \;\Longrightarrow\; \frac{\partial \ell}{\partial \Omega} = q \times p \,, \quad \frac{dq}{dt} = \Omega \times q \,, \quad \frac{dp}{dt} = \Omega \times p \,, \quad \Big\langle p, \delta q \Big\rangle \Big|_a^b = 0.$$

$$\frac{d}{dt}\frac{\partial \ell}{\partial \Omega} = -\Omega \times (q \times p) = -\Omega \times \frac{\partial \ell(\Omega)}{\partial \Omega} \;\Longrightarrow\; \frac{d\Pi}{dt} = -\Omega \times \Pi \quad \text{for} \quad \Pi = \frac{\partial \ell}{\partial \Omega} \;\Longrightarrow\; \frac{d|\Pi|^2}{dt} = 0 \,.$$

**Theorem 17** (Clebsch form of Hamilton's principle for the rigid body)**.**
*For $Q \in SO(3)$, the Euler-Lagrange equations become Euler-Poincaré rigid-body equations in matrix commutator form,*

$$\frac{d}{dt}\frac{\partial l}{\partial \widehat{\Omega}} = -\left[\widehat{\Omega},\, \frac{\partial l}{\partial \widehat{\Omega}}\right] \quad \textit{or, for} \quad \widehat{\Pi} := \frac{\partial l}{\partial \widehat{\Omega}}, \quad \textit{equivalently} \quad \frac{d\widehat{\Pi}}{dt} = -\widehat{\Omega}\widehat{\Pi} + \widehat{\Pi}\widehat{\Omega} = -\big[\widehat{\Omega},\, \widehat{\Pi}\big], \tag{10}$$

*with (body, left-invariant) angular velocity $\widehat{\Omega} = Q^{-1}\dot{Q} = -\widehat{\Omega}^T \in \mathfrak{so}(3) = T_eSO(3)$ and body angular momentum $\widehat{\Pi} := \partial l/\partial\widehat{\Omega}$. The commutator equation* (10) *emerges from the constrained Hamilton's principle, $\delta S = 0$ with constrained action integral*

$$S(\widehat{\Omega}, Q, P) = \int_a^b l(\widehat{\Omega}) + \Big\langle P\,,\, \dot{Q} - Q\widehat{\Omega}\Big\rangle\, dt = \int_a^b l(\widehat{\Omega}) + \mathrm{tr}\Big(P^T\big(\dot{Q} - Q\widehat{\Omega}\big)\Big)\, dt = \int_a^b l(\widehat{\Omega}) + \mathrm{tr}\Big((Q^TP)^T\big(Q^{-1}\dot{Q} - \widehat{\Omega}\big)\Big)\, dt\,, \tag{11}$$

*for $(Q, P) \in T^*SO(3)$. Stationarity $(\delta S = 0)$ leads to the following variational conditions*

$$\widehat{\Pi} = \frac{\delta l}{\delta\widehat{\Omega}} = \frac{1}{2}(P^TQ - Q^TP) \in \mathfrak{so}(3)^*\,, \quad \Big\langle P\,,\, \dot{Q} - Q\widehat{\Omega}\Big\rangle := \mathrm{tr}\Big(P^T\,\big(\dot{Q} - Q\widehat{\Omega}\big)\Big) = \mathrm{tr}\Big((Q^TP)^T\big(Q^{-1}\dot{Q} - \widehat{\Omega}\big)\Big)\,,$$

*and the quantities $(Q, P) \in T^*SO(3)$ satisfy the following symmetric equations,*

$$\dot{Q} = Q\widehat{\Omega} \quad \textit{and} \quad \dot{P} = P\widehat{\Omega}\,, \tag{12}$$

*as a result of the constraints. These equations have Lie-Poisson Hamiltonian form,*

$$\frac{dF}{dt} = \Big\{F\,,\, H\Big\} = -\left\langle \Pi\,,\, \left[\frac{\partial F}{\partial \Pi}\,,\, \frac{\partial H}{\partial \Pi}\right]\right\rangle. \tag{13}$$

*Proof.* The variations of the constrained action $S$ in (11) are given by

$$
\begin{aligned}
\delta S &= \int_a^b \Big\langle \frac{\delta l}{\delta\widehat{\Omega}} \,,\, \delta\widehat{\Omega} \Big\rangle - \Big\langle P \,,\, Q\,\delta\widehat{\Omega} \Big\rangle + \Big\langle \delta P \,,\, \dot{Q} - Q\widehat{\Omega} \Big\rangle + \Big\langle P \,,\, \delta\dot{Q} - (\delta Q)\,\widehat{\Omega} \Big\rangle \, dt \\
&= \int_a^b \Big\{ \mathrm{tr}\Big[\Big(\widehat{\Pi}^T - \frac{1}{2}(P^TQ - Q^TP)\Big)\delta\widehat{\Omega}\Big] \\
&\qquad + \mathrm{tr}\Big[\delta P^T\Big(\dot{Q} - Q\widehat{\Omega}\Big)\Big] - \mathrm{tr}\Big[\Big(\dot{P}^T + \widehat{\Omega}P^T\Big)\delta Q\Big]\Big\} dt + \Big\langle P \,,\, \delta Q \Big\rangle \Big|_a^b .
\end{aligned}
$$

Thus, stationarity of this ***implicit variational principle*** implies the following set of equations

$$
\widehat{\Pi} = \frac{\delta l}{\delta\widehat{\Omega}} = \frac{1}{2}(P^TQ - Q^TP)\,, \quad \dot{Q} = Q\widehat{\Omega} \quad \text{and} \quad \dot{P} = P\widehat{\Omega}\,. \tag{14}
$$

The commutator form of the rigid-body equations in (10) emerges from these, upon elimination of $Q$ and $P$, as

$$
\begin{aligned}
\frac{d\widehat{\Pi}}{dt} &= \frac{1}{2}(\dot{P}^TQ + P^T\dot{Q} - \dot{Q}^TP - Q^T\dot{P}) \\
&= \frac{1}{2}\widehat{\Omega}(Q^TP - P^TQ) - \frac{1}{2}(P^TQ - Q^TP)\widehat{\Omega} \\
&= -\,\widehat{\Omega}\,\widehat{\Pi} + \widehat{\Pi}\,\widehat{\Omega} = -\,\big[\widehat{\Omega},\,\widehat{\Pi}\big]\,.
\end{aligned}
$$

These are Euler's equations for the rigid body on $T^*SO(3) \simeq so(3)^*$.

We Legendre transform the Lagrangian $l(\widehat{\Omega})$ to the Hamiltonian $H(\widehat{\Pi})$, as

$$H(\widehat{\Pi}) = \Big\langle \widehat{\Pi}\,,\,\widehat{\Omega} \Big\rangle - l(\widehat{\Omega}) \quad \text{with} \quad dH = \Big\langle d\widehat{\Pi}\,,\,\widehat{\Omega} \Big\rangle + \Big\langle \widehat{\Pi} - \frac{\partial l}{\partial \widehat{\Omega}}\,,\,d\widehat{\Omega} \Big\rangle\,.$$

Then, by using $\widehat{\Omega} = \partial H/\partial \widehat{\Pi}$ and $\widehat{\Omega}^T = -\widehat{\Omega}$ we find the following *Lie-Poisson bracket* for the Hamiltonian formulation of the rigid body dynamics,

$$\begin{aligned}
\frac{dF}{dt} = \Big\langle \frac{\partial F}{\partial \widehat{\Pi}}\,,\,\frac{d\widehat{\Pi}}{dt} \Big\rangle &= \Big\langle \frac{\partial F}{\partial \widehat{\Pi}}\,,\,\Big[\widehat{\Pi}\,,\,\frac{\partial H}{\partial \widehat{\Pi}}\Big] \Big\rangle \\
&= \operatorname{tr}\left( \frac{\partial F}{\partial \widehat{\Pi}} \Big[\widehat{\Pi}\,,\,\frac{\partial H}{\partial \widehat{\Pi}}\Big]^T \right) = -\operatorname{tr}\left( \widehat{\Pi}^T \Big[\frac{\partial F}{\partial \widehat{\Pi}}\,,\,\frac{\partial H}{\partial \widehat{\Pi}}\Big] \right) \\
&= -\Big\langle \widehat{\Pi}\,,\,\Big[\frac{\partial F}{\partial \widehat{\Pi}}\,,\,\frac{\partial H}{\partial \widehat{\Pi}}\Big] \Big\rangle =: \Big\{ F\,,\,H \Big\}.
\end{aligned}$$

In the ad-ad$^*$ notation, with $[\xi, \eta] =: \mathrm{ad}_\xi \eta$ for $\xi, \eta \in \mathfrak{g}$, where in this case $\mathfrak{g} = \mathfrak{so}(3)$, the Lie-Poisson bracket is written as

$$\frac{dF}{dt} = -\Big\langle \widehat{\Pi}\,,\,\Big[\frac{\partial F}{\partial \widehat{\Pi}}\,,\,\frac{\partial H}{\partial \widehat{\Pi}}\Big] \Big\rangle =: \Big\langle \widehat{\Pi}\,,\,\mathrm{ad}_{\frac{\partial H}{\partial \widehat{\Pi}}} \frac{\partial F}{\partial \widehat{\Pi}} \Big\rangle =: \Big\langle \mathrm{ad}^*_{\frac{\partial H}{\partial \widehat{\Pi}}} \widehat{\Pi}\,,\,\frac{\partial F}{\partial \widehat{\Pi}} \Big\rangle\,.$$

From this Lie Poisson equation, one verifies that

$$\frac{d\widehat{\Pi}}{dt} = \mathrm{ad}^*_{\frac{\partial H}{\partial \widehat{\Pi}}} \widehat{\Pi} = -\Big[\frac{\partial H}{\partial \widehat{\Pi}}\,,\,\widehat{\Pi}\Big]\,.$$

$\square$

## 5.8 Hamilton-Pontryagin principle for the Rigid Body equations

The Hamilton-Pontryagin constrained variation principle is more direct than the Clebsch variational principle, although it does not reveal the momentum map associated with the reduction by symmetry of the Lagrangian.

Consider the following constrained left-invariant action integral,

$$
\begin{aligned}
0 = \delta S(\widehat{\Omega}, O, \dot{O}) &= \delta \int_a^b l(\widehat{\Omega}) + \langle\, \widehat{\Pi}\,,\, O^{-1}\dot{O} - \widehat{\Omega}\,\rangle \, dt \\
&= \int_a^b \Big\langle\, \frac{\delta l}{\delta \widehat{\Omega}} - \widehat{\Pi}\,,\, \delta\widehat{\Omega} \Big\rangle + \langle\, \delta\widehat{\Pi}\,,\, O^{-1}\dot{O} - \widehat{\Omega}\,\rangle + \langle\, \widehat{\Pi}\,,\, \delta(O^{-1}\dot{O})\,\rangle \, dt\,,
\end{aligned}
$$

with Frobenius pairing of skew symmetric matrices $\langle\, \widehat{\Pi}\,,\, \widehat{\Omega}\rangle = \mathrm{tr}(\widehat{\Pi}^T\widehat{\Omega})$.

Denote $\delta(\,\cdot\,) = \frac{\partial}{\partial\epsilon}|_{\epsilon=0}(\,\cdot\,) = (\,\cdot\,)'$ as well as $\widehat{\Omega} = O^{-1}\dot{O}$ and $\widehat{\Xi} = O^{-1}O'$, then compute that

$$
\begin{aligned}
(O^{-1}\dot{O})' = \widehat{\Omega}' &= -(O^{-1}O')(O^{-1}\dot{O}) + \dot{O}' = -\widehat{\Xi}\widehat{\Omega} + \dot{O}' \\
(O^{-1}O')^{\cdot} = \widehat{\Xi}^{\,\cdot} &= -(O^{-1}\dot{O})(O^{-1}O') + O'^{\,\cdot} = -\widehat{\Omega}\widehat{\Xi} + \dot{O}'\,.
\end{aligned}
$$

Subtracting these two equations yields

$$
\widehat{\Omega}' = \delta\widehat{\Omega} = \widehat{\Xi}^{\,\cdot} + \widehat{\Omega}\widehat{\Xi} - \widehat{\Xi}\widehat{\Omega} =: \widehat{\Xi}^{\,\cdot} + \Big[\widehat{\Omega}\,,\, \widehat{\Xi}\Big] =: \widehat{\Xi}^{\,\cdot} + \mathrm{ad}_{\widehat{\Omega}}\,\widehat{\Xi}\,.
$$

Substitution then yields

$$
\int_a^b \langle\, \widehat{\Pi}\,,\, \delta(O^{-1}\dot{O})\,\rangle \, dt = \int_a^b \langle\, \widehat{\Pi}\,,\, \frac{d\widehat{\Xi}}{dt} + \mathrm{ad}_{\widehat{\Omega}}\,\widehat{\Xi}\,\rangle \, dt + \langle\, \widehat{\Pi}\,,\, \widehat{\Xi}\,\rangle\Big|_a^b = \int_a^b \Big\langle - \frac{d\widehat{\Pi}}{dt} + \mathrm{ad}^*_{\widehat{\Omega}}\widehat{\Pi}\,,\, \widehat{\Xi}\,\Big\rangle \, dt + \langle\, \widehat{\Pi}\,,\, \widehat{\Xi}\,\rangle\Big|_a^b\,,
$$

where $\mathrm{ad}_{\widehat{\Omega}}\widehat{\Xi} = [\widehat{\Omega}\,,\, \widehat{\Xi}]$ and $\mathrm{ad}^*_{\widehat{\Omega}}\widehat{\Pi} = -[\widehat{\Omega}\,,\, \widehat{\Pi}]$ via the Frobenius pairing $\langle\, \widehat{\Pi}, \mathrm{ad}_{\widehat{\Omega}}\,\widehat{\Xi}\rangle = \langle\, \mathrm{ad}^*_{\widehat{\Omega}}\widehat{\Pi}, \widehat{\Xi}\,\rangle$.
Consequently, we recover Euler's rigid body equation, $\frac{d\widehat{\Pi}}{dt} = -[\widehat{\Omega}\,,\, \widehat{\Pi}]$.

### 5.9 Hamilton-Pontryagin principle for the Euler–Poincaré equations

**Theorem 18** (Hamilton–Pontryagin principle for the Euler–Poincaré equations)**.**
*The Euler–Poincaré equation*

$$\frac{d}{dt}\frac{\delta l}{\delta \xi} = \mathrm{ad}^*_\xi \frac{\delta l}{\delta \xi} \tag{15}$$

*on the dual Lie algebra* $\mathfrak{g}^*$ *is equivalent to the following variational principle,*

$$\delta S(\xi, g, \dot g) = \delta \int_a^b l(\xi, g, \dot g)\, dt = 0, \tag{16}$$

*for a constrained left-invariant action integral*

$$S(\xi, g, \dot g) = \int_a^b l(\xi, g, \dot g)\, dt = \int_a^b \Big[ l(\xi) + \langle\, \mu\,,\, (g^{-1}\dot g - \xi)\,\rangle \Big]\, dt\,.$$

*Proof.* The variations of $S$ in formula (16) are given by

$$\delta S = \int_a^b \Big\langle\, \frac{\delta l}{\delta \xi} - \mu\,,\, \delta\xi \,\Big\rangle + \Big\langle\, \delta\mu\,,\, (g^{-1}\dot g - \xi) \,\Big\rangle + \Big\langle\, \mu\,,\, \delta(g^{-1}\dot g) \,\Big\rangle\, dt\,.$$

Substituting $\delta(g^{-1}\dot g) = \dot\eta + \mathrm{ad}_\xi\, \eta$ obtained from $\delta(\dot g) = (\delta g)^{\cdot}$ with $\eta := g^{-1}\delta g$ into the last term produces

$$\begin{aligned} \int_a^b \Big\langle\, \mu\,,\, \delta(g^{-1}\dot g) \,\Big\rangle\, dt &= \int_a^b \Big\langle\, \mu\,,\, \dot\eta + \mathrm{ad}_\xi\, \eta \,\Big\rangle\, dt \\ &= \int_a^b \Big\langle\, -\dot\mu + \mathrm{ad}^*_\xi\, \mu\,,\, \eta \,\Big\rangle\, dt + \Big\langle\, \mu\,,\, \eta \,\Big\rangle\Big|_a^b, \end{aligned}$$

where $\eta = g^{-1}\delta g$ vanishes at the endpoints in time. Thus, stationarity $\delta S = 0$ of the Hamilton–Pontryagin variational principle yields the following set of equations:

$$\frac{\delta l}{\delta \xi} = \mu\,, \quad g^{-1}\dot{g} = \xi\,, \quad \dot{\mu} = \mathrm{ad}^*_\xi\,\mu\,. \tag{17}$$

$\square$

**Legendre transformation.** After the Legendre transformation to

$$h(\mu) = \langle\, \mu\,,\, \xi\,\rangle - \ell(\xi) \tag{18}$$

we have the differential relations

$$dh = \Big\langle\, \frac{\partial h}{\partial \mu}\,,\, d\mu \,\Big\rangle = \langle\, d\mu\,,\, \xi\,\rangle + \Big\langle\, \mu - \frac{\partial l}{\partial \xi}\,,\, \delta\xi \,\Big\rangle \tag{19}$$

so that $\partial h/\partial\mu = \xi$, which leads to the Hamiltonian formulation of the Hamilton–Pontryagin equations (20)

$$\dot{\mu} = \mathrm{ad}^*_{\partial h/\partial\mu}\,\mu\,, \quad \frac{dF}{dt} = \Big\langle\, \mathrm{ad}^*_{\partial h/\partial\mu}\,\mu\,,\, \frac{\partial F}{\partial \mu} \,\Big\rangle = \Big\langle\, \mu\,,\, \mathrm{ad}_{\partial h/\partial\mu}\,\frac{\partial F}{\partial \mu} \,\Big\rangle = -\Big\langle\, \mu\,,\, \Big[\frac{\partial F}{\partial \mu}\,,\, \frac{\partial H}{\partial \mu}\Big] \,\Big\rangle =: \Big\{\, F\,,\, H \Big\}\,. \tag{20}$$

**Exercise.** Recalculate the Hamilton-Pontryagin variational principle and derive its associated Lie-Poisson bracket for a constrained *right-invariant* action integral

$$S(\xi, g, \dot{g}) = \int_a^b \Big[ l(\xi) + \langle\, \mu\,,\, (\dot{g}g^{-1} - \xi)\,\rangle \Big]\, dt\,.$$

★

**Answer.**

$$\frac{\delta l}{\delta \xi} = \mu\,, \quad \dot{g}g^{-1} = \xi\,, \quad \dot{\mu} = -\operatorname{ad}^*_\xi \mu\,. \quad \text{Note the minus sign for right-invariance, cf. (20).}$$

We will need the right-invariant Euler-Poincaré and Lie-Poisson equations when we study fluid dynamics. ▲

**Exercise.** Suppose left-invariance of the previous action principle is broken by the presence in the Lagrangian of a parameter $a_0$ which transforms under the group $g$ as $a_t = g_t^{-1}a_0$, whose definition implies that it satisfies the auxiliary equation,

$$\frac{da_t}{dt} = \frac{d(g_t^{-1}a_0)}{dt} = -(g_t^{-1}\dot{g}_t g_t^{-1})a_0 = -g_t^{-1}\dot{g}_t a_t = -\xi a_t$$

The reduced Lagrangian becomes $l(\xi) \to l(\xi, g^{-1}a_0)$ with $\xi = \dot{g}g^{-1}$ and $g \in G$. The symmetry of the Lagrangian is thus reduced from the group $G$ to the isotropy subgroup $G_{a_0} \subset G$ which leaves $a_0$ invariant under left action. That is, $ga_0 = 0$ for all $g \in G_{a_0}$. In the presence of this broken symmetry, the previous constrained action integral becomes

$$S(\xi, a_0, g, \dot{g}) = \int_a^b l(\xi, g^{-1}a_0, g, \dot{g})\, dt = \int_a^b \Big[ l(\xi, g^{-1}a_0) + \langle\, \mu\,,\, (g^{-1}\dot{g} - \xi)\,\rangle \Big]\, dt\,.$$

Show that the Euler–Poincaré equation (21) changes to

$$\frac{d}{dt}\frac{\delta l}{\delta \xi} = \operatorname{ad}^*_\xi \frac{\delta l}{\delta \xi} + \frac{\partial l}{\partial a_t} \diamond a_t \tag{21}$$

where the $\diamond$ operator notation is defined by

$$\left\langle \frac{\partial l}{\partial a_t} \diamond a_t\,,\, \eta \right\rangle = \left\langle \frac{\partial l}{\partial a_t}\,,\, -\eta a_t \right\rangle$$

★

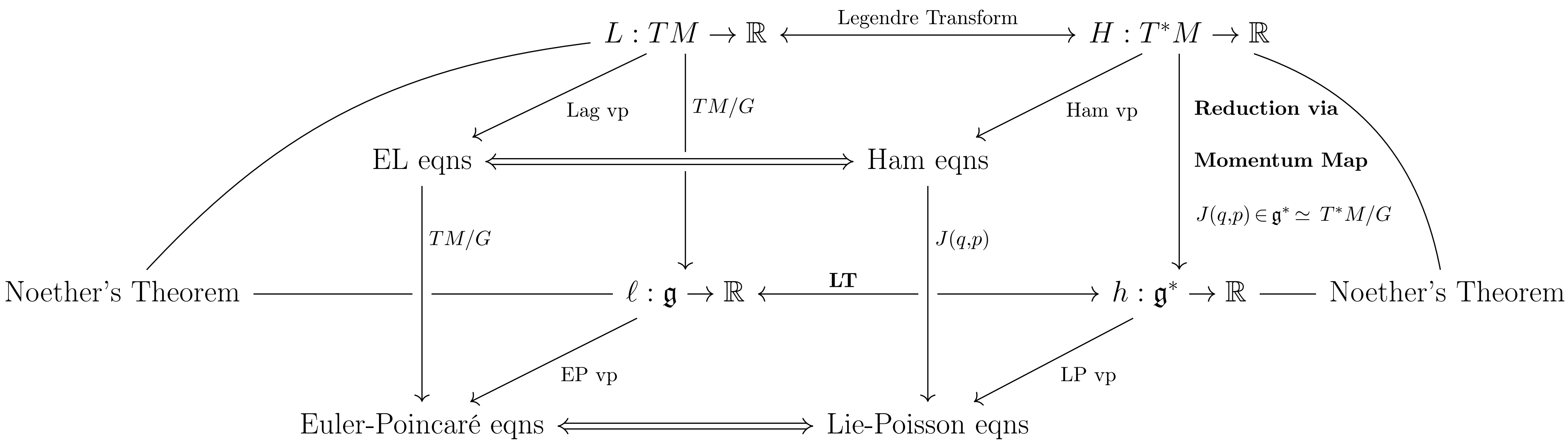



Figure 9: Framework for Geometric Mechanics

## Summary for the rigid body

- Relation between left-invariant Lagrangians:

$$L(Q, \dot{Q}) = L(e, Q^{-1}\dot{Q}) = \ell(\Omega)$$

- Poisson brackets:

$$\{F, H\} = -\Pi \cdot \frac{\partial F}{\partial \Pi} \times \frac{\partial H}{\partial \Pi} = -\left\langle \Pi, \left[\frac{\partial F}{\partial \Pi}, \frac{\partial H}{\partial \Pi}\right]\right\rangle$$

$$\{\Pi, H\} = \Pi \times \mathbb{I}^{-1}\Pi, \qquad (\Pi = \mathbb{I}\Omega)$$

$$
\begin{tikzcd}
 & & L(Q,\dot{Q}) : TSO(3) \to \mathbb{R} \arrow[dl, "\mathrm{LagHvp}"] \arrow[dd, "\Omega = Q^{-1}\dot{Q}"] \arrow[rr, leftrightarrow, "\mathbf{LT}\ \ P = \partial L/\partial \dot{Q}"] & & H(Q,P) : T^*SO(3) \to \mathbb{R} \arrow[dl, "\mathrm{HamHvp}"'] \arrow[dd, "\Pi"] \arrow[ddr, dash] & \\
 & \begin{array}{l} \frac{d}{dt}\frac{\partial L}{\partial \dot{Q}} - \frac{\partial L}{\partial Q} = 0 \\ \text{Euler-Lagrange} \end{array} \arrow[dd, "\Omega = Q^{-1}\dot{Q}"] \arrow[rr, Leftrightarrow] & & \begin{array}{l} \frac{dF}{dt}(Q,P) = \{F,H\}_{\mathrm{can}} \\ \text{Hamilton} \end{array} \arrow[dd, "\Pi"] & & \\
\begin{array}{l} \text{Lagrangian} \\ \text{Noether} \end{array} \arrow[rr, dash] \arrow[uurr, dash, bend left] & & \ell(\Omega) : \mathfrak{so}(3) \to \mathbb{R} \arrow[dl, "\mathrm{EPvp}"] \arrow[rr, leftrightarrow, "\mathbf{LT}\ \ \Pi = \partial \ell/\partial \Omega"] & & h(\Pi) : \mathfrak{so}^*(3) \to \mathbb{R} \arrow[dl, "\mathrm{LPvp}"] \arrow[r, dash] & \begin{array}{l} \text{Hamiltonian} \\ \text{Noether} \end{array} \\
 & \begin{array}{l} \frac{d}{dt}\frac{\partial \ell}{\partial \Omega} = \frac{\partial \ell}{\partial \Omega} \times \Omega \\ \text{Euler-Poincaré} \end{array} \arrow[rr, Leftrightarrow] & & \begin{array}{l} \frac{d\Pi}{dt} = \{\Pi, H\} \\ \text{Lie-Poisson} \end{array} & &
\end{tikzcd}
$$

Figure 10: Rigid body dynamics

- Momentum map

  For $Q \in SO(3)$ and $\delta Q = \Phi_\xi(Q) \in TSO(3)$, the Noether quantity is

$$J_\xi(P,Q) := \big\langle P, \Phi_\xi(Q) \big\rangle_{TSO(3)} = \big\langle \Pi, \xi \big\rangle_{\mathfrak{so}(3)}$$

### 5.10 The reduced Kepler problem: Newton (1686)

The ***reduced Kepler problem*** of planetary motion arises from Hamilton's principle for the Lagrangian

$$L(\mathbf{r},\dot{\mathbf{r}}) = \frac{1}{2}|\dot{\mathbf{r}}|^2 - V(r) = \frac{1}{2}\big(|\dot{r}|^2 + r^2\dot{\theta}^2\big) - V(r) \quad \text{with} \quad V(r) = -\frac{\mu}{r}\,.$$

For this Lagrangian, Hamilton's Principle implies Newton's equation of motion,

$$\ddot{\mathbf{r}} + \frac{\mu\mathbf{r}}{r^3} = 0\,, \tag{22}$$

in which $\mu$ is a constant and $r = |\mathbf{r}|$ with $\mathbf{r} \in \mathbb{R}^3$.

Scale invariance of this equation under the changes $R \to s^2 R$ and $T \to s^3 T$ in the units of space $R$ and time $T$ for any constant $(s)$ means that it admits families of solutions whose space and time scales are related by $T^2/R^3 = const.$ This is ***Kepler's third law***. Newton (1686) showed that his equation (22) implied that $T^2/a^3 = (2\pi)^2/\mu = constant$, and thereby founded celestial mechanics.

1. The scalar (resp. vector) product of equation (22) with $\mathbf{r}$ shows conservation of the energy $E$ and (resp.) specific angular momentum $\mathbf{L}$, given by

$$\begin{aligned} E &= \frac{1}{2}|\dot{\mathbf{r}}|^2 - \frac{\mu}{r} \quad \text{(energy)}\,, \ \text{or } H(r,p_r,p_\theta) = \frac{1}{2}\Big(p_r^2 + \frac{p_\theta^2}{r^2}\Big) - \frac{\mu}{r} \quad \text{(Hamiltonian)} \\ \mathbf{L} &= \mathbf{r}\times\dot{\mathbf{r}} \quad \text{(specific angular momentum)}\,. \end{aligned}$$

   Since $\mathbf{r}\cdot\mathbf{L} = 0$, the planetary motion in $\mathbb{R}^3$ takes place in a plane to which vector $\mathbf{L}$ is perpendicular. This is the orbital plane. Constancy of magnitude $L$ means the orbit sweeps out equal areas in equal times (Kepler's second law). In the orbital plane, one may specify plane polar coordinates $(r,\theta)$ with unit vectors $(\hat{\mathbf{r}},\,\hat{\boldsymbol{\theta}})$ in the plane and $\hat{\mathbf{r}}\times\hat{\boldsymbol{\theta}} = \widehat{\mathbf{L}}$ normal to it. In particular,

$$\mathbf{L} = \mathbf{r}\times\dot{\mathbf{r}} = r\hat{\mathbf{r}}\times(\dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}) = r^2\dot{\theta}\,\hat{\mathbf{r}}\times\hat{\boldsymbol{\theta}} = r^2\dot{\theta}\,\widehat{\mathbf{L}} = p_\theta\,\widehat{\mathbf{L}}\,,$$

   so the magnitude of the angular momentum is $L = |\mathbf{L}| = r^2\dot{\theta} = p_\theta$.

2. The unit vectors for polar coordinates in the orbital plane are $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$. These vectors satisfy

$$\frac{d\hat{\mathbf{r}}}{dt} = \dot{\theta}\,\widehat{\mathbf{L}}\times\hat{\mathbf{r}} = \dot{\theta}\,\hat{\boldsymbol{\theta}} \quad\text{and}\quad \frac{d\hat{\boldsymbol{\theta}}}{dt} = \dot{\theta}\,\widehat{\mathbf{L}}\times\hat{\boldsymbol{\theta}} = -\,\dot{\theta}\,\hat{\mathbf{r}}\,, \quad \text{where} \quad \dot{\theta} = \frac{L}{r^2}\,.$$

Newton's equation of motion (22) for the Kepler problem may now be written equivalently using $\dot{\theta}/L = 1/r^2$ and $\frac{d\hat{\boldsymbol{\theta}}}{dt} = -\,\dot{\theta}\,\hat{\mathbf{r}}$, as

$$0 = \ddot{\mathbf{r}} + \frac{\mu\mathbf{r}}{r^3} = \ddot{\mathbf{r}} + \frac{\mu}{L}\dot{\theta}\,\hat{\mathbf{r}} = \frac{d}{dt}\Big(\dot{\mathbf{r}} - \frac{\mu}{L}\hat{\boldsymbol{\theta}}\Big)\,.$$

This equation implies conservation of the following vector *in the plane of motion*:

$$\mathbf{K} = \dot{\mathbf{r}} - \frac{\mu}{L}\hat{\boldsymbol{\theta}} \quad (\textit{\textbf{Hamilton's vector}})\,.$$

The cross product of the two conserved vectors $\mathbf{K}$ and $\mathbf{L}$ yields another conserved vector in the plane of motion

$$\mathbf{J} = \mathbf{K}\times\mathbf{L} = \dot{\mathbf{r}}\times\mathbf{L} - \mu\hat{\mathbf{r}} \quad (\textit{\textbf{Laplace–Runge–Lenz vector}})\,.$$

3. Note immediately that $\mathbf{J}\cdot\mathbf{L} = 0 = \mathbf{J}\cdot\mathbf{K}$, $J^2 = 2EL^2 + \mu^2 = K^2L^2$ and the dimensions of $\mathbf{J}$ are given by $[J] = [\mu] = [r]^3[t]^{-2}$, the same as Kepler's Third Law! Thus, from their definitions, these conserved quantities are related by

$$K^2 = 2E + \frac{\mu^2}{L^2} = \frac{J^2}{L^2}\,, \quad \text{upon using} \quad K^2 = \Big|\dot{\mathbf{r}} - \frac{\mu}{L}\hat{\boldsymbol{\theta}}\Big|^2 = |\dot{\mathbf{r}}|^2 - \frac{2\mu}{L}\dot{\mathbf{r}}\cdot\hat{\boldsymbol{\theta}} + \frac{\mu^2}{L^2} = |\dot{\mathbf{r}}|^2 - \frac{2\mu}{r} + \frac{\mu^2}{L^2}\,,$$

since $\dot{\mathbf{r}} = \dot{r}\,\hat{\mathbf{r}} + r\dot{\theta}\,\hat{\boldsymbol{\theta}}$ and $L = r^2\dot{\theta}$. Equivalently,

$$L^2 + \frac{J^2}{(-2E)} = \frac{\mu^2}{(-2E)} \quad\Longrightarrow\quad -2E = \frac{\mu^2 - J^2}{L^2} \quad \text{and} \quad \mathbf{J}\cdot\mathbf{K}\times\mathbf{L} = K^2L^2 = J^2\,,$$

where $J^2 := |\mathbf{J}|^2$, etc. and $-2E > 0$ for bounded orbits. Hence, the motion $(\mathbf{r},\dot{\mathbf{r}}) \in T\mathbb{R}^3 \simeq \mathbb{R}^3\times\mathbb{R}^3$ takes place in 6 dimensions on the intersections of level sets of $E$, $J^2 - 2EL^2 = \mu^2$ and $\mathbf{J}\cdot\mathbf{L} = 0$.

4. Orient the conserved Laplace–Runge–Lenz vector $\mathbf{J}$ in the orbital plane to point along the reference line for the measurement of the polar angle $\theta$, say from the centre of the orbit (Sun) to the perihelion (point of nearest approach, on midsummer's day). The scalar product of $\mathbf{r}$ and $\mathbf{J}$ then yields an elegant result for the Kepler orbit in plane polar coordinates:

$$\mathbf{r}\cdot\mathbf{J} = rJ\cos\theta = \mathbf{r}\cdot(\dot{\mathbf{r}}\times\mathbf{L} - \mu\,\mathbf{r}/r) = \mathbf{r}\cdot(\dot{\mathbf{r}}\times\mathbf{L}) - \mu\,r\,,$$

which implies

$$r(\theta) = \frac{L^2}{\mu + J\cos\theta} = \frac{l_\perp}{1 + e\cos\theta}\,. \tag{23}$$

As expected, the orbit $r(\theta)$ is a ***conic section*** whose origin is at one of the two foci. This is ***Kepler's first law***.

Let $a$ and $b$ be respectively the semi-major and semi-minor axes of the ellipse drawn with a string of length $2a$ attached at foci $\pm e$. One may form two right angles with the string to discover that $e^2 + b^2 = a^2$ and $l_\perp = b^2/a$, by Pythagoras' theorem. The eccentricity vanishes ($e = 0$) for a circle and correspondingly $K = 0$ implies that $\dot{\mathbf{r}} = \mu\,\hat{\boldsymbol{\theta}}/L$. The eccentricity takes values $0 < e < 1$ for an ellipse, $e = 1$ for a parabola and $e > 1$ for a hyperbola.

In summary: the Laplace–Runge–Lenz vector $\mathbf{J}$ is directed from the focus of the orbit to its perihelion (point of closest approach). The eccentricity of the elliptical orbit is $e = J/\mu = KL/\mu = \sqrt{a^2 - b^2}$ and its semi-latus rectum (normal distance from the line through the foci to the orbit) is $l_\perp = L^2/\mu = b^2/a$.

5. One may use the conservation of $\mathbf{L}$ in $\mathbf{L}dt = \mathbf{r}\times d\mathbf{r}$ or $L$ in $Ldt = r^2 d\theta$ to show that the constancy of magnitude $L = |\mathbf{L}|$ means the orbit sweeps out equal areas in equal times. This is ***Kepler's second law***.

   For an elliptical orbit, the integral $LT = \int_0^T Ldt = \int_0^{2\pi} r(\theta)^2 d\theta = 2A$ yields the period in terms of angular momentum and the area; namely, $T = 2A/L$. Hence, $4A^2/T^2 = L^2 = \mu l_\perp$

6. One may use the result of part 5 and the geometric properties of ellipses to show that the period of the orbit is given by

$$\left(\frac{T}{2\pi}\right)^2 = \frac{a^3}{\mu} = \frac{\mu^2}{(-2E)^3}\,. \tag{24}$$

The relation $T^2/a^3 = (2\pi)^2/\mu = constant$ comprises Kepler's third law, which reflects the scale invariance of Newton's equation. The constant $\mu$ is Newton's universal constant of gravitational attraction.

This is a profound relation! Time and space are linked! The period and aphelion-to-perihelion distance of a planetary orbit determines the fundamental gravitational property which holds the universe together!

7. One may check that the Poisson brackets amongst the components of the vectors $\mathbf{L} = \mathbf{r}\times\mathbf{p}$ and $\mathbf{J} = \mathbf{p}\times\mathbf{L} - \mu\mathbf{r}/r$ satisfy the following relations:

$$\begin{aligned}
&\{L_i,\, L_j\} = \epsilon_{ijk}L_k\,,\\
&\{L_i,\, J_j\} = \epsilon_{ijk}J_k\,, \quad\Longrightarrow\quad \{L_i,\, \bar{J}_j\} = \epsilon_{ijk}\bar{J}_k \quad \text{with} \quad \bar{J}_k := J_k/\sqrt{-2H}\,,\\
&\{J_i,\, J_j\} = -\,2H\epsilon_{ijk}L_k\,, \quad\Longrightarrow\quad \{\bar{J}_i,\, \bar{J}_j\} = \epsilon_{ijk}L_k \quad \text{after using} \quad \{L_i,\, H\} = 0 = \{J_i,\, H\}\,.
\end{aligned}$$

In tabular form, this is

$$\{(L,J)\,,\,(L,J)\} = \begin{array}{|c|cc|}\hline \{\,\cdot\,,\,\cdot\,\} & L & J \\ \hline L & L\times & J\times \\ J & J\times & -2H\,L\times \\ \hline\end{array}\,. \tag{25}$$

Importantly, these relations imply that the Poisson bracket with $J_i$ alters both the eccentricity and the width of an elliptical orbit, as one finds upon using $\{J^2, J_i\} = 4H(\mathbf{J}\times\mathbf{L})_i = 2H\{L^2, J_i\}$ in the following sequence,

$$\{J_i,\, e^2\}\mu^2 = \{J_i,\, J^2\} = -\,4H\epsilon_{ijk}J_jL_k = -\,4H(\mathbf{J}\times\mathbf{L})_i = -\,2H\{L^2,\, J_i\} = -\,2H\mu\{l_\perp,\, J_i\}\,.$$

The conservation laws $\{L^2, H\} = 0$ and $\{J_i, H\} = 0$ allow the use of formula (25) to check the properties of the previous Poisson bracket relations. In particular, two *Casimir functions* Poisson commute with any other functions $F(\mathbf{J}, \mathbf{L})$ on any level surface of $H$, and therefore are constants of the $(\mathbf{J}, \mathbf{L})$ motions. Namely,

$$C_1 = J^2 - 2H\,L^2 \quad \text{and} \quad C_2 = \mathbf{J}\cdot\mathbf{L} \Longrightarrow \{J_i, C_a\} = 0 = \{L_i, C_a\}\,, \quad i = 1,2,3, \quad a = 1,2\,.$$

For the Kepler problem, the level sets of the Casmirs take the physically meaningful values $C_1 = \mu^2$ and $C_2 = 0$. In summary, the motion $(\mathbf{r}, \mathbf{p}) \in T^*\mathbb{R}^3 \simeq \mathbb{R}^3 \times \mathbb{R}^3$ takes place in 6 dimensions on the intersections of level sets of phase space functions $H$, $C_1 = J^2 - 2EL^2 = \mu^2$ and $C_2 = \mathbf{J}\cdot\mathbf{L} = 0$, all in involution under Poisson brackets. The dimensions drop as $6 - 2 - 2 = 2$. Hence, this Hamiltonian system is integrable.

8. Upon referring to the relationships between the orbital parameters and the conservation laws derived in (23), explain how the canonical transformations generated by $\mathbf{J}$ affect the (i) energy, (ii) eccentricity and (iii) width of the orbit.

   The canonical transformations generated by $\mathbf{J}$ change both the eccentricity $e$ and the width $l_\perp$ of the orbit, but in conserving a certain combination of them, *they map closed ellipses into closed ellipses*, without any precession. Namely,

$$\big\{J_i, \big(J^2 - 2H\,L^2\big)\big\} = (-2H\mu)\left\{J_i, \frac{e^2\mu}{-2H} + l_\perp\right\} = 0\,.$$

   Note that because of equation (24) the canonical transformations generated by the Runge-Lenz vector $\mathbf{J}$ preserve the period and the semi-major axis of the elliptical Kepler orbit, while changing its area $(A = \pi ab = LT/2)$ and shape $(b/a)$ with $b = \sqrt{a^2 - e^2} = \sqrt{l_\perp a}$, by altering its eccentricity $(e = J/\mu)$ and its semi-latus rectum

($l_\perp = L^2/\mu$) with $J^2/L^2 = K^2 = e^2\mu/l_\perp$, according to

$$\{e^2, J_i\} = \frac{1}{\mu^2}\{J^2, J_i\} = \frac{4H}{\mu^2}(\mathbf{J}\times\mathbf{L})_i = -\frac{4HL^2}{\mu^2}K_i = -\frac{4Hl_\perp}{\mu}K_i\,,$$
$$\{l_\perp, J_i\} = \frac{1}{\mu}\{L^2, J_i\} = \frac{2}{\mu}(\mathbf{J}\times\mathbf{L})_i = -\frac{2L^2}{\mu}K_i = -2l_\perp K_i\,.$$

Finally, if we define $\mathbf{M}^\pm := \sqrt{-2H}\,\mathbf{L}\pm\mathbf{J}$, with $|\mathbf{M}^+|^2 = (J^2 - 2HL^2) = \mu^2 = |\mathbf{M}^-|^2$ since $\mathbf{L}\cdot\mathbf{J} = 0$ then we find the following Poisson bracket relations for negative energy (-2H>0),

$$\{M_i^+, M_j^-\} = 0\,,\quad \{M_i^+, M_j^+\} = 2\epsilon_{ijk}M_k^+\,,\quad \{M_i^-, M_j^-\} = 2\epsilon_{ijk}M_k^-\,,$$

which one may recognize as the Lie-Poisson brackets on the dual of the Lie algebra of $so(3)\times so(3)\simeq so(4)$, for which $|\mathbf{M}^\pm|^2$ are Casimirs.

**Exercise.** How do these Poisson bracket relations change when $H \ge 0$? ★

# 6 Transformation Theory

| | | |
|---|---|---|
| motion | linearisation | differential, $d$ |
| motion equation | infinitesimal transformation | differential $k$-form |
| vector field | pull-back | wedge product, $\wedge$ |
| diffeomorphism | push-forward | Lie derivative, $\pounds_Q$ |
| flow | Jacobian matrix | product rule |
| fixed point | directional derivative | fluid dynamics |
| equilibrium | commutator | other flows |

## 6.1 Motions, pull-backs, push-forwards, commutators & differentials

- A ***motion*** is defined as a smooth curve $q(t) \in M$ parameterised by $t \in \mathbb{R}$ that solves the ***motion equation***, which is a system of differential equations

$$\dot{q}(t) = \frac{dq}{dt} = f(q) \in TM\,, \tag{26}$$

or in components

$$\dot{q}^i(t) = \frac{dq^i}{dt} = f^i(q) \quad i = 1, 2, \ldots, n\,, \tag{27}$$

- The map $f : q \in M \to f(q) \in T_qM$ is a ***vector field***.

  According to standard theorems about differential equations that are not proven in this course, the solution, or integral curve, $q(t)$ exists, provided $f$ is sufficiently smooth, which will always be assumed to hold.

- Vector fields can also be defined as ***differential operators*** that act on functions, as

$$\frac{d}{dt}G(q) = \dot{q}^i(t)\frac{\partial G}{\partial q^i} = f^i(q)\frac{\partial G}{\partial q^i} \quad i = 1, 2, \ldots, n, \quad \text{(sum on repeated indices)} \tag{28}$$

  for any smooth function $G(q) : M \to \mathbb{R}$.

- To indicate the dependence of the solution of its initial condition $q(0) = q_0$, we write the motion as a smooth transformation

$$q(t) = \phi_t(q_0)\,.$$

  Because the vector field $f$ is independent of time $t$, for any fixed value of $t$ we may regard $\phi_t$ as mapping from $M$ into itself that satisfies the composition law

$$\phi_t \circ \phi_s = \phi_{t+s}$$

  and

$$\phi_0 = \mathrm{Id}\,.$$

  Setting $s = -t$ shows that $\phi_t$ has a smooth inverse. A smooth mapping that has a smooth inverse is called a ***diffeomorphism***. Geometric mechanics deals with diffeomorphisms.

- The smooth mapping $\phi_t : \mathbb{R} \times M \to M$ that determines the solution $\phi_t \circ q_0 = q(t) \in M$ of the motion equation (26) with initial condition $q(0) = q_0$ is called the ***flow*** of the vector field $Q$.

  A point $q^\star \in M$ at which $f(q^\star) = 0$ is called a ***fixed point*** of the flow $\phi_t$, or an ***equilibrium***.

  Vice versa, the vector field $f$ is called the ***infinitesimal transformation*** of the mapping $\phi_t$, since

$$\frac{d}{dt}\bigg|_{t=0}(\phi_t \circ q_0) = f(q)\,.$$

That is, $f(q)$ is the ***linearisation*** of the flow map $\phi_t$ at the point $q \in M$.

More generally, the ***directional derivative*** of the function $h$ along the vector field $f$ is given by the action of a differential operator, as

$$\frac{d}{dt}\bigg|_{t=0} h \circ \phi_t = \left[ \frac{\partial h}{\partial \phi_t} \frac{d}{dt} (\phi_t \circ q_0) \right]_{t=0} = \frac{\partial h}{\partial q^i} \dot{q}^i = \frac{\partial h}{\partial q^i} f^i(q) =: Qh \,.$$

- Under a smooth change of variables $q = c(r)$ the vector field $Q$ in the expression $Qh$ transforms as

$$Q = f^i(q) \frac{\partial}{\partial q^i} \quad \mapsto \quad R = g^j(r) \frac{\partial}{\partial r^j} \quad \text{with} \quad g^j(r) \frac{\partial c^i}{\partial r^j} = f^i(c(r)) \quad \text{or} \quad g = c_r^{-1} f \circ c \,, \tag{29}$$

where $c_r$ is the ***Jacobian matrix*** of the transformation. That is, since $h(q)$ is a function of $q$,

$$(Qh) \circ c = R(h \circ c) \,.$$

We express the transformation between the vector fields as $R = c^* Q$ and write this relation as

$$(Qh) \circ c =: c^* Q(h \circ c) \,. \tag{30}$$

The expression $c^*Q$ is called the ***pull-back*** of the vector field $Q$ by the map $c$. Two vector fields are equivalent under a map $c$, if one is the pull-back of the other, and fixed points are mapped into fixed points.

The inverse of the pull-back is called the ***push-forward***. It is the pull-back by the inverse map.

- The ***commutator***

$$QR - RQ =: \big[ Q,\, R \big]$$

of two vector fields $Q$ and $R$ defines another vector field. Indeed, if

$$Q = f^i(q) \frac{\partial}{\partial q^i} \quad \text{and} \quad R = g^j(q) \frac{\partial}{\partial q^j}$$

then

$$\left[Q,\,R\right] = \left( f^i(q)\frac{\partial g^j(q)}{\partial q^i} - g^i(q)\frac{\partial f^j(q)}{\partial q^i} \right) \frac{\partial}{\partial q^j}$$

because the second-order derivative terms cancel. By the pull-back relation (30) we have

$$c^*\big[Q,\,R\big] = \big[c^*Q,\,c^*R\big] \tag{31}$$

under a change of variables defined by a smooth map, $c$. This means the definition of the vector field commutator is independent of the choice of coordinates. As we shall see, the ***tangent*** to the relation $c_t^*\big[Q,\,R\big] = \big[c_t^*Q,\,c_t^*R\big]$ at the identity $t = 0$ is the ***Jacobi condition*** for the vector fields to form an algebra.

- The ***differential*** of a smooth function $f : M \to M$ is defined as

$$df = \frac{\partial f}{\partial q^i} dq^i \,.$$

- Under a smooth change of variables $s = \phi \circ q = \phi(q)$ the differential of the composition of functions $d(f \circ \phi)$ transforms according to the chain rule as

$$df = \frac{\partial f}{\partial q^i} dq^i \,, \quad d(f\circ\phi) = \frac{\partial f}{\partial \phi^j(q)}\frac{\partial \phi^j}{\partial q^i} dq^i = \frac{\partial f}{\partial s^j} ds^j \quad \Longrightarrow \quad d(f\circ\phi) = (df)\circ\phi \tag{32}$$

That is, the differential $d$ commutes with the pull-back $\phi^*$ of a smooth transformation $\phi$,

$$d(\phi^* f) = \phi^* df \,. \tag{33}$$

In a moment, this pull-back formula will give us the rule for transforming differential forms of any order.

### 6.2 Wedge products

- Differential $k$-forms on an $n$-dimensional manifold are defined in terms of the differential $d$ and the antisymmetric ***wedge product*** ($\wedge$) satisfying

$$dq^i \wedge dq^j = - dq^j \wedge dq^i \,, \quad \text{for} \quad i, j = 1, 2, \ldots, n \tag{34}$$

By using wedge product, any $k$-form $\alpha \in \Lambda^k$ on $M$ may be written locally at a point $q \in M$ in the differential basis $dq^j$ as

$$\alpha_m = \alpha_{i_1 \ldots i_k}(m) dq^{i_1} \wedge \cdots \wedge dq^{i_k} \in \Lambda^k \,, \quad i_1 < i_2 < \cdots < i_k \,, \tag{35}$$

where the sum over repeated indices is ordered, so that it must be taken over all $i_j$ satisfying $i_1 < i_2 < \cdots < i_k$. Roughly speaking differential forms $\Lambda^k$ are objects that can be integrated. As we shall see, vector fields also act on differential forms in interesting ways.

- Pull-backs of other differential forms may be built up from their basis elements, the $dq^{i_k}$. By equation (33),

**Theorem 19** (Pull-back of a wedge product)**.** *The pull-back of a wedge product of two differential forms is the wedge product of their pull-backs:*

$$\phi_t^*(\alpha \wedge \beta) = \phi_t^* \alpha \wedge \phi_t^* \beta \,. \tag{36}$$

### 6.3 Lie derivatives

**Definition 20** (Lie derivative of a differential $k$-form)**.** *The **Lie derivative** of a differential $k$-form $\Lambda^k$ by a vector field $Q \in \mathfrak{X}$ is defined by linearising its flow $\phi_t$ around the identity $t = 0$,*

$$\pounds_Q \Lambda^k = \frac{d}{dt}\Big|_{t=0} \phi_t^* \Lambda^k \quad \textit{maps} \quad \mathfrak{X} \times \Lambda^k \mapsto \Lambda^k \,.$$

Hence, by equation (36), the Lie derivative satisfies the product rule for the wedge product.

**Corollary 21** (Product rule for the Lie derivative of a wedge product)**.**

$$\pounds_Q(\alpha \wedge \beta) = \pounds_Q \alpha \wedge \beta + \alpha \wedge \pounds_Q \beta \,. \tag{37}$$

- Pullbacks of vector fields lead to Lie derivative expressions, too.

**Definition 22** (Lie derivative of a vector field)**.** *The **Lie derivative** of a vector field $Y \in \mathfrak{X}$ by another vector field $X \in \mathfrak{X}$ is defined by linearising the flow $\phi_t$ of $X$ around the identity $t = 0$,*

$$\pounds_X Y = \frac{d}{dt}\Big|_{t=0} \phi_t^* Y \quad \textit{maps} \quad \pounds_X \in \mathfrak{X} \mapsto \mathfrak{X} \,.$$

**Theorem 23.** *The Lie derivative $\pounds_X Y$ of a vector field $Y$ by a vector field $X$ satisfies*

$$\pounds_X Y = \frac{d}{dt}\Big|_{t=0} \phi_t^* Y = [X,\, Y] \,, \tag{38}$$

*where $[X,\, Y] = XY - YX$ is the commutator of the vector fields $X$ and $Y$.*

*Proof.* Denote the vector fields in components as

$$X = X^i(q)\frac{\partial}{\partial q^i} = \frac{d}{dt}\bigg|_{t=0} \phi_t^* \quad \text{and} \quad Y = Y^j(q)\frac{\partial}{\partial q^j}\,.$$

Then, by the pull-back relation (30) a direct computation yields, on using the matrix identity $dM^{-1} = -\,M^{-1}dM M^{-1}$,

$$\begin{aligned}
\pounds_X Y &= \frac{d}{dt}\bigg|_{t=0} \phi_t^* Y = \frac{d}{dt}\bigg|_{t=0} \left( Y^j(\phi_t q)\frac{\partial}{\partial(\phi_t q)^j}\right) \\
&= \frac{d}{dt}\bigg|_{t=0} \left( Y^j(\phi_t q) \left[\frac{\partial(\phi_t q)}{\partial q}^{-1}\right]^k_j \frac{\partial}{\partial q^k}\right) \\
&= \left( X^j \frac{\partial Y^k}{\partial q^j} - Y^j \frac{\partial X^k}{\partial q^j}\right) \frac{\partial}{\partial q^k} \\
&= [X,\, Y]\,.
\end{aligned}$$

□

**Corollary 24.** *The Lie derivative of the relation (31) for the pull-back of the commutator* $c_t^*\big[Y,\, Z\big] = \big[c_t^* Y,\, c_t^* Z\big]$ *yields the* ***Jacobi condition*** *for the vector fields to form an algebra.*

*Proof.* By the product rule and the definition of the Lie bracket (38) we have

$$\frac{d}{dt}\bigg|_{t=0} \phi_t^*\big[Y,\, Z\big] = \big[X,\, \big[Y,\, Z\big]\big] = \big[[X, Y],\, Z\big] + \big[Y,\, [X, Z]\big] = \frac{d}{dt}\bigg|_{t=0} \big[\phi_t^* Y,\, \phi_t^* Z\big]$$

This is the ***Jacobi identity*** for vector fields. □

Use the hat map and the relation $R_t(\mathbf{x} \times \mathbf{y}) = R_t\mathbf{x} \times R_t\mathbf{y}$ to show that the same argument gives the Jacobi identity for the cross product of vectors in $\mathbb{R}^3$, when $\phi_t^*$ is a rotation.

### 6.4 Contraction

**Definition 25** (Contraction)**.** *In exterior calculus, the operation of* ***contraction*** *denoted as* $\lrcorner$ *introduces a pairing between vector fields and differential forms. Contraction is also called* ***substitution*** *of a vector field into a differential form. For basis elements in phase space, contraction defines* ***duality relations****,*

$$\partial_q \lrcorner\, dq = 1 = \partial_p \lrcorner\, dp\,, \quad \textit{and} \quad \partial_q \lrcorner\, dp = 0 = \partial_p \lrcorner\, dq\,, \tag{39}$$

*so that differential forms are linear functions of vector fields. A* ***Hamiltonian vector field****,*

$$X_H = \dot{q}\frac{\partial}{\partial q} + \dot{p}\frac{\partial}{\partial p} = H_p\partial_q - H_q\partial_p = \{\,\cdot\,,\,H\,\}\,, \tag{40}$$

*satisfies the intriguing linear functional relations with the basis elements in phase space,*

$$X_H \lrcorner\, dq = H_p \quad \textit{and} \quad X_H \lrcorner\, dp = -\,H_q\,. \tag{41}$$

**Definition 26** (Contraction rules with higher forms)**.** *The rule for contraction or substitution of a vector field into a differential form is to sum the substitutions of $X_H$ over the permutations of the factors in the differential form that bring the corresponding dual basis element into its leftmost position. For example, substitution of the Hamiltonian vector field $X_H$ into the symplectic form $\omega = dq \wedge dp$ yields*

$$X_H \lrcorner\, \omega = X_H \lrcorner\, (dq \wedge dp) = (X_H \lrcorner\, dq)\, dp - (X_H \lrcorner\, dp)\, dq\,.$$

*In this example,* $X_H \,\lrcorner\, dq = H_p$ *and* $X_H \,\lrcorner\, dp = -\,H_q$*, so*

$$X_H \,\lrcorner\, \omega = H_p dp + H_q dq = dH \,,$$

*which follows from the duality relations (39).*

This calculation has proved the following.

**Theorem 27** (Hamiltonian vector field)**.** *The Hamiltonian vector field* $X_H = \{\,\cdot\,,\,H\,\}$ *satisfies*

$$X_H \,\lrcorner\, \omega \;=\; dH \quad \textit{with} \quad \omega = dq \wedge dp \,. \tag{42}$$

**Remark 28.**
The purely geometric nature of relation (42) argues for it to be taken as the definition of a Hamiltonian vector field.

**Lemma 29.** $d^2 = 0$ *for smooth phase-space functions.*

*Proof.* For any smooth phase-space function $H(q,p)$, one computes

$$dH = H_q dq + H_p dp$$

and taking the second exterior derivative yields

$$\begin{aligned} d^2 H &= H_{qp}\, dp \wedge dq + H_{pq}\, dq \wedge dp \\ &= (H_{pq} - H_{qp})\, dq \wedge dp = 0 \,. \end{aligned}$$

□

Relation (42) also implies the following.

**Corollary 30** (Poincaré's theorem)**.** *The flow of $X_H$ preserves the exact two-form $\omega$ for any Hamiltonian $H$.*

*Proof.* Preservation of $\omega$ may be verified first by a formal calculation using (42). Along

$$X_H = (dq/dt, dp/dt) = (\dot{q}, \dot{p}) = (H_p, -H_q)\,,$$

for a solution of Hamilton's equations, we have

$$\begin{aligned}
\pounds_{X_H}\omega &= \pounds_{X_H}(dq \wedge dp) \\
&= \frac{d}{dt}\Big|_{t=0} g_t^*(dq \wedge dp) \\
&= \frac{d}{dt}\Big|_{t=0} (g_t^* dq \wedge g_t^* dp) \\
&= d\dot{q} \,\wedge\, dp + dq \,\wedge\, d\dot{p} \\
&= dH_p \,\wedge\, dp - dq \,\wedge\, dH_q \\
&= d(H_p\, dp + H_q\, dq) \\
&= d(X_H \,\lrcorner\, \omega) \\
&= d(dH) = 0\,.
\end{aligned}$$

The first two steps use the product rule for Lie derivatives of differential forms

$$\begin{aligned}
\pounds_{X_H}(dq \wedge dp) &= \frac{d}{dt}\Big|_{t=0} g_t^*(dq \wedge dp) = \frac{d}{dt}\Big|_{t=0} (g_t^* dq \wedge g_t^* dp) \\
&= \Big[\frac{d}{dt} g_t^* dq \wedge g_t^* dp + g_t^* dq \wedge \frac{d}{dt} g_t^* dp\Big]_{t=0} \\
&= \pounds_{X_H} dq \wedge dp + dq \wedge \pounds_{X_H} dp
\end{aligned} \tag{43}$$

and the third-to-the-last and last steps use the property of the exterior derivative $d$ that $d^2 = 0$ for continuous forms. The latter is due to the equality of cross derivatives $H_{pq} = H_{qp}$ and antisymmetry of the wedge product $dq \,\wedge\, dp = -\, dp \,\wedge\, dq$. $\square$

**Definition 31** (Symplectic flow)**.** *A flow is* ***symplectic*** *if it preserves the phase-space area or symplectic two-form,* $\omega = dq \wedge dp$.

According to this definition, Corollary 30 may be simply re-stated as

**Corollary 32** (Poincaré's theorem)**.** *The flow of a Hamiltonian vector field is symplectic.*

**Definition 33** (Canonical transformations)**.** *A smooth invertible map $g$ of the phase space $T^*M$ is called a* ***canonical transformation*** *if it preserves the canonical symplectic form $\omega$ on $T^*M$, i.e., $g^*\omega = \omega$, where $g^*\omega$ denotes the pull-back of $\omega$ under the map $g$.*

**Remark 34** (Criterion for a canonical transformation)**.**
Suppose in original coordinates $(p, q)$ the symplectic form is expressed as $\omega = dq \wedge dp$. A transformation $g : T^*M \mapsto T^*M$ written as $(Q, P) = (Q(p, q), P(p, q)$ is canonical if the direct computation shows that $dQ \wedge dP = g^*(dq \wedge dp) = c\, dq \wedge dp$, up to a constant factor $c$. (Such a constant factor $c$ is unimportant, since it may be absorbed into the units of time in Hamilton's canonical equations.)

**Remark 35.**
By Corollary 32 (Poincaré's Theorem), the Hamiltonian phase flow $g_t$ is a one-parameter group of canonical transformations.

**Theorem 36** (Preservation of Hamiltonian form)**.** *Canonical transformations preserve the Hamiltonian form.*

*Proof.* The coordinate-free relation $X_H \,\lrcorner\, \omega = dH$ with $\omega = dq \wedge dp$ keeps its form if

$$dQ \wedge dP = g^*(dq \wedge dp) = c\, dq \wedge dp\,,$$

up to the constant factor $c$. Hence, Hamilton's equations re-emerge in canonical form in the new coordinates, up to a rescaling by $c$ which may be absorbed into the units of time. □

### 6.5 Summary of natural operations on differential forms

Besides the wedge product, three basic operations are commonly applied to differential forms. These are contraction, exterior derivative and Lie derivative.

- Contraction $\rfloor$ with a vector field $X$ lowers the degree:

$$X \rfloor \Lambda^k \mapsto \Lambda^{k-1}\,.$$

- Exterior derivative $d$ raises the degree:

$$d\Lambda^k \mapsto \Lambda^{k+1}\,.$$

- Lie derivative $\pounds_X$ by vector field $X$ preserves the degree:

$$\pounds_X \Lambda^k \mapsto \Lambda^k\,, \quad \text{where} \quad \pounds_X \Lambda^k = \frac{d}{dt}\bigg|_{t=0} \phi_t^* \Lambda^k\,,$$

  in which $\phi_t$ is the flow of the vector field $X$. In analogy with fluids one may write $\pounds_X \Lambda^k = \frac{d}{dt}\Lambda^k$ along $\frac{dx}{dt} = X$.

- Lie derivative $\pounds_X$ satisfies ***Cartan's formula***: (The proof is a direct calculation.)

$$\pounds_X \alpha = X \rfloor d\alpha + d(X \rfloor \alpha) \quad \text{for} \quad \alpha = \alpha_{i_1 \dots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \in \Lambda^k\,.$$

**Remark 37.**
Note also that the Lie derivative commutes with the exterior derivative. That is,

$$d(\pounds_X \alpha) = \pounds_X d\alpha\,, \quad \text{for} \quad \alpha \in \Lambda^k(M) \quad \text{and} \quad X \in \mathfrak{X}(M)\,.$$

## 6.6 Examples of contraction, or interior product

**Definition 38** (Contraction, or interior product)**.** *Let $\alpha \in \Lambda^k$ be a $k$-form on a manifold $M$,*

$$\alpha = \alpha_{i_1 \dots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \in \Lambda^k \,, \quad \textit{with} \quad i_1 < i_2 < \cdots < i_k \,,$$

*and let $X = X^j \partial_j$ be a vector field. The contraction or interior product $X \lrcorner\, \alpha$ of a vector field $X$ with a $k$-form $\alpha$ is defined by*

$$X \lrcorner\, \alpha = X^j \alpha_{j i_2 \dots i_k} dx^{i_2} \wedge \cdots \wedge dx^{i_k} \,. \tag{44}$$

*Note that*

$$\begin{aligned} X \lrcorner\, (Y \lrcorner\, \alpha) &= X^l Y^m \alpha_{m l i_3 \dots i_k} dx^{i_3} \wedge \cdots \wedge dx^{i_k} \\ &= - Y \lrcorner\, (X \lrcorner\, \alpha) \,, \end{aligned}$$

*by antisymmetry of $\alpha_{m l i_3 \dots i_k}$, particularly in its first two indices.*

**Remark 39** (Examples of contraction)**.**

1. A mnemonic device for keeping track of signs in contraction or substitution of a vector field into a differential form is to sum the substitutions of $X = X^j \partial_j$ over the permutations that bring the corresponding dual basis element into the leftmost position in the $k$-form $\alpha$. For example, in two dimensions, contraction of the vector field $X = X^j \partial_j = X^1 \partial_1 + X^2 \partial_2$ into the two-form $\alpha = \alpha_{jk} dx^j \wedge dx^k$ with $\alpha_{21} = -\, \alpha_{12}$ yields

$$X \lrcorner\, \alpha = X^j \alpha_{j i_2} dx^{i_2} = X^1 \alpha_{12} dx^2 + X^2 \alpha_{21} dx^1 \,.$$

Likewise, in three dimensions, contraction of the vector field $X = X^1 \partial_1 + X^2 \partial_2 + X^3 \partial_3$ into the three-form

$\alpha = \alpha_{123} dx^1 \wedge dx^2 \wedge dx^3$ with $\alpha_{213} = -\alpha_{123}$, etc. yields

$$
\begin{aligned}
X \,\lrcorner\, \alpha &= X^1 \alpha_{123} dx^2 \wedge dx^3 + \text{cyclic permutations} \\
&= X^j \alpha_{j i_2 i_3} dx^{i_2} \wedge dx^{i_3} \quad \text{with} \quad i_2 < i_3 \,.
\end{aligned}
$$

2. The rule for contraction of a vector field with a differential form develops from the relation

$$\partial_j \,\lrcorner\, dx^k = \delta_j^k \,,$$

in the coordinate basis $e_j = \partial_j := \partial/\partial x^j$ and its dual basis $e^k = dx^k$. Contraction of a vector field with a one-form yields the dot product, or inner product, between a covariant vector and a contravariant vector is given by

$$X^j \partial_j \,\lrcorner\, v_k dx^k = v_k \delta_j^k X^j = v_j X^j \,,$$

or, in vector notation,

$$X \,\lrcorner\, \mathbf{v} \cdot d\mathbf{x} = \mathbf{v} \cdot \mathbf{X} \,.$$

This is the ***dot product of vectors*** $\mathbf{v}$ and $\mathbf{X}$.

3. By the linearity of its definition (44), contraction of a vector field $X$ with a differential $k$-form $\alpha$ satisfies

$$(hX) \,\lrcorner\, \alpha = h(X \,\lrcorner\, \alpha) = X \,\lrcorner\, h\alpha \,.$$

Our previous calculations for two-forms and three-forms provide the following additional expressions for contraction of a vector field with a differential form, which may be written in vector notation as:

$$
\begin{aligned}
X \lrcorner\, \mathbf{B} \cdot d\mathbf{S} &= -\,\mathbf{X} \times \mathbf{B} \cdot d\mathbf{x}\,, \\
X \lrcorner\, d^{\,3}x &= \mathbf{X} \cdot d\mathbf{S}\,, \\
d(X \lrcorner\, d^{\,3}x) &= d(\mathbf{X} \cdot d\mathbf{S}) = (\mathrm{div}\,\mathbf{X})\, d^{\,3}x\,.
\end{aligned}
$$

**Remark 40** (Physical examples of contraction)**.**
The first of these contraction relations represents the Lorentz, or Coriolis force, when $\mathbf{X}$ is particle velocity and $\mathbf{B}$ is either magnetic field, or rotation rate, respectively. The second contraction relation is the flux of the vector $\mathbf{X}$ through a surface element. The third is the exterior derivative of the second, thereby yielding the divergence of the vector $\mathbf{X}$.

**Exercise.** Show that

$$X \lrcorner\, (X \lrcorner\, \mathbf{B} \cdot d\mathbf{S}) = 0$$

and

$$(X \lrcorner\, \mathbf{B} \cdot d\mathbf{S}) \wedge \mathbf{B} \cdot d\mathbf{S} = 0\,,$$

for any vector field $X$ and two-form $\mathbf{B} \cdot d\mathbf{S}$. ★

**Proposition 41** (Contracting through wedge product)**.** *Let $\alpha$ be a $k$-form and $\beta$ be a one-form on a manifold $M$ and let $X = X^j\partial_j$ be a vector field. Then the contraction of $X$ through the wedge product $\alpha \wedge \beta$ satisfies*

$$X \lrcorner\, (\alpha \wedge \beta) = (X \lrcorner\, \alpha) \wedge \beta + (-1)^k \alpha \wedge (X \lrcorner\, \beta)\,. \tag{45}$$

*Proof.* The proof is a straightforward calculation using the definition of contraction. The exponent $k$ in the factor $(-1)^k$ counts the number of exchanges needed to get the one-form $\beta$ to the left most position through the $k$-form $\alpha$. □

**Proposition 42.** *[Contraction is natural under pull-back]*
*That is,*

$$\phi^*(X(m) \,\lrcorner\, \alpha) = X(\phi(m)) \,\lrcorner\, \phi^*\alpha = \phi^* X \,\lrcorner\, \phi^*\alpha \,. \tag{46}$$

*Proof.* Direct verification using the relation between pull-back of forms and push-forward of vector fields. Note the implication, $\pounds_X(Y \,\lrcorner\, \alpha) = [X\,,\, Y] \,\lrcorner\, \alpha + Y \,\lrcorner\, (\pounds_X \alpha)$. □

**Definition 43** (Alternative notations for contraction)**.** *Besides the hook notation with* $\lrcorner$*, one also finds in the literature the following two alternative notations for contraction of a vector field* $X$ *with* $k$*-form* $\alpha \in \Lambda^k$ *on a manifold* $M$*:*

$$X \,\lrcorner\, \alpha = i_X \alpha = \alpha(X, \underbrace{\,\cdot\,, \cdot\,, \ldots\,, \cdot\,}_{k-1 \ slots}) \in \Lambda^{k-1} \,. \tag{47}$$

*In the last alternative, one leaves a dot* $(\,\cdot\,)$ *in each remaining slot of the form that results after contraction. For example, contraction of the Hamiltonian vector field* $X_H = \{\,\cdot\,,\, H\}$ *with the symplectic two-form* $\omega \in \Lambda^2$ *produces the one-form*

$$X_H \,\lrcorner\, \omega = \omega(X_H,\, \cdot\,) = -\,\omega(\,\cdot\,,\, X_H) = dH \,.$$

In this alternative notation, the proof of formula (46) in Proposition 42 may be written, as follows.

*Proof.* Since forms are multilinear maps to the real numbers, one may define the pull-back of a $k$-form, $\alpha$, by

$$\phi^*\alpha(X_1, X_2, ...) := \alpha(\phi_* X, \phi_* X_2, ...) \,.$$

Therefore, we are able to use the following proof.

$$
\begin{aligned}
\phi^* X \,\lrcorner\, \phi^* \alpha(X_1, X_2, ....) &= \phi^* \alpha(\phi^* X, X_1, X_2, ...) \\
&= \alpha(\phi_* \phi^* X, \phi_* X_1, \phi_* X_2, ...) \\
&= \alpha(X, \phi_* X_1, \phi_* X_2, ...) \\
&= (X \,\lrcorner\, \alpha)(\phi_* X_1, \phi_* X_2, ...) \\
&= \phi^* (X \,\lrcorner\, \alpha)(X_1, X_2, ...)
\end{aligned}
$$

Now, if we allow $X_1, X_2, \dots$ to be arbitrary, then formula (46) in Proposition 42 follows. □

**Proposition 44** (Hamiltonian vector field definitions)**.** *The two definitions of Hamiltonian vector field $X_H$*

$$
dH = X_H \,\lrcorner\, \omega \quad \textit{and} \quad X_H = \{\,\cdot\,,\, H\}
$$

*are equivalent.*

*Proof.* The symplectic Poisson bracket satisfies $\{F, H\} = \omega(X_F, X_H)$, because

$$
\omega(X_F, X_H) := X_H \,\lrcorner\, X_F \,\lrcorner\, \omega = X_H \,\lrcorner\, dF = -X_F \,\lrcorner\, dH = \{F,\, H\}\,.
$$

□

**Remark 45.**
The relation $\{F,\, H\} = \omega(X_F, X_H)$ means that the Hamiltonian vector field defined via the symplectic form coincides exactly with the Hamiltonian vector field defined using the Poisson bracket.

## 6.7 Exercises in exterior calculus operations

**Vector notation for differential basis elements** One denotes differential basis elements $dx^i$ and $dS_i = \frac{1}{2}\epsilon_{ijk}dx^j \wedge dx^k$, for $i, j, k = 1, 2, 3$ in vector notation as

$$
\begin{aligned}
d\mathbf{x} &:= (dx^1, dx^2, dx^3)\,,\\
d\mathbf{S} &= (dS_1, dS_2, dS_3)\\
&:= (dx^2 \wedge dx^3,\ dx^3 \wedge dx^1,\ dx^1 \wedge dx^2)\,,\\
dS_i &:= \frac{1}{2}\epsilon_{ijk}dx^j \wedge dx^k\,,\\
d^{\,3}x &= dVol := dx^1 \wedge dx^2 \wedge dx^3\\
&= \frac{1}{6}\epsilon_{ijk}dx^i \wedge dx^j \wedge dx^k\,.
\end{aligned}
$$

**Exercise. (Vector calculus operations)** Show that contraction $\;\lrcorner\; : \mathfrak{X} \times \Lambda^k \to \Lambda^{k-1}$ of the vector field $X = X^j\partial_j =: \mathbf{X}\cdot\nabla$ with the differential basis elements $d\mathbf{x}$, $d\mathbf{S}$ and $d^{\,3}x$ recovers the following familiar operations among vectors:

$$
\begin{aligned}
X \lrcorner\, d\mathbf{x} &= \mathbf{X}\,,\\
X \lrcorner\, d\mathbf{S} &= \mathbf{X} \times d\mathbf{x}\,,\\
(\text{or,}\quad X \lrcorner\, dS_i &= \epsilon_{ijk}X^j dx^k)\\
Y \lrcorner\, X \lrcorner\, d\mathbf{S} &= \mathbf{X} \times \mathbf{Y}\,,\\
X \lrcorner\, d^{\,3}x &= \mathbf{X}\cdot d\mathbf{S} = X^k dS_k\,,\\
Y \lrcorner\, X \lrcorner\, d^{\,3}x &= \mathbf{X} \times \mathbf{Y}\cdot d\mathbf{x} = \epsilon_{ijk}X^iY^jdx^k\,,\\
Z \lrcorner\, Y \lrcorner\, X \lrcorner\, d^{\,3}x &= \mathbf{X} \times \mathbf{Y}\cdot\mathbf{Z}\,.
\end{aligned}
$$

★

**Exercise. (Exterior derivatives in vector notation)** Show that the exterior derivative and wedge product satisfy the following relations in components and in three-dimensional vector notation:

$$
\begin{aligned}
df &= f_{,j}\, dx^j =: \nabla f \cdot d\mathbf{x}\,, \\
0 = d^2 f &= f_{,jk}\, dx^k \wedge dx^j\,, \\
df \wedge dg &= f_{,j}\, dx^j \wedge g_{,k}\, dx^k \\
&=: (\nabla f \times \nabla g) \cdot d\mathbf{S}\,, \\
df \wedge dg \wedge dh &= f_{,j}\, dx^j \wedge g_{,k}\, dx^k \wedge h_{,l}\, dx^l \\
&=: (\nabla f \cdot \nabla g \times \nabla h)\, d^{\,3}x\,.
\end{aligned}
$$

★

**Exercise. (Vector calculus formulas)** Show that the exterior derivative yields the following vector calculus formulas:

$$
\begin{aligned}
df &= \nabla f \cdot d\mathbf{x}\,, \\
d(\mathbf{v} \cdot d\mathbf{x}) &= (\operatorname{curl} \mathbf{v}) \cdot d\mathbf{S}\,, \\
d(\mathbf{A} \cdot d\mathbf{S}) &= (\operatorname{div} \mathbf{A})\, d^{\,3}x\,.
\end{aligned}
$$

The compatibility condition $d^2 = 0$ is written for these forms as

$$
\begin{aligned}
0 = d^2 f = d(\nabla f \cdot d\mathbf{x}) &= (\operatorname{curl} \operatorname{grad} f) \cdot d\mathbf{S}\,, \\
0 = d^2(\mathbf{v} \cdot d\mathbf{x}) = d\big((\operatorname{curl} \mathbf{v}) \cdot d\mathbf{S}\big) &= (\operatorname{div} \operatorname{curl} \mathbf{v})\, d^{\,3}x\,.
\end{aligned}
$$

The product rule is written for these forms as

$$
\begin{aligned}
d\big(f(\mathbf{A} \cdot d\mathbf{x})\big) &= df \wedge \mathbf{A} \cdot d\mathbf{x} + f \operatorname{curl} \mathbf{A} \cdot d\mathbf{S} \\
&= \big(\nabla f \times \mathbf{A} + f \operatorname{curl} \mathbf{A}\big) \cdot d\mathbf{S} \\
&= \operatorname{curl}(f\mathbf{A}) \cdot d\mathbf{S}\,,
\end{aligned}
$$

$$\begin{aligned}
d\big((\mathbf{A}\cdot d\mathbf{x})\wedge(\mathbf{B}\cdot d\mathbf{x})\big) &= (\mathrm{curl}\,\mathbf{A})\cdot d\mathbf{S}\wedge\mathbf{B}\cdot d\mathbf{x} - \mathbf{A}\cdot d\mathbf{x}\wedge(\mathrm{curl}\,\mathbf{B})\cdot d\mathbf{S}\\
&= \big(\mathbf{B}\cdot\mathrm{curl}\,\mathbf{A} - \mathbf{A}\cdot\mathrm{curl}\,\mathbf{B}\big)d^{\,3}x\\
&= d\big((\mathbf{A}\times\mathbf{B})\cdot d\mathbf{S}\big)\\
&= \mathrm{div}(\mathbf{A}\times\mathbf{B})\,d^{\,3}x\,.
\end{aligned}$$

These calculations yield familiar formulas from vector calculus for quantities curl(grad), div(curl), curl($f\mathbf{A}$) and div($\mathbf{A}\times\mathbf{B}$). ★

## 6.8 Integral calculus formulas

**Exercise. (Integral calculus formulas)** Show that the Stokes' theorem for the vector calculus formulas yields the following familiar results in $\mathbb{R}^3$:

- The ***fundamental theorem of calculus***, upon integrating $df$ along a curve in $\mathbb{R}^3$ starting at point $a$ and ending at point $b$:

$$\int_a^b df \;=\; \int_a^b \nabla f\cdot d\mathbf{x} = f(b) - f(a)\,.$$

- The ***classical Stokes theorem***, for a compact surface $S$ with boundary $\partial S$:

$$\int_S(\mathrm{curl}\,\mathbf{v})\cdot d\mathbf{S} \;=\; \oint_{\partial S}\mathbf{v}\cdot d\mathbf{x}\,.$$

  (For a planar surface $S\in\mathbb{R}^2$, this is ***Green's theorem***.)

- The ***Gauss divergence theorem***, for a compact spatial domain $D$ with boundary $\partial D$:

$$\int_D(\mathrm{div}\,\mathbf{A})\,d^{\,3}x \;=\; \oint_{\partial D}\mathbf{A}\cdot d\mathbf{S}\,.$$

★

These exercises illustrate the following,

**Theorem 46** (Stokes' theorem)**.** *Suppose $M$ is a compact oriented $k$-dimensional manifold with boundary $\partial M$ and $\alpha$ is a smooth $(k-1)$-form on $M$. Then*

$$\int_M d\alpha = \oint_{\partial M} \alpha \,.$$

## 6.9 Summary and an exercise

### Summary

The pull-back $\phi_t^*$ of a smooth flow $\phi_t$ generated by a smooth vector field $X$ on a smooth manifold $M$ commutes with the exterior derivative $d$, wedge product $\wedge$ and contraction $\,\lrcorner\,$.

That is, for $k$-forms $\alpha$, $\beta \in \Lambda^k(M)$, and $m \in M$, the pull-back $\phi_t^*$ satisfies

$$\begin{aligned} d(\phi_t^*\alpha) &= \phi_t^* d\alpha \,, \\ \phi_t^*(\alpha \wedge \beta) &= \phi_t^*\alpha \wedge \phi_t^*\beta \,, \\ \phi_t^*(X \,\lrcorner\, \alpha) &= \phi_t^* X \,\lrcorner\, \phi_t^*\alpha \,. \end{aligned}$$

In addition, the Lie derivative $\pounds_X\alpha$ of a $k$-form $\alpha \in \Lambda^k(M)$ by the vector field $X$ tangent to the flow $\phi_t$ on $M$ is defined either dynamically or geometrically (by Cartan's formula) as

$$\pounds_X \alpha \;=\; \frac{d}{dt}\Big|_{t=0} (\phi_t^* \alpha) = X \mathbin{\lrcorner} d\alpha + d(X \mathbin{\lrcorner} \alpha), \tag{48}$$

in which the last equality is Cartan's geometric formula in (57) for the Lie derivative.

**Definition 47. (Lie derivative pull-back formula)**
*The tangent to the pull-back $\phi_t^*\alpha$ of a differential form $\alpha$ is the pull-back of the Lie derivative of $\alpha$ wrt the vector field $X$ that generates the flow, $\phi_t$:*

$$\frac{d}{dt}(\phi_t^*\alpha) = \phi_t^*\big(\pounds_X \alpha\big)\,.$$

*Likewise, for the push-forward, which is the pull-back by the inverse, we have*

$$\frac{d}{dt}((\phi_t^{-1})^*\alpha) = -(\phi_t^{-1})^*\big(\pounds_X \alpha\big)\,.$$

**Definition 48. (Advected quantity)**
*A quantity which is invariant along a flow trajectory satisfies $\alpha_0(x_0) = \alpha_t(x_t) = (\phi_t^*\alpha_t)(x_0)$, so that*

$$0 = \frac{d}{dt}\alpha_0(x_0) = \frac{d}{dt}(\phi_t^*\alpha_t)(x_0) = \phi_t^*(\partial_t + \pounds_X)\alpha_t(x_0) = (\partial_t + \pounds_X)\alpha_t(x_t)$$

*Or vice versa*

$$\alpha_t(x_t) = (\alpha_0 \circ \phi_t^{-1})(x_t) = ((\phi_t)_*\alpha_0)(x_t)$$

*satisfies*

$$\frac{d}{dt}\alpha_t(x_t) = \frac{d}{dt}(\phi_t)_*\alpha_0 = -\pounds_X\alpha_t\,.$$

**Exercise.**

(a) Verify the formula $[X\,,\,Y]\,\lrcorner\,\alpha = \pounds_X(Y\,\lrcorner\,\alpha) - Y\,\lrcorner\,(\pounds_X\alpha)$.

(b) Use (a) to verify $\pounds_{[X\,,Y]}\alpha = \pounds_X\pounds_Y\alpha - \pounds_Y\pounds_X\alpha$.

(c) Use (b) to verify the Jacobi identity.

(d) Use (c) to verify that the divergence-free vector fields are closed under commutation.

(e) For a top-form $\alpha$ show divergence-free vector fields that

$$[X\,,\,Y]\,\lrcorner\,\alpha = d\big(X\,\lrcorner\,(Y\,\lrcorner\,\alpha)\big)\,. \tag{49}$$

(f) Write the equivalent of equation (49) as a formula in vector calculus.

★

**Answer.**

(a) The required formula follows immediately from the product rule in (43) for the dynamical definition of the Lie derivative. Since pull-back commutes with contraction, insertion of a vector field into a $k$-form transforms under the flow $\phi_t$ of a smooth vector field $Y$ as

$$\phi_t^*(Y\,\lrcorner\,\alpha) = \phi_t^*Y\,\lrcorner\,\phi_t^*\alpha\,.$$

A direct computation using the dynamical definition of the Lie derivative $\pounds_Y\alpha = \frac{d}{dt}\big|_{t=0}(\phi_t^*\alpha)$, then yields

$$
\begin{aligned}
\frac{d}{dt}\Big|_{t=0}\phi_t^*\big(Y \lrcorner\, \alpha\big) \;=\; & \Big(\frac{d}{dt}\Big|_{t=0}\phi_t^*Y\Big) \lrcorner\, \alpha \\
& + Y \lrcorner \Big(\frac{d}{dt}\Big|_{t=0}\phi_t^*\alpha\Big) .
\end{aligned}
$$

Hence, we recognise that the desired formula is the ***product rule*** met earlier in equation (43):

$$
\pounds_X(Y \lrcorner\, \alpha) = (\pounds_X Y) \lrcorner\, \alpha + Y \lrcorner\, (\pounds_X\alpha) .
$$

(b) Insert $\pounds_X Y = [X\,,\,Y]$ into the product rule formula in part (a). Then

$$
[X\,,\,Y] \lrcorner\, \alpha = \pounds_X(Y \lrcorner\, \alpha) - Y \lrcorner\, (\pounds_X\alpha).
$$

Now use Cartan's formula in (57)

$$
\pounds_X\alpha \;=\; \frac{d}{dt}\Big|_{t=0}(\phi_t^*\alpha) = X \lrcorner\, d\alpha + d(X \lrcorner\, \alpha),
$$

to compute the required result, as

$$
\begin{aligned}
\pounds_{[X\,,\,Y]}\alpha &= d([X\,,\,Y] \lrcorner\, \alpha) + [X\,,\,Y] \lrcorner\, d\alpha \\
&= d\Big( \pounds_X(Y \lrcorner\, \alpha) - Y \lrcorner\, (\pounds_X\alpha) \Big) \\
&\quad + \pounds_X(Y \lrcorner\, d\alpha) - Y \lrcorner\, (\pounds_X d\alpha) \\
&= \pounds_X d(Y \lrcorner\, \alpha) - d(Y \lrcorner\, (\pounds_X\alpha) \\
&\quad + \pounds_X(Y \lrcorner\, d\alpha) - Y \lrcorner\, d(\pounds_X\alpha) \\
&= \pounds_X(\pounds_Y\alpha) - \pounds_Y(\pounds_X\alpha)\,.
\end{aligned}
$$

Can you think of an alternative proof based on the dynamical definition of the Lie derivative?

(c) Applying part (b), ($\pounds_{[X\,,\,Y]}\alpha = \pounds_X\pounds_Y\alpha - \pounds_Y\pounds_X\alpha$) to $\alpha = d^3x$ proves that $\pounds_{[X\,,\,Y]}d^3x = 0$; since both $\pounds_Y d^3x = 0 = \pounds_X d^3x$, because, e.g., $\pounds_Y d^3x = (\mathrm{div}Y)\, d^3x$.

(d) As a consequence of part (b),

$$
\begin{aligned}
\pounds_{[\,Z\,,\,[X\,,\,Y]\,]}\alpha &= \pounds_Z(\pounds_X\pounds_Y - \pounds_Y\pounds_X)\alpha - (\pounds_X\pounds_Y - \pounds_Y\pounds_X)\pounds_Z\alpha \\
&= \pounds_Z\pounds_X\pounds_Y\alpha - \pounds_Z\pounds_Y\pounds_X\alpha - \pounds_X\pounds_Y\pounds_Z\alpha + \pounds_Y\pounds_X\pounds_Z\alpha\,,
\end{aligned}
$$

and summing over cyclic permutations verifies that

$$
\pounds_{[\,Z\,,\,[X\,,\,Y]\,]}\,\alpha + \pounds_{[\,X\,,\,[Y\,,\,Z]\,]}\,\alpha + \pounds_{[\,Y\,,\,[Z\,,\,X]\,]}\,\alpha = 0\,.
$$

This is the ***Jacobi identity for the Lie derivative***.

(e) Substituting the relation $\pounds_X Y = [X,\, Y]$ into the product rule above in part (b) and rearranging yields

$$
[X,\, Y] \lrcorner\, \alpha = \pounds_X(Y \lrcorner\, \alpha) - Y \lrcorner\, (\pounds_X\alpha)\,, \tag{50}
$$

as required, for an arbitrary $k$-form $\alpha$.

From formula (50), we have

$$
\begin{aligned}
[X\,,\,Y] \lrcorner\, \alpha &= \pounds_X (Y \lrcorner\, \alpha) - Y \lrcorner\, (\pounds_X \alpha) \\
&= d\big(X \lrcorner\, (Y \lrcorner\, \alpha) + X \lrcorner\, d(Y \lrcorner\, \alpha)\big) - Y \lrcorner\, (\pounds_X \alpha) \\
&= d\big(X \lrcorner\, (Y \lrcorner\, \alpha)\big) + X \lrcorner\, (\pounds_Y \alpha - Y \lrcorner\, d\alpha) - Y \lrcorner\, (\pounds_X \alpha) \\
&= d\big(X \lrcorner\, (Y \lrcorner\, \alpha)\big) + X \lrcorner\, (\pounds_Y \alpha) - Y \lrcorner\, (\pounds_X \alpha) \\
[X\,,\,Y] \lrcorner\, \alpha &= d\big(X \lrcorner\, (Y \lrcorner\, \alpha)\big) + (\mathrm{div}\,\mathbf{Y}) X \lrcorner\, \alpha - (\mathrm{div}\,\mathbf{X}) Y \lrcorner\, \alpha .
\end{aligned}
\tag{51}
$$

The last two steps to obtain (51) follow, because $d\alpha = 0$ and $\pounds_X \alpha = (\mathrm{div}\,\mathbf{X})\,\alpha$ for a top-form $\alpha$.

For divergence-free vectors $\mathbf{X}$ and $\mathbf{Y}$, the last result takes the elegant form,

$$
[X\,,\,Y] \lrcorner\, \alpha = d\big(X \lrcorner\, (Y \lrcorner\, \alpha)\big),
\tag{52}
$$

when $\mathrm{div}\,\mathbf{X} = 0 = \mathrm{div}\,\mathbf{Y}$.

(f) The vector calculus formula to which equation (51) is equivalent may be found by writing its left and right sides in a coordinate basis, as

$$
\begin{aligned}
[X\,,\,Y] \lrcorner\, \alpha &= (\mathbf{X}\cdot\nabla\mathbf{Y} - \mathbf{Y}\cdot\nabla\mathbf{X})\cdot d\mathbf{S} \\
d\big(X \lrcorner\, (Y \lrcorner\, \alpha)\big) + X \lrcorner\, (\pounds_Y \alpha) - Y \lrcorner\, (\pounds_X \alpha) &= -\,\mathrm{curl}\,(\mathbf{X}\times\mathbf{Y})\cdot d\mathbf{S} + (\mathrm{div}\,\mathbf{Y})\,\mathbf{X}\cdot d\mathbf{S} - (\mathrm{div}\,\mathbf{X})\,\mathbf{Y}\cdot d\mathbf{S}
\end{aligned}
$$

Thus, equation (51) for a top-form $\alpha\, d^n x$ is equivalent to the well-known vector calculus identity

$$
(\mathbf{X}\cdot\nabla\mathbf{Y} - \mathbf{Y}\cdot\nabla\mathbf{X}) = -\,\mathrm{curl}\,(\mathbf{X}\times\mathbf{Y}) + (\mathrm{div}\,\mathbf{Y})\,\mathbf{X} - (\mathrm{div}\,\mathbf{X})\,\mathbf{Y}\,.
$$

▲

**Exercise.**

(a) Starting from

$$[u\,,\,v] \,\lrcorner\, \alpha = \pounds_u(v \,\lrcorner\, \alpha) - v \,\lrcorner\, (\pounds_u \alpha)$$

prove the following

$$\begin{aligned}
\pounds_u(v \,\lrcorner\, \alpha) - \pounds_v(u \,\lrcorner\, \alpha) &= 2[u,v] \,\lrcorner\, \alpha + v \,\lrcorner\, \pounds_u \alpha - u \,\lrcorner\, \pounds_v \alpha \\
&= [u,v] \,\lrcorner\, \alpha - u \,\lrcorner\, (v \,\lrcorner\, \alpha) + d(u \,\lrcorner\, (v \,\lrcorner\, \alpha))
\end{aligned}$$

(b) Evaluate the last equation for a $k$-form $\alpha$ with $k = 3, 2, 1$, in terms of vector calculus expressions.

★

**Answer.**

(a)

$$\begin{aligned}
[u,v] \,\lrcorner\, \alpha &= \pounds_u(v \,\lrcorner\, \alpha) - v \,\lrcorner\, \pounds_u \alpha \\
&= d(u \,\lrcorner\, (v \,\lrcorner\, \alpha)) + u \,\lrcorner\, d(v \,\lrcorner\, \alpha) - v \,\lrcorner\, \pounds_u \alpha \\
&= d(u \,\lrcorner\, (v \,\lrcorner\, \alpha)) + u \,\lrcorner\, (\pounds_v \alpha - v \,\lrcorner\, d\alpha) - v \,\lrcorner\, \pounds_u \alpha \\
&= d(u \,\lrcorner\, (v \,\lrcorner\, \alpha)) + u \,\lrcorner\, \pounds_v \alpha - u \,\lrcorner\, (v \,\lrcorner\, d\alpha) - v \,\lrcorner\, \pounds_u \alpha \\
[u,v] \,\lrcorner\, \alpha + u \,\lrcorner\, (v \,\lrcorner\, \alpha) &= d(u \,\lrcorner\, (v \,\lrcorner\, \alpha)) + (u \,\lrcorner\, \pounds_v \alpha - v \,\lrcorner\, \pounds_u \alpha)
\end{aligned}$$

$$
\begin{aligned}
u \,\lrcorner\, \pounds_v \alpha - v \,\lrcorner\, \pounds_u \alpha &= [u,v] \,\lrcorner\, \alpha + u \,\lrcorner\, (v \,\lrcorner\, \alpha) - d(u \,\lrcorner\, (v \,\lrcorner\, \alpha)) \\
v \,\lrcorner\, \pounds_u \alpha - u \,\lrcorner\, \pounds_v \alpha &= [v,u] \,\lrcorner\, \alpha + v \,\lrcorner\, (u \,\lrcorner\, \alpha) - d(v \,\lrcorner\, (u \,\lrcorner\, \alpha)) \\
&= -[u,v] \,\lrcorner\, \alpha - u \,\lrcorner\, (v \,\lrcorner\, \alpha) + d(u \,\lrcorner\, (v \,\lrcorner\, \alpha))
\end{aligned}
$$

$$
\begin{aligned}
\pounds_u (v \,\lrcorner\, \alpha) - \pounds_v (u \,\lrcorner\, \alpha) &= 2[u,v] \,\lrcorner\, \alpha + v \,\lrcorner\, \pounds_u \alpha - u \,\lrcorner\, \pounds_v \alpha \\
&= [u,v] \,\lrcorner\, \alpha - u \,\lrcorner\, (v \,\lrcorner\, d\alpha) + d(u \,\lrcorner\, (v \,\lrcorner\, \alpha))
\end{aligned}
$$

(b) For a 3-form $\alpha = d^3x$ (top form in 3D) one again finds the vector calculus identity in the previous exercise which is antisymmetric under exchange of $\mathbf{u}$ and $\mathbf{v}$,

$$
(\mathrm{div}\mathbf{v})\mathbf{u} - (\mathrm{div}\mathbf{u})\mathbf{v} - \mathrm{curl}(\mathbf{u} \times \mathbf{v}) = (\mathbf{u} \cdot \nabla)\mathbf{v} - (\mathbf{v} \cdot \nabla)\mathbf{u} =: [\mathbf{u}, \mathbf{v}]
$$

For a 2-form $\alpha = \boldsymbol{\alpha} \cdot d\mathbf{S}$ in 3D one finds the vector calculus identity

$$
-\mathbf{u} \times \mathrm{curl}(\boldsymbol{\alpha} \times \mathbf{v}) + \mathbf{v} \times \mathrm{curl}(\boldsymbol{\alpha} \times \mathbf{u}) = \boldsymbol{\alpha} \times [\mathbf{u}, \mathbf{v}] + (\mathrm{div}\boldsymbol{\alpha})(\mathbf{u} \times \mathbf{v}) + \nabla(\boldsymbol{\alpha} \cdot \mathbf{u} \times \mathbf{v})
$$

in which we denote as in the previous vector calculus identity

$$
[\mathbf{u}, \mathbf{v}] := (\mathbf{u} \cdot \nabla)\mathbf{v} - (\mathbf{v} \cdot \nabla)\mathbf{u} = -\,\mathrm{curl}(\mathbf{u} \times \mathbf{v}) - (\mathrm{div}\mathbf{u})\mathbf{v} + (\mathrm{div}\mathbf{v})\mathbf{u}
$$

After the substitution of this expression for $[\mathbf{u}, \mathbf{v}]$ obtained in the case of the 3-form $\alpha = d^3x$, one sees that the vector calculus identity for a 2-form $\alpha = \boldsymbol{\alpha} \cdot d\mathbf{S}$ has cyclic permutation symmetry

$$
\mathbf{u} \times \mathrm{curl}(\mathbf{v} \times \boldsymbol{\alpha}) + \mathbf{v} \times \mathrm{curl}(\boldsymbol{\alpha} \times \mathbf{u}) + \boldsymbol{\alpha} \times \mathrm{curl}(\mathbf{u} \times \mathbf{v}) = (\mathrm{div}\mathbf{u})(\mathbf{v} \times \boldsymbol{\alpha}) + (\mathrm{div}\mathbf{v})(\boldsymbol{\alpha} \times \mathbf{u}) + (\mathrm{div}\boldsymbol{\alpha})(\mathbf{u} \times \mathbf{v}) + \nabla(\boldsymbol{\alpha} \cdot \mathbf{u} \times \mathbf{v})
$$

Also, in the divergence-free case this reduces to

$$
\mathrm{curl}\Big(\mathbf{u} \times \mathrm{curl}(\mathbf{v} \times \boldsymbol{\alpha}) + \mathbf{v} \times \mathrm{curl}(\boldsymbol{\alpha} \times \mathbf{u}) + \boldsymbol{\alpha} \times \mathrm{curl}(\mathbf{u} \times \mathbf{v})\Big) = 0.
$$

For a 1-form $\alpha = \boldsymbol{\alpha} \cdot d\mathbf{x}$, the result turns out to be trivial.

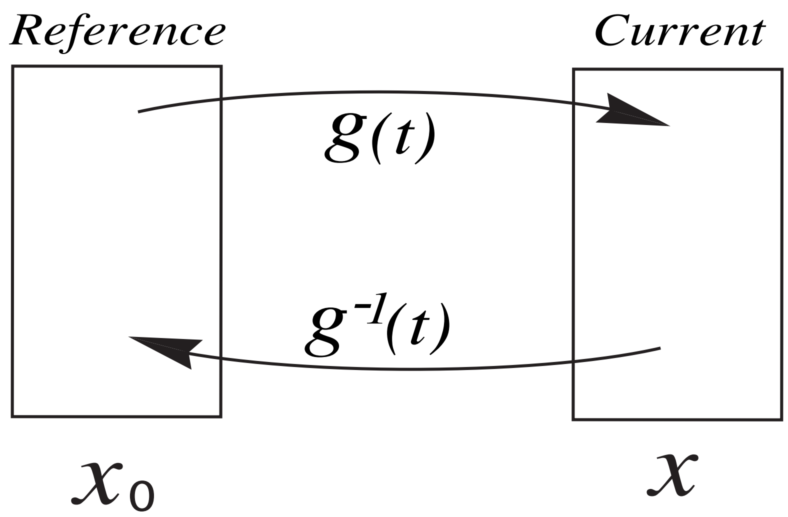
▲

# 7 Hamilton's principle for fluid dynamics

## 7.1 Advected quantities in fluid dynamics

We regard fluid flow as a smooth invertible time-dependent transformation of initial conditions $x_0$ regarded as fluid labels taking values in a configuration manifold $M$ acted on by smooth invertible maps Diff$(M)$. Thus, we lift the motion of fluid parcels $x_t \in M$ with initial condition $x_0 \in M$ to the manifold of diffeomorphisms by identifying it with a time-dependent curve $g_t \in \text{Diff}(M)$ with $g_0 = Id$, whose action from the left generates the motion $x_t$,

$$x_t = g_t x_0 \quad \text{with} \quad \dot{x}_t = \dot{g}_t x_0 = (u_t \circ g_t) x_0$$



**Advected quantity.** A quantity $a_t(x_t) = a_0(x_0)$ which remains invariant under the flow is said to be *advected* by the flow. In terms of the group action, advected quantities satisfy

$$a_0(x_0) = a_t(x_t) = (a_t \circ g_t)(x_0) = (g_t^* a_t)(x_0)$$

where $g_t^* a_t$ is the *pull-back* of $a_t$ by $g_t$. Invariance of an advected quantity implies an evolution equation

$$0 = \frac{d}{dt} a_0(x_0) = \frac{d}{dt}(g_t^* a_t)(x_0) = g_t^* \big( (\partial_t + \pounds_u) a_t \big)(x_0) = (\partial_t + \pounds_u) a_t(x_t)$$

where $\pounds_u$ denotes the Lie derivative with respect to the vector field $u = \dot{g} g^{-1}$ which generates the flow $g_t$.

Vice versa, we have the *push-forward* relation

$$\frac{d}{dt} a_t(x_t) = \frac{d}{dt}(a_0 g_t^{-1})(x_t) = \frac{d}{dt}\big( (g_t)_* a_0 \big)(x_t) = -(\pounds_u a_t)(x_t)\,.$$

The previous formula will be useful in taking variations of advected quantities in Hamilton's principle, since it implies the following formula for the variation of an advected quantity, $a_t$ at fixed $t$,

$$\delta a_t(x_t) = a_t'(x_t) := \left( \frac{d}{d\epsilon}\Big|_{\epsilon=0} a_{t,\epsilon} \right)(x_t) = -(\pounds_{v_t} a_t)(x_t) \quad \text{where} \quad v_t = \left( \frac{d g_{t,\epsilon}}{d\epsilon} g_{t,\epsilon}^{-1} \right)\Big|_{\epsilon=0} = g' g^{-1}$$

where $\pounds_v$ denotes the Lie derivative with respect to the vector field $v = [g_\epsilon{}' g_\epsilon^{-1}]_{\epsilon=0}$ which generates the flow $g_\epsilon$.

Equality of cross derivatives in $t$ and $\epsilon$ implies the following pair of relations

$$
\begin{aligned}
(\dot{g})' \circ g^{-1} &= (u \circ g)' \circ g^{-1} = (\partial_x u) g' \circ g^{-1} + (u' \circ g) \circ g^{-1} \\
&= (\partial_x u) v + u' \\
(g'\,)^{\cdot} \circ g^{-1} &= (\partial_x v) u + \dot{v}\,,
\end{aligned}
$$

from which we conclude upon substituting $u = \dot{g}g^{-1}$ that

$$
\delta(\dot{g}g^{-1}) = (\dot{g}g^{-1})' = \dot{v} + (\partial_x v)u - (\partial_x u)v = \dot{v} - [u, v] = \dot{v} - \mathrm{ad}_u v = \dot{v} - \mathrm{ad}_{\dot{g}g^{-1}} v
$$

Now we are ready to compute the Euler-Poincaré equations for fluid dynamics.

### 7.2 Euler-Poincaré equations for fluid dynamics

We shall compute the compute the Euler-Poincaré equations for fluid dynamics using the Hamilton-Pontryagin principle,

$$
\begin{aligned}
0 = \delta S = \delta \int_0^T & \ell(u, a_0 g_t^{-1}) + \langle m\,,\, \dot{g}g^{-1} - u \rangle \, dt \\
= \int_0^T & \Big\langle \frac{\delta \ell}{\delta u} - m\,,\, \delta u \Big\rangle + \Big\langle \frac{\delta \ell}{\delta a}\,,\, -\pounds_v a \Big\rangle + \Big\langle m\,,\, \partial_t v - \mathrm{ad}_{\dot{g}g^{-1}} v \Big\rangle + \Big\langle \delta m\,,\, \dot{g}g^{-1} - u \Big\rangle \, dt \\
= \int_0^T & \Big\langle \frac{\delta \ell}{\delta u} - m\,,\, \delta u \Big\rangle + \Big\langle \frac{\delta \ell}{\delta a} \diamond a - \partial_t m - \mathrm{ad}^*_{\dot{g}g^{-1}} m\,,\, v \Big\rangle + \Big\langle \delta m\,,\, \dot{g}g^{-1} - u \Big\rangle \, dt + \langle m\,,\, v \rangle \Big|_0^T ,
\end{aligned}
$$

where we have used $\delta(a_0 g_t^{-1}) = -\pounds_v a$ and have defined the diamond operator $(\diamond)$ as

$$\diamond : V^* \times V \to \mathfrak{X}^* \quad \text{defined by} \quad \left\langle \frac{\delta \ell}{\delta a} \diamond a \,,\, v \right\rangle := \left\langle \frac{\delta \ell}{\delta a} \,,\, -\pounds_v a \right\rangle$$

and the $\mathrm{ad}^*$ operation as

$$\mathrm{ad}^* : \mathfrak{X} \times \mathfrak{X}^* \to \mathfrak{X}^* \quad \text{defined by} \quad \langle \mathrm{ad}^*_u m \,,\, v \rangle = \Big\langle m \,,\, \mathrm{ad}_u v \Big\rangle$$

In particular, $\mathrm{ad}^*_u m = \pounds_u m$, so that the fluid motion equation for $m = \mathbf{m} \cdot d\mathbf{x} \otimes d^3x$ and advection equations become

$$(\partial_t + \pounds_u) m = \frac{\delta \ell}{\delta a} \diamond a \quad \text{and} \quad (\partial_t + \pounds_u) a = 0$$

In general, fluid motion advects mass, so that $D_t(x_t) d^3x_t = D_0(x_0) d^3x_0$, which implies the continuity equation

$$0 = (\partial_t + \pounds_u)(D_t(x_t) d^3x_t) = \Big(\partial_t D + \mathrm{div}(D\mathbf{u})\Big) d^3x$$

Consequently, the motion equation may be rewritten as

$$\Big(\partial_t + \pounds_u\Big)(D^{-1}\mathbf{m} \cdot d\mathbf{x}) = \frac{1}{D}\frac{\delta \ell}{\delta a} \diamond a$$

in which $\frac{1}{D}\frac{\delta \ell}{\delta a} \diamond a$ is a 1-form. Integrating this relation around a material loop $c_t$ moving with the fluid yields

$$\frac{d}{dt} \oint_{c_t} (D^{-1}\mathbf{m} \cdot d\mathbf{x}) = \oint_{c_t} \frac{1}{D}\frac{\delta \ell}{\delta a} \diamond a$$

This is the Kelvin-Noether theorem, which arises from relabelling symmetry of the Lagrangian fluid parcels.

### 7.3 Euler's fluid equations

Euler's equations for the incompressible motion of an ideal flow of a fluid of unit density and velocity $\mathbf{u}$ satisfying $\operatorname{div}\mathbf{u} = 0$ in a rotating frame with Coriolis parameter $\operatorname{curl}\mathbf{R} = 2\boldsymbol{\Omega}$ are given in the form of Newton's law of force by

$$\underbrace{\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}}_{\text{acceleration}} = \underbrace{\mathbf{u}\times 2\boldsymbol{\Omega}}_{\text{Coriolis}} - \underbrace{\nabla p}_{\text{pressure}} . \tag{53}$$

**Exercise.** Prove that Euler's equations in a rotating frame arise as Euler-Poincaré equations from Hamilton's variational principle for the following action integral.

$$0 = \delta S = \int_0^T \frac{1}{2}D|\mathbf{u}|^2 + D\mathbf{u}\cdot\mathbf{R} - p(D-1)\,d^3x\,dt$$

★

The Newton's law equation for Euler fluid motion in (53) may be rearranged into an alternative form,

$$\partial_t \mathbf{v} - \mathbf{u}\times\boldsymbol{\omega} + \nabla\Big(p + \frac{1}{2}|\mathbf{u}|^2\Big) = 0\,, \tag{54}$$

by denoting

$$\mathbf{v} \equiv \mathbf{u} + \mathbf{R}\,, \quad \boldsymbol{\omega} = \operatorname{curl}\mathbf{v} = \operatorname{curl}\mathbf{u} + 2\boldsymbol{\Omega}\,, \tag{55}$$

and using the fundamental vector calculus identity of fluid dynamics

$$\mathbf{u}\cdot\nabla\mathbf{v} + v_j\nabla u^j = -\,\mathbf{u}\times\operatorname{curl}\mathbf{v} + \nabla(\mathbf{u}\cdot\mathbf{v})\,. \tag{56}$$

This identity follows from equality of the dynamic and geometricdefinitions of the Lie derivative $\pounds_u\alpha$ of a $k$-form $\alpha\in\Lambda^k(M)$ by the vector field $u=\dot{g}g^{-1}$ tangent to the flow $g_t$ on $M$ as

$$\pounds_u\alpha \;=\; \frac{d}{dt}\bigg|_{t=0}(g_t^*\alpha) = u \,\lrcorner\, d\alpha + d(u \,\lrcorner\, \alpha), \tag{57}$$

in which the last equality is Cartan's geometric formula for the Lie derivative.

For the case of the circulation 1-form $\alpha = \mathbf{v}\cdot d\mathbf{x}$, this becomes

$$\begin{aligned}
\pounds_{\mathbf{u}}(\mathbf{v}\cdot d\mathbf{x}) &= \big(\mathbf{u}\cdot\nabla\mathbf{v} + v_j\nabla u^j\big)\cdot d\mathbf{x} \\
&= u \,\lrcorner\, d(\mathbf{v}\cdot d\mathbf{x}) + d(u \,\lrcorner\, \mathbf{v}\cdot d\mathbf{x}) \\
&= u \,\lrcorner\, (\operatorname{curl}\mathbf{v}\cdot d\mathbf{S}) + d(\mathbf{u}\cdot\mathbf{v}) \\
&= \big(-\;\mathbf{u}\times\operatorname{curl}\mathbf{v} + \nabla(\mathbf{u}\cdot\mathbf{v})\big)\cdot d\mathbf{x}\,,
\end{aligned} \tag{58}$$

and the identity (55) emerges. This identity and the calculation (58) recasts Euler's fluid motion equation into the following geometric form:

$$
\begin{aligned}
\left(\frac{\partial}{\partial t} + \pounds_{\mathbf{u}}\right)(\mathbf{v}\cdot d\mathbf{x}) &= \big(\partial_t \mathbf{v} - \mathbf{u}\times \operatorname{curl}\mathbf{v} + \nabla(\mathbf{u}\cdot\mathbf{v})\big)\cdot d\mathbf{x} \\
&= -\nabla\left(p + \frac{1}{2}|\mathbf{u}|^2 - \mathbf{u}\cdot\mathbf{v}\right)\cdot d\mathbf{x} \\
&= -d\left(p + \frac{1}{2}|\mathbf{u}|^2 - \mathbf{u}\cdot\mathbf{v}\right).
\end{aligned}
\tag{59}
$$

Requiring preservation of the divergence-free (volume-preserving) constraint $\nabla\cdot\mathbf{u} = 0$ results in a Poisson equation for pressure $p$, which may be written in several equivalent forms,

$$
\begin{aligned}
-\Delta p &= \operatorname{div}\big(\mathbf{u}\cdot\nabla\mathbf{u} - \mathbf{u}\times 2\boldsymbol{\Omega}\big) \\
&= u_{i,j}u_{j,i} - \operatorname{div}\big(\mathbf{u}\times 2\boldsymbol{\Omega}\big) \\
&= \operatorname{tr}\mathsf{S}^2 - \frac{1}{2}|\operatorname{curl}\mathbf{u}|^2 - \operatorname{div}\big(\mathbf{u}\times 2\boldsymbol{\Omega}\big),
\end{aligned}
\tag{60}
$$

where $\mathsf{S} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$ is the ***strain-rate tensor***.

We introduce the ***Lamb vector***,

$$
\boldsymbol{\ell} := -\,\mathbf{u}\times\boldsymbol{\omega}\,,
\tag{61}
$$

which represents the nonlinearity in Euler's fluid equation (54). The Poisson equation (60) for pressure $p$ may now be expressed in terms of the divergence of the Lamb vector,

$$
-\,\Delta\left(p + \frac{1}{2}|\mathbf{u}|^2\right) = \operatorname{div}(-\,\mathbf{u}\times\operatorname{curl}\mathbf{v}) = \operatorname{div}\boldsymbol{\ell}\,.
\tag{62}
$$

**Remark 49** (***Boundary conditions***)**.**
Because the velocity $\mathbf{u}$ must be tangent to any fixed boundary, the normal component of the motion equation must vanish. This requirement produces a Neumann condition for pressure given by

$$\partial_n \left( p + \frac{1}{2}|\mathbf{u}|^2 \right) + \hat{\mathbf{n}} \cdot \boldsymbol{\ell} = 0\,, \tag{63}$$

at a fixed boundary with unit outward normal vector $\hat{\mathbf{n}}$.

**Remark 50** (***Helmholtz vorticity dynamics***)**.**
Taking the curl of the Euler fluid equation (54) yields the ***Helmholtz vorticity equation***

$$\partial_t \boldsymbol{\omega} - \mathrm{curl}(\mathbf{u} \times \boldsymbol{\omega}) = 0\,, \tag{64}$$

whose geometrical meaning will emerge in discussing Stokes' Theorem 66 for the vorticity of a rotating fluid.

The rotation terms have now been fully integrated into both the dynamics and the boundary conditions. In this form, the ***Kelvin circulation theorem*** and the ***Stokes vorticity theorem*** will emerge naturally together as geometrical statements.

### 7.4 Kelvin's circulation theorem

**Theorem 51** (***Kelvin's circulation theorem***). *The Euler equations (53) preserve the circulation integral $I(t)$ defined by*

$$I(t) = \oint_{c(\mathbf{u})} \mathbf{v} \cdot d\mathbf{x} \,, \tag{65}$$

*where $c(\mathbf{u})$ is a closed circuit moving with the fluid at velocity $\mathbf{u}$.*

*Proof.* The dynamical definition of the Lie derivative in (57) yields the following for the time rate of change of this circulation integral:

$$\begin{aligned} \frac{d}{dt} \oint_{c(\mathbf{u})} \mathbf{v} \cdot d\mathbf{x} &= \oint_{c(\mathbf{u})} \left( \frac{\partial}{\partial t} + \pounds_{\mathbf{u}} \right) (\mathbf{v} \cdot d\mathbf{x}) \\ &= \oint_{c(\mathbf{u})} \left( \frac{\partial \mathbf{v}}{\partial t} + \frac{\partial \mathbf{v}}{\partial x^j} u^j + v_j \frac{\partial u^j}{\partial \mathbf{x}} \right) \cdot d\mathbf{x} \\ &= - \oint_{c(\mathbf{u})} \nabla \left( p + \frac{1}{2} |\mathbf{u}|^2 - \mathbf{u} \cdot \mathbf{v} \right) \cdot d\mathbf{x} \\ &= - \oint_{c(\mathbf{u})} d \left( p + \frac{1}{2} |\mathbf{u}|^2 - \mathbf{u} \cdot \mathbf{v} \right) = 0 \,. \end{aligned} \tag{66}$$

The last step in the proof follows, because the integral of an exact differential around a closed loop vanishes. □

The exterior derivative of the Euler fluid equation in the form (59) yields Stokes' theorem, after using the commutativity of the exterior and Lie derivatives $[d,\, \pounds_{\mathbf{u}}] = 0$,

$$
\begin{aligned}
d\pounds_{\mathbf{u}}(\mathbf{v}\cdot d\mathbf{x}) &= d\Big(-\mathbf{u}\times \mathrm{curl}\mathbf{v}\cdot d\mathbf{x} + d(\mathbf{u}\cdot\mathbf{v})\Big)\\
&= \pounds_{\mathbf{u}}(\mathrm{curl}\,\mathbf{v}\cdot d\mathbf{S})\\
&= -\,\mathrm{curl}\Big(\mathbf{u}\times \mathrm{curl}\,\mathbf{v}\Big)\cdot d\mathbf{S}\\
&= \Big[\mathbf{u}\cdot\nabla \mathrm{curl}\,\mathbf{v} + \mathrm{curl}\,\mathbf{v}(\mathrm{div}\,\mathbf{u}) - (\mathrm{curl}\,\mathbf{v})\cdot\nabla\mathbf{u}\Big]\cdot d\mathbf{S}\,,\\
(\text{by } \mathrm{div}\,\mathbf{u}=0)\quad &= \Big[\mathbf{u}\cdot\nabla \mathrm{curl}\,\mathbf{v} - (\mathrm{curl}\,\mathbf{v})\cdot\nabla\mathbf{u}\Big]\cdot d\mathbf{S}\\
&=: [\,u,\, \mathrm{curl}\,v\,]\cdot d\mathbf{S}\,,
\end{aligned}
\tag{67}
$$

where $[\,u,\, \mathrm{curl}\,v\,]$ denotes (minus) the ***Jacobi–Lie bracket*** of the vector fields $u$ and $\mathrm{curl}\,v$.

This calculation proves the following.

**Theorem 52.** *Euler's fluid equations (54) imply that*

$$
\frac{\partial\omega}{\partial t} = -\,[\,u,\, \omega\,]
\tag{68}
$$

*where* $[\,u,\, \omega\,]$ *denotes the Jacobi–Lie bracket of the divergenceless vector fields* $u$ *and* $\omega := \mathrm{curl}\,v$.

The exterior derivative of Euler's equation in its geometric form (59) is equivalent to the curl of its vector form (54). That is,

$$
d\left(\frac{\partial}{\partial t} + \pounds_{\mathbf{u}}\right)(\mathbf{v}\cdot d\mathbf{x}) = \left(\frac{\partial}{\partial t} + \pounds_{\mathbf{u}}\right)(\mathrm{curl}\,\mathbf{v}\cdot d\mathbf{S}) = 0\,.
\tag{69}
$$

Hence from the calculation in (67) and the Helmholtz vorticity equation (69) we have

$$\Big(\frac{\partial}{\partial t}+\pounds_{\mathbf{u}}\Big)(\operatorname{curl}\mathbf{v}\cdot d\mathbf{S})=\Big(\partial_t\,\boldsymbol{\omega}-\operatorname{curl}(\mathbf{u}\times\boldsymbol{\omega})\Big)\cdot d\mathbf{S}=0\,, \tag{70}$$

in which one denotes $\boldsymbol{\omega}:=\operatorname{curl}\mathbf{v}$. This Lie-derivative version of the Helmholtz vorticity equation may be used to prove the following form of Stokes' theorem for the Euler equations in a rotating frame.

**Theorem 53.** ***[Kelvin/Stokes' theorem for vorticity of a rotating fluid]***

$$\begin{aligned}\frac{d}{dt}\oint_{c(\mathbf{u})}\mathbf{v}\cdot d\mathbf{x}&=\frac{d}{dt}\iint_{S(\mathbf{u})}\operatorname{curl}\mathbf{v}\cdot d\mathbf{S}\\&=\iint_{S(\mathbf{u})}\Big(\frac{\partial}{\partial t}+\pounds_{\mathbf{u}}\Big)(\operatorname{curl}\mathbf{v}\cdot d\mathbf{S})\\&=\iint_{S(\mathbf{u})}\Big(\partial_t\,\boldsymbol{\omega}-\operatorname{curl}\,(\mathbf{u}\times\boldsymbol{\omega})\Big)\cdot d\mathbf{S}=0\,,\end{aligned} \tag{71}$$

*where the surface $S(\mathbf{u})$ is bounded by an arbitrary circuit $\partial S=c(\mathbf{u})$ moving with the fluid.*

### 7.5 Steady solutions: Lamb surfaces

According to Theorem 52, Euler's fluid equations (54) imply that

$$\frac{\partial\omega}{\partial t}=-\,[\,u,\,\omega\,]\,. \tag{72}$$

Consequently, the vector fields $u$, $\omega$ in *steady* Euler flows, which satisfy $\partial_t\omega = 0$, also satisfy the condition necessary for the Frobenius theorem to hold – namely, that their Jacobi–Lie bracket vanishes. That is, in smooth steady, or equilibrium, solutions of Euler's fluid equations, the flows of the two divergenceless vector fields $u$ and $\omega$ commute with each other and lie on a surface in three dimensions.

A sufficient condition for this commutation relation is that the ***Lamb vector*** $\boldsymbol{\ell} := -\,\mathbf{u}\times\operatorname{curl}\mathbf{v}$ in (61) satisfies

$$\boldsymbol{\ell} := -\,\mathbf{u}\times\operatorname{curl}\mathbf{v} = \nabla H(\mathbf{x})\,, \tag{73}$$

for some smooth function $H(\mathbf{x})$. This condition means that the flows of vector fields $u$ and $\operatorname{curl} v$ (which are *steady* flows of the Euler equations) are both confined to the same surface $H(\mathbf{x}) = const$. Such a surface is called a ***Lamb surface***.

The vectors of velocity ($\mathbf{u}$) and total vorticity ($\operatorname{curl}\mathbf{v}$) for a steady Euler flow are both perpendicular to the normal vector to the Lamb surface along $\nabla H(\mathbf{x})$. That is, the Lamb surface is invariant under the flows of both vector fields, *viz*

$$\pounds_u H = \mathbf{u}\cdot\nabla H = 0 \quad\text{and}\quad \pounds_{\operatorname{curl} v} H = \operatorname{curl}\mathbf{v}\cdot\nabla H = 0\,. \tag{74}$$

The Lamb surface condition (73) has the following coordinate-free representation.

**Theorem 54** (***Lamb surface condition)***. *The Lamb surface condition (73) is equivalent to the following double substitution of vector fields into the volume form,*

$$dH = u \,\lrcorner\, \operatorname{curl} v \,\lrcorner\, d^{\,3}x\,. \tag{75}$$

*Proof.* Recall that the contraction of vector fields with forms yields the following useful formula for the surface element:

$$\nabla \,\lrcorner\, d^{\,3}x = d\mathbf{S}\,. \tag{76}$$

Then using results from previous exercises in vector calculus operations one finds by direct computation that

$$\begin{aligned} u \,\lrcorner\, \mathrm{curl}\, v \,\lrcorner\, d^{\,3}x &= u \,\lrcorner\, (\mathrm{curl}\,\mathbf{v}\cdot d\mathbf{S}) \\ &= -\,\big(\mathbf{u}\times \mathrm{curl}\,\mathbf{v}\big)\cdot d\mathbf{x} \\ &= \nabla H \cdot d\mathbf{x} \\ &= dH\,. \end{aligned} \tag{77}$$

□

**Remark 55.**
Formula (77)

$$u \,\lrcorner\, (\mathrm{curl}\,\mathbf{v}\cdot d\mathbf{S}) = dH$$

is to be compared with

$$X_h \,\lrcorner\, \omega = dH\,,$$

in the definition of a Hamiltonian vector field in Equation (42) of Theorem 27. Likewise, the stationary case of the Helmholtz vorticity equation (69), namely,

$$\pounds_{\mathbf{u}}(\mathrm{curl}\,\mathbf{v}\cdot d\mathbf{S}) = 0\,, \tag{78}$$

is to be compared with the proof of Poincaré's theorem in Corollary 30

$$\pounds_{X_h}\omega = d(X_h \,\lrcorner\, \omega) = d^2 H = 0\,.$$

Thus, the two-form $\mathrm{curl}\,\mathbf{v}\cdot d\mathbf{S}$ plays the same role for stationary Euler fluid flows as the symplectic form $dq\wedge dp$ plays for canonical Hamiltonian flows. We seek the corresponding symplectic coordinates.

**Definition 56.** *The* ***Clebsch representation*** *of the one-form* $\mathbf{v}\cdot d\mathbf{x}$ *is defined by*

$$\mathbf{v}\cdot d\mathbf{x} = -\,\Pi\, d\Xi + d\Psi\,. \tag{79}$$

*The functions* $\Xi$*,* $\Pi$ *and* $\Psi$ *are called* ***Clebsch potentials*** *for the vector* $\mathbf{v}$*.*[4]

In terms of the Clebsch representation (79) of the one-form $\mathbf{v}\cdot d\mathbf{x}$, the total vorticity flux $\mathrm{curl}\,\mathbf{v}\cdot d\mathbf{S} = d(\mathbf{v}\cdot d\mathbf{x})$ is the exact two-form,

$$\mathrm{curl}\,\mathbf{v}\cdot d\mathbf{S} = d\Xi\wedge d\Pi\,. \tag{80}$$

This amounts to writing the flow lines of the *vector field* of the total vorticity $\mathrm{curl}\,v$ as the intersections of level sets of surfaces $\Xi = const$ and $\Pi = const$. In other words,

$$\mathrm{curl}\,\mathbf{v} = \nabla\Xi\times\nabla\Pi\,, \tag{81}$$

with the assumption that these level sets foliate $\mathbb{R}^3$. That is, one assumes that any point in $\mathbb{R}^3$ along the flow of the total vorticity vector field $\mathrm{curl}\,v$ may be assigned to a regular intersection of these level sets. The main result of this assumption is the following theorem.

[4]The Clebsch representation is another example of a cotangent lift momentum map.

**Theorem 57** (***Lamb surfaces are symplectic manifolds*****).** *[ArKh1992, ArKh1998] The steady flow of the vector field* $u$ *satisfying the symmetry relation given by the vanishing of the commutator* $[\,u,\,\mathrm{curl}\,v\,] = 0$ *on a three-dimensional manifold* $M \in \mathbb{R}^3$ *reduces to incompressible flow on a two-dimensional symplectic manifold whose canonically conjugate coordinates* $(\Xi,\,\Pi)$ *are provided by the total vorticity flux*

$$\mathrm{curl}\, v \,\lrcorner\, d^{\,3}x = \mathrm{curl}\,\mathbf{v}\cdot d\mathbf{S} = d\Xi\wedge d\Pi\,.$$

*The reduced flow is canonically Hamiltonian on this symplectic manifold. Furthermore, the reduced Hamiltonian is precisely the restriction of the invariant* $H$ *onto the reduced phase space.*

*Proof.* Restricting formula (77) to coordinates on a total vorticity flux surface (80) yields the exterior derivative of the Hamiltonian,

$$\begin{aligned} dH(\Xi,\,\Pi) &= u \,\lrcorner\, (\mathrm{curl}\,\mathbf{v}\cdot d\mathbf{S}) \\ &= u \,\lrcorner\, (d\Xi\wedge d\Pi) \\ &= (\mathbf{u}\cdot\nabla\Xi)\, d\Pi - (\mathbf{u}\cdot\nabla\Pi)\, d\Xi \\ &=: \frac{d\Xi}{dT}\, d\Pi - \frac{d\Pi}{dT}\, d\Xi \\ &= \frac{\partial H}{\partial \Pi}\, d\Pi + \frac{\partial H}{\partial \Xi}\, d\Xi\,, \end{aligned} \tag{82}$$

where $T\in\mathbb{R}$ is the time parameter along the flow lines of the steady vector field $u$, which carries the Lagrangian fluid parcels. On identifying corresponding terms, the steady flow of the fluid velocity

$\mathbf{u}$ is found to obey the canonical Hamiltonian equations,

$$(\mathbf{u}\cdot\nabla\Xi) = \pounds_u\Xi \;=:\; \frac{d\Xi}{dT} = \frac{\partial H}{\partial \Pi} = \big\{\Xi,\, H\big\}\,, \tag{83}$$

$$(\mathbf{u}\cdot\nabla\Pi) = \pounds_u\Pi \;=:\; \frac{d\Pi}{dT} = -\,\frac{\partial H}{\partial \Xi} = \big\{\Pi,\, H\big\}\,, \tag{84}$$

where $\{\,\cdot\,,\,\cdot\,\}$ is the canonical Poisson bracket for the symplectic form $d\Xi\wedge d\Pi$. □

**Corollary 58.** *The vorticity flux $d\Xi\wedge d\Pi$ is invariant under the flow of the velocity vector field $u$.*

*Proof.* By (82), one verifies

$$\pounds_u(d\Xi\wedge d\Pi) = d\big(u \,\lrcorner\, (d\Xi\wedge d\Pi)\big) = d^2H = 0\,.$$

This is the standard computation in the proof of Poincaré's theorem in Corollary 30 for the preservation of a symplectic form by a canonical transformation. Its interpretation here is that the steady Euler flows preserve the total vorticity flux, $\mathrm{curl}\,\mathbf{v}\cdot d\mathbf{S} = d\Xi\wedge d\Pi$. □

### 7.6 The conserved helicity of ideal incompressible flows

**Definition 59** (Helicity)**.** *The helicity* $\Lambda[\mathrm{curl}\,\mathbf{v}]$ *of a divergence-free vector field* $\mathrm{curl}\,v$ *that is tangent to the boundary $\partial D$ of a simply connected domain $D\in\mathbb{R}^3$ is defined as*

$$\Lambda[\mathrm{curl}\,\mathbf{v}] = \int_D \mathbf{v}\cdot\mathrm{curl}\,\mathbf{v}\,d^{\,3}x\,, \tag{85}$$

*where* $\mathbf{v}$ *is a divergence-free vector-potential for the field* $\mathrm{curl}\,\mathbf{v}$.

**Remark 60.**
The helicity is unchanged by adding a gradient to the vector $\mathbf{v}$. Thus, $\mathbf{v}$ is not unique and $\operatorname{div}\mathbf{v} = 0$ is not a restriction for simply connected domains in $\mathbb{R}^3$, provided $\operatorname{curl}\mathbf{v}$ is tangent to the boundary $\partial D$.

The helicity of a vector field $\operatorname{curl} v$ measures the total linking of its field lines, or their relative winding. (For details and mathematical history, see [ArKh1998].) The idea of helicity goes back to Helmholtz and Kelvin in the 19th century. The principal feature of this concept for fluid dynamics is embodied in the following theorem.

**Theorem 61** (***Euler flows preserve helicity***)**.** *When homogeneous or periodic boundary conditions are imposed, Euler's equations for an ideal incompressible fluid flow in a rotating frame with Coriolis parameter* $\operatorname{curl}\mathbf{R} = 2\boldsymbol{\Omega}$ *preserves the helicity*

$$\Lambda[\operatorname{curl}\mathbf{v}] = \int_D \mathbf{v}\cdot\operatorname{curl}\mathbf{v}\, d^{\,3}x\,, \tag{86}$$

*with* $\mathbf{v} = \mathbf{u} + \mathbf{R}$*, for which* $\mathbf{u}$ *is the divergenceless fluid velocity* $(\operatorname{div}\mathbf{u} = 0)$ *and* $\operatorname{curl}\mathbf{v} = \operatorname{curl}\mathbf{u} + 2\boldsymbol{\Omega}$ *is the total vorticity.*

*Proof.* Rewrite the geometric form of the Euler equations (59) for rotating incompressible flow with unit mass density in terms of the circulation one-form $v := \mathbf{v}\cdot d\mathbf{x}$ as

$$(\partial_t + \pounds_u)v = -\,d\left(p + \frac{1}{2}|\mathbf{u}|^2 - \mathbf{u}\cdot\mathbf{v}\right) =: -\,d\,\varpi\,, \tag{87}$$

and $\pounds_u \, d^{\,3}x = 0$. Here, $\varpi$ is an augmented pressure variable,

$$\varpi := p + \frac{1}{2}|\mathbf{u}|^2 - \mathbf{u}\cdot\mathbf{v}\,. \tag{88}$$

The fluid velocity vector field is denoted as $u = \mathbf{u}\cdot\nabla$ with $\operatorname{div}\mathbf{u} = 0$. Then the ***helicity density***, defined as

$$v \wedge dv = \mathbf{v}\cdot\operatorname{curl}\mathbf{v}\, d^{\,3}x = \lambda\, d^{\,3}x\,, \quad \text{with} \quad \lambda = \mathbf{v}\cdot\operatorname{curl}\mathbf{v}\,, \tag{89}$$

obeys the dynamics it inherits from the Euler equations,

$$\big(\partial_t + \pounds_u\big)(v \wedge dv) = -d\varpi \wedge dv - v \wedge d^2\varpi = -d(\varpi\, dv)\,, \tag{90}$$

after using $d^2\varpi = 0$ and $d^2 v = 0$. In vector form, this result may be expressed as a conservation law,

$$\big(\partial_t\lambda + \operatorname{div}\lambda\mathbf{u}\big)d^{\,3}x = -\operatorname{div}(\varpi\operatorname{curl}\mathbf{v})\, d^{\,3}x\,. \tag{91}$$

Consequently, the time derivative of the integrated helicity in a domain $D$ obeys

$$\begin{aligned}\frac{d}{dt}\Lambda[\operatorname{curl}\mathbf{v}] &= \int_D \partial_t\lambda\, d^{\,3}x = -\int_D \operatorname{div}(\lambda\mathbf{u} + \varpi\operatorname{curl}\mathbf{v})\, d^{\,3}x \\ &= -\oint_{\partial D}(\lambda\mathbf{u} + \varpi\operatorname{curl}\mathbf{v})\,\cdot\, d\mathbf{S}\,,\end{aligned} \tag{92}$$

which vanishes when homogeneous, or periodic, or even Neumann boundary conditions are imposed on the values of $\mathbf{u}$ and $\operatorname{curl}\mathbf{v}$ at the boundary $\partial D$. $\square$

**Remark 62.**
This result means the ***helicity integral***

$$\Lambda[\operatorname{curl}\mathbf{v}] = \int_D \lambda\, d^{\,3}x$$

is conserved in periodic domains, or in all of $\mathbb{R}^3$ with vanishing boundary conditions at spatial infinity. However, if either the velocity or total vorticity at the boundary possesses a nonzero normal component, then the boundary is a *source* of helicity (that is, it causes winding of field lines of $\operatorname{curl}\mathbf{v}$). For a fixed impervious boundary, the normal component of velocity does vanish, but no such condition is imposed on the total vorticity by the physics of fluid flow. Thus, we have the following.

**Corollary 63.** *A flux of total vorticity* $\operatorname{curl}\mathbf{v}$ *into the domain is a source of helicity.*

> **Exercise.** Use Cartan's formula in (57) to compute $\pounds_u(v \wedge dv)$ in Equation (90). ★
>
> **Exercise.** Compute the helicity for the one-form $v = \mathbf{v}\cdot d\mathbf{x}$ in the Clebsch representation (79). What does this mean for the linkage of the vortex lines that admit the Clebsch representation? ★

**Theorem 64** (Diffeomorphisms preserve helicity)**.** *The helicity* $\Lambda[\xi]$ *of any divergenceless vector field* $\xi$ *is preserved under the action on* $\xi$ *of any volume-preserving diffeomorphism of the manifold* $M$ *[ArKh1998].*

**Remark 65** (Helicity is a topological invariant)**.**
The helicity $\Lambda[\xi]$ is a topological invariant, not a dynamical invariant, because its invariance is independent of which diffeomorphism acts on $\xi$. This means the invariance of helicity is independent of which Hamiltonian flow produces the diffeomorphism. This is the hallmark of a Casimir function. Although it is defined above with the help of a metric, every volume-preserving diffeomorphism carries a divergenceless vector field $\xi$ into another such field with the same helicity. However, independently of any metric properties, the action of diffeomorphisms does not create or destroy linkages of the characteristic curves of divergenceless vector fields.

### 7.7 Ertel theorem for potential vorticity

**Euler–Boussinesq equations** The Euler–Boussinesq equations for the incompressible motion of an ideal flow of a stratified fluid and velocity $\mathbf{u}$ satisfying $\operatorname{div}\mathbf{u} = 0$ in a rotating frame with Coriolis parameter $\operatorname{curl}\mathbf{R} = 2\boldsymbol{\Omega}$ are given by

$$\underbrace{\partial_t\,\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}}_{\text{acceleration}} = \underbrace{-\,gb\nabla z}_{\text{buoyancy}} + \underbrace{\mathbf{u}\times 2\boldsymbol{\Omega}}_{\text{Coriolis}} - \underbrace{\nabla p}_{\text{pressure}} \tag{93}$$

where $-g\nabla z$ is the constant downward acceleration of gravity and $b$ is the buoyancy, a scalar function of space and time which satisfies the ***advection relation***,

$$\partial_t\, b + \mathbf{u}\cdot\nabla b = 0\,. \tag{94}$$

As for Euler's equations without buoyancy, requiring preservation of the divergence-free (volume-preserving) constraint $\nabla \cdot \mathbf{u} = 0$ results in a Poisson equation for pressure $p$,

$$- \Delta \left( p + \frac{1}{2}|\mathbf{u}|^2 \right) = \operatorname{div}(- \mathbf{u} \times \operatorname{curl} \mathbf{v}) + g\partial_z b \,, \tag{95}$$

which satisfies a Neumann boundary condition because the velocity $\mathbf{u}$ must be tangent to the boundary. where we denote

$$\mathbf{v} \equiv \mathbf{u} + \mathbf{R} \,, \quad \boldsymbol{\omega} = \operatorname{curl} \mathbf{v} = \operatorname{curl} \mathbf{u} + 2\boldsymbol{\Omega} \,, \tag{96}$$

The Newton's law form of the Euler–Boussinesq equations (93) may be rearranged as

$$\partial_t \mathbf{v} - \mathbf{u} \times \operatorname{curl} \mathbf{v} + gb\nabla z + \nabla \left( p + \frac{1}{2}|\mathbf{u}|^2 \right) = 0 \,, \tag{97}$$

where $\mathbf{v} \equiv \mathbf{u} + \mathbf{R}$ and $\nabla \cdot \mathbf{u} = 0$.

**Exercise.** Prove that the Euler–Boussinesq equations in (93) emerge as Euler-Poincaré equations from Hamilton's variational principle for the following action integral.

$$0 = \delta S = \delta \int_0^T \frac{1}{2}D|\mathbf{u}|^2 + D\mathbf{u} \cdot \mathbf{R} - Dbz - p(D-1)\, d^3x\, dt$$

★

**Theorem 66.** ***[The Kelvin/Stokes' theorem for vorticity of a stratified, rotating fluid]***

$$
\begin{aligned}
\frac{d}{dt}\oint_{c(\mathbf{u})} \mathbf{v}\cdot d\mathbf{x} &= \frac{d}{dt}\iint_{S(\mathbf{u})} \operatorname{curl}\mathbf{v}\cdot d\mathbf{S} \\
&= \iint_{S(\mathbf{u})} \Big(\frac{\partial}{\partial t} + \pounds_{\mathbf{u}}\Big)(\operatorname{curl}\mathbf{v}\cdot d\mathbf{S}) \\
&= \iint_{S(\mathbf{u})} \Big(\partial_t\,\boldsymbol{\omega} - \operatorname{curl}(\mathbf{u}\times\boldsymbol{\omega})\Big)\cdot d\mathbf{S} \\
&= \iint_{S(\mathbf{u})} \Big(-\,g\nabla b\times\nabla z\Big)\cdot d\mathbf{S}\,,
\end{aligned}
\tag{98}
$$

*where the surface $S(\mathbf{u})$ is bounded by an arbitrary circuit $\partial S = c(\mathbf{u})$ moving with the fluid. Thus, non-alignment of the gradient of buoyancy $\nabla b$ with the vertical $\nabla z$ creates circulation. Compare this result with equation (71) in the absence of stratification.*

Geometrically, equation (97) may be written as

$$
\big(\partial_t + \pounds_u\big)v + gbdz + d\varpi = 0\,, \tag{99}
$$

where $\varpi$ is defined in (88). In addition, the buoyancy satisfies

$$
\big(\partial_t + \pounds_u\big)b = 0\,, \quad \text{with} \quad \pounds_u\, d^{\,3}x = 0\,. \tag{100}
$$

The fluid velocity vector field is denoted as $u = \mathbf{u}\cdot\nabla$ and the circulation one-form as $v = \mathbf{v}\cdot d\mathbf{x}$. The exterior derivatives of the two equations in (99) are written as

$$\big(\partial_t + \pounds_u\big)dv = -g db\wedge dz \quad \text{and} \quad \big(\partial_t + \pounds_u\big)db = 0\,. \tag{101}$$

Consequently, one finds from the product rule for Lie derivatives (43) that

$$\big(\partial_t + \pounds_u\big)(dv\wedge db) = 0 \quad \text{or} \quad \partial_t\, q + \mathbf{u}\cdot\nabla q = 0\,, \tag{102}$$

in which the quantity

$$q = \nabla b\cdot \operatorname{curl}\mathbf{v}\,, \tag{103}$$

is called ***potential vorticity*** and is abbreviated as PV. The potential vorticity is an important diagnostic for many processes in geophysical fluid dynamics. Conservation of PV on fluid parcels is called ***Ertel's theorem***.

**Remark 67** (***Ertel's theorem for the vorticity vector field).***
Writing the vorticity vector field $\omega = \boldsymbol{\omega}\cdot\nabla$, we have

$$\big(\partial_t + \pounds_u\big)\omega = \partial_t\omega + [u,\,\omega] = g\nabla z\times\nabla b\cdot\nabla\,.$$

Thus, conservation of the potential vorticity may also be proved by the product rule, as

$$\big(\partial_t + \pounds_u\big)q = \big(\partial_t + \pounds_u\big)(\boldsymbol{\omega}\cdot\nabla b) = \big(\partial_t + \pounds_u\big)(\omega b) = \Big(\big(\partial_t + \pounds_u\big)\omega\Big)b + \omega\big(\partial_t + \pounds_u\big)b = 0\,.$$

**Remark 68** (***Material derivative formulation).***
Denoting

$$\frac{D}{Dt} = \partial_t + \pounds_u \quad \text{and} \quad \omega = \boldsymbol{\omega}\cdot\nabla$$

provides an intuitive expression of the Ertel theorem (102) that helps understand it in terms of the time derivative $\frac{D}{Dt}$ following the flow of the fluid particles. Namely, it suggests writing in vector form

$$\frac{D}{Dt}(\boldsymbol{\omega}\cdot\nabla) = g\nabla z\times\nabla b\cdot\nabla \quad \text{and} \quad \frac{Db}{Dt} = 0\,,$$

so that the product rule for derivatives yields conservation of PV on fluid parcels, as

$$\frac{Dq}{Dt} = \frac{D}{Dt}(\boldsymbol{\omega}\cdot\nabla b) = \Big(\frac{D}{Dt}(\boldsymbol{\omega}\cdot\nabla)\Big)b + (\boldsymbol{\omega}\cdot\nabla)\frac{Db}{Dt} = g\nabla z\times\nabla b\cdot\nabla b + (\boldsymbol{\omega}\cdot\nabla)\frac{Db}{Dt} = 0\,.$$

**Remark 69** (***The conserved quantities associated with Ertel's theorem).***
The constancy of the scalar quantities $b$ and $q$ on fluid parcels implies conservation of the spatially integrated quantity,

$$C_\Phi = \int_D \Phi(b,q)\,d^{\,3}x\,, \tag{104}$$

for any smooth function $\Phi$ for which the integral exists.

*Proof.*

$$\frac{d}{dt}C_\Phi = \int_D \Phi_b \partial_t b + \Phi_q \partial_t q \, d^{\,3}x = -\int_D \Phi_b \mathbf{u}\cdot\nabla b + \Phi_q \mathbf{u}\cdot\nabla q \, d^{\,3}x$$
$$= -\int_D \mathbf{u}\cdot\nabla\Phi(b,q)\, d^{\,3}x = -\int_D \nabla\cdot\big(\mathbf{u}\,\Phi(b,q)\big)\, d^{\,3}x = -\oint_{\partial D} \Phi(b,q)\,\mathbf{u}\cdot\hat{\mathbf{n}}\, d\,S = 0\,,$$

when the normal component of the velocity $\mathbf{u}\cdot\hat{\mathbf{n}}$ vanishes at the boundary $\partial D$.

□

**Remark 70** (***Energy conservation*).**
In addition to $C_\Phi$, the Euler–Boussinesq fluid equations (97) also conserve the total energy

$$E = \int_D \frac{1}{2}|\mathbf{u}|^2 + bz \; d^{\,3}x\,, \tag{105}$$

which is the sum of the kinetic and potential energies.

We do not develop the Hamiltonian formulation of the three-dimensional stratified rotating fluid equations studied here. However, one may imagine that the conserved quantity $C_\Phi$ with the arbitrary function $\Phi$ would play an important role. For more explanation in the framework of Geometric Mechanics, see [Ho2011GM] and references therein.

These issues will be discussed in Spring term in M3-4-5A34, *Geometry, Mechanics and Symmetry.*

### 7.8 Rotating shallow water (RSW) equations

Consider dynamics of rotating shallow water (RSW) on a two dimensional domain with horizontal planar coordinates $\mathbf{x} = (x, y)$. This RSW motion is governed by the following nondimensional equations for variables depending on $(\mathbf{x}, t)$ comprising the horizontal fluid velocity vector $\mathbf{u} = (u, v)$ and the total depth $\eta$,

$$\epsilon \frac{d}{dt}\mathbf{u} + f(\mathbf{x})\hat{\mathbf{z}} \times \mathbf{u} + \nabla h = 0\,, \qquad \frac{\partial \eta}{\partial t} + \nabla \cdot (\eta \mathbf{u}) = 0\,, \tag{106}$$

with notation

$$\frac{d}{dt} := \left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right) \quad \text{and} \quad h := \left(\frac{\eta - B}{\epsilon \mathcal{F}}\right),$$

where $\epsilon \ll 1$ and $\mathcal{F} = O(1)$ are nondimensional constants. These equations include spatially variable Coriolis parameter $f(\mathbf{x})\hat{\mathbf{z}} = \mathrm{curl}\mathbf{R}(\mathbf{x})$ and mean depth $B = B(\mathbf{x})$.

**Exercise.**

(i) Show that the RSW equations in (106) follow as Euler-Poincaré equations

$$\big(\partial_t + \pounds_u\big)\frac{1}{\eta}\frac{\delta l}{\delta u} = \frac{1}{\eta}\frac{\delta l}{\delta \eta} \diamond \eta \quad \text{and} \quad \big(\partial_t + \pounds_u\big)\big(\eta\, d^2x\big) = 0\,,$$

from Hamilton's variational principle for the following action integral.

$$0 = \delta S \quad \text{with} \quad S = \int_0^T l(\mathbf{u}, \eta) dt \quad \text{and} \quad l(\mathbf{u}, \eta) = \int \frac{\epsilon}{2}\eta |\mathbf{u}|^2 + \eta \mathbf{u}\cdot\mathbf{R}(\mathbf{x}) - \frac{(\eta - B(\mathbf{x}))^2}{2\epsilon\mathcal{F}}\, d^2x\,.$$

in which $\eta(\mathbf{x}, t)\, d^2x$ is an advected quantity. Recall that $\diamond : V^* \times V \to \mathfrak{X}^*$ is defined by $\langle \frac{\delta \ell}{\delta a} \diamond a \,,\, v \rangle := \langle \frac{\delta \ell}{\delta a} \,,\, -\pounds_v a \rangle$ for vector field $v \in \mathfrak{X}$ and $L^2$ pairing $\langle \,\cdot\,,\,\cdot\, \rangle$.

(ii) Use the Euler-Poincaré equations to show that the RSW equations satisfy Kelvin's circulation theorem

$$\frac{d}{dt} \oint_{c_t} \mathbf{v} \cdot d\mathbf{x} = 0\,,$$

with $\mathbf{v} = \epsilon \mathbf{u} + \mathbf{R}(\mathbf{x})$.

(iii) Use the Euler-Poincaré equations to show that the RSW equations satisfy

$$\big(\partial_t + \pounds_u\big) d\big(\mathbf{v} \cdot d\mathbf{x}\big) = 0\,,$$

with $\mathbf{v} = \epsilon \mathbf{u} + \mathbf{R}(\mathbf{x})$.

(iv) Show that $d(\mathbf{v} \cdot d\mathbf{x}) = \omega\, d^2x$, with $\omega := \hat{\mathbf{z}} \cdot \mathrm{curl}\mathbf{v}$.

(v) Use $(\partial_t + \pounds_u)(\omega\, d^2x) = 0$ obtained in the previous two parts to derive conservation of potential vorticity on fluid particles.

★

**Answer.**

1. The Euler-Poincaré equations are

$$\big(\partial_t + \pounds_u\big) \frac{1}{\eta} \frac{\delta l}{\delta u} = \frac{1}{\eta} \frac{\delta l}{\delta \eta} \diamond \eta \quad \text{and} \quad \big(\partial_t + \pounds_u\big) \big(\eta\, d^2x\big) = 0\,,$$

where $\eta^{-1}\frac{\delta l}{\delta u} = (\epsilon\mathbf{u} + \mathbf{R}(\mathbf{x}))\cdot d\mathbf{x} =: \mathbf{v}\cdot d\mathbf{x}$ and $\eta^{-1}\frac{\delta l}{\delta\eta}\diamond\eta = d(\frac{\epsilon}{2}|\mathbf{u}|^2 + \mathbf{u}\cdot\mathbf{R} - h)$. Thus,

$$\big(\partial_t + \pounds_u\big)(\mathbf{v}\cdot d\mathbf{x}) = d\Big(\frac{\epsilon}{2}|\mathbf{u}|^2 + \mathbf{u}\cdot\mathbf{R} - h\Big)$$

with $\mathbf{v} = \epsilon\mathbf{u} + \mathbf{R}(\mathbf{x})$.

2. Integrating the previous equation around a loop moving with the fluid produces

$$\frac{d}{dt}\oint_{c_t}\mathbf{v}\cdot d\mathbf{x} = \oint_{c_t} d\Big(\frac{\epsilon}{2}|\mathbf{u}|^2 + \mathbf{u}\cdot\mathbf{R} - h\Big) = 0\,,$$

with $\mathbf{v} = \epsilon\mathbf{u} + \mathbf{R}(\mathbf{x})$.

3. The differential of the Euler-Poincaré equation yields with $\omega := \hat{\mathbf{z}}\cdot\mathrm{curl}\mathbf{v}$

$$\big(\partial_t + \pounds_u\big)(\omega d^2x) = \big(\partial_t + \pounds_u\big)d(\mathbf{v}\cdot d\mathbf{x}) = d^2\Big(\frac{\epsilon}{2}|\mathbf{u}|^2 + \mathbf{u}\cdot\mathbf{R} - h\Big) = 0$$

upon commuting the differential $d$ with the Lie derivative and using $d^2 = 0$.

4. By direct computation,

$$d(\mathbf{v}\cdot d\mathbf{x}) = v_{i,j}dx^j\wedge dx^i = v_{1,2}dx^2\wedge dx^1 + v_{2,1}dx^1\wedge dx^2 = (v_{2,1} - v_{1,2})\,d^2x = \hat{\mathbf{z}}\cdot\mathrm{curl}\mathbf{v}\,d^2x = \omega\,d^2x$$

5. We have $(\partial_t+\pounds_u)(\omega d^2x)$ and $(\partial_t+\pounds_u)(\eta d^2x)$. Therefore, by the product rule for the evolutionary operator $(\partial_t + \pounds_u)$ we have

$$0 = (\partial_t + \pounds_u)\Big(\frac{\omega}{\eta}(\eta d^2x)\Big) = \Big((\partial_t + \pounds_u)\frac{\omega}{\eta}\Big)(\eta d^2x) + \frac{\omega}{\eta}(\partial_t + \pounds_u)(\eta d^2x)$$

Since the second term vanishes via the continuity equation, $(\partial_t + \pounds_u)(\eta d^2x)$, the first term yields

$$0 = (\partial_t + \pounds_u)\frac{\omega}{\eta} = \left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)\frac{\omega}{\eta}\,. \quad \text{Hence,} \quad \frac{dq}{dt} = 0\,, \quad \text{with} \quad q := \omega/\eta\,.$$

This is conservation of potential vorticity on fluid particles.

▲

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