M 3-4-5 A16 Notes: Geometric Mechanics, Part I

Professor Darryl D Holm Autumn term 2012 Imperial College London d.holm@ic.ac.uk http://www.ma.ic.ac.uk/~dholm/

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Lecture 1, Thurs 4 Oct 2012: Introduction to the Course and to Smooth Manifolds

Geometric Mechanics, Part I



Figure 1: Geometric Mechanics has involved many great mathematicians!

1 What is Geometric Mechanics?

Text for the course M3-4-5 A16:

Geometric Mechanics I: Dynamics and Symmetry, by Darryl D Holm World Scientific: Imperial College Press, Singapore, Second edition (2011). ISBN 978-1-84816-195-5

1.1 Geometric Mechanics is a framework for many modern applications

Molecular oscillations Astroid pairs Satellites with tethers Molecular strands Elasticity Image registration Robotics Peakons Solitons Fluid dynamics Turbulence models Complex fluids Liquid crystals Superfluids Plasmas Magnetohydrodynamics Geophysical Fluid Dynamics Global warming General relativity Field theory (GIMMSY!) Lie groupoids and algebroids Snakeboards Swarming motion Telecommunications

1.2 What are the next directions for GM?

Information geometry?	Hybrid fluids/kinetic?	Geometric quantum mechanics?
Nanoscience?	Information communication?	Data assimilation?
DNA folding?	Salsa dancing robots?	Things that don't yet have names!

The rest of this handout introduces a few key concepts that will be studied further in the course.

2 Motion on smooth manifolds

2.1 Variational principles

smooth manifold	Riemannian metric	Legendre transformation
tangent space	geodesic	momentum
tangent bundle	Lagrangian	fibre derivative
tangent lift	Hamilton's principle	pairing
kinetic energy	variational derivative	

• Let M be a **smooth manifold** dim M = n. That is, M is a smooth space that is locally \mathbb{R}^n (e.g., Riemann's map $S^2 \to \mathbb{R}^2$).



Figure 2: A manifold is defined by the disjoint union (or, atlas) of local charts, each of which is isomorphic to $\mathbb{R}^{\dim M}$.



Figure 3: The circle S^1 is an example of a manifold that can be covered with two charts that are each locally \mathbb{R}^1 .



Figure 4: The Riemann map shows that the unit sphere S^2 is a manifold that can be covered with two charts that are each locally \mathbb{R}^2 .

Remark. There is much more than this to say about manifolds, but it must wait until the next term.

2.2 Curves on manifolds, tangent spaces, & Hamilton's principle

• The tangent space T_qM contains velocity $v_q = \dot{q}(t) \in T_qM$, tangent to curve $q(t) \in M$ at point q. The coordinates are $(q, v_q) \in TM_q$. Note, dim $T_qM = 2n$ and subscript q reminds us that v_q is an element of the tangent space at the point q and that on manifolds we must keep track of base points.



Figure 5: This is a sketch of the tangent bundle TS^1 of the circle S^1 .

The union of tangent spaces $TM := \bigcup_{q \in M} T_q M$ is also called the **tangent bundle** of the manifold M. The curve q(t) describes the **motion** on manifold M. The curve $\dot{q}(t) \in T_q M$ is called the **tangent lift** of the curve $q(t) \in M$.

- Define kinetic energy $KE: TM \to \mathbb{R}$, via a Riemannian metric $g_q(\cdot, \cdot): TM \times TM \to \mathbb{R}$. Explicitly, $KE = \frac{1}{2}g_q(\dot{q}, \dot{q}) =: \frac{1}{2} ||\dot{q}||^2$.
- Choose the Lagrangian $L: TM \to \mathbb{R}$. (For example, one could choose L to be KE.)
- **Hamilton's principle** is $\delta S = 0$ with $S = \int_a^b L(q, \dot{q}) dt$, for a family of curves $q(t, \epsilon)$ parameterised smoothly by $(t, \epsilon) \in \mathbb{R} \times \mathbb{R}$. The linearisation

$$\delta S := \frac{d}{d\epsilon} \bigg|_{\epsilon=0} \int_{a}^{b} L(q(t,\epsilon), \dot{q}(t,\epsilon)) dt \quad \text{with} \quad \delta q(t) := \frac{dq(t,\epsilon)}{d\epsilon} \bigg|_{\epsilon=0}$$

defines the variational derivative δS of S near the identity $\epsilon = 0$. The variations in q are assumed to vanish at endpoints in time, so that $q(a, \epsilon) = q(a)$ and $q(b, \epsilon) = q(b)$.



Figure 6: This is a sketch of variations of a family of curves on a manifold.

3 Euler–Lagrange equation

Theorem 1 (Hamilton 1835, Euler 1750, Lagrange 1756). Hamilton's principle $\delta S = 0$ with $S = \int_{a}^{b} L(q, \dot{q}) dt$ implies the **Euler-Lagrange (EL) equation:** $\frac{d}{dt} \frac{\partial L(q, \dot{q})}{\partial \dot{q}} = \frac{\partial L(q, \dot{q})}{\partial q}, \quad \text{for any } L(q, \dot{q}).$

 \mathbf{Proof}

$$\delta S = \delta \int_{a}^{b} L(q, \dot{q}) dt = \int_{a}^{b} \delta L(q, \dot{q}) dt = \int_{a}^{b} \left\langle \frac{\partial L}{\partial \dot{q}}, \delta \dot{q} \right\rangle + \left\langle \frac{\partial L}{\partial q}, \delta q \right\rangle dt \\ = \int_{a}^{b} \left\langle \underbrace{-\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} + \frac{\partial L}{\partial q}}_{\text{EL equation}}, \delta q \right\rangle dt + \underbrace{\left\langle \frac{\partial L}{\partial \dot{q}}, \delta q \right\rangle}_{\text{Endpoint term}}^{b} \Box$$

3.1 Hamilton's principle

Ten examples for Simple Mechanical Systems: $L(q, \dot{q}) = T(\dot{q}) - V(q) = KE - PE$. For example,

- 1. Planar isotropic oscillator, $(\mathbf{x}, \dot{\mathbf{x}}) \in T\mathbb{R}^2$: $L = \frac{m}{2} |\dot{\mathbf{x}}|^2 \frac{k}{2} |\mathbf{x}|^2 \implies \ddot{\mathbf{x}} = -\omega^2 \mathbf{x}$ with $\omega^2 = k/m$
- 2. Planar anisotropic oscillator, $(\mathbf{x}, \dot{\mathbf{x}}) \in T\mathbb{R}^2$: $L = \frac{m}{2} |\dot{\mathbf{x}}|^2 \frac{k_1}{2} x_1^2 \frac{k_2}{2} x_2^2 \implies \ddot{x}_i = -\omega_i^2 x_i \text{ with } \omega_i^2 = k_i/m \quad i = 1, 2$

3. Planar pendulum in polar coordinates, $(\theta, \dot{\theta}) \in TS^1$: $L = \frac{m}{2}R^2\dot{\theta}^2 - mgR(1 - \cos\theta) \implies \ddot{\theta} = -\omega^2\sin\theta \text{ with } \omega^2 = g/R$

- 4. Planar pendulum, $(\mathbf{x}, \dot{\mathbf{x}}) \in T\mathbb{R}^2$, constrained to $TS^1 = \{\mathbf{x}, \dot{\mathbf{x}} \in T\mathbb{R}^2 | 1 |\mathbf{x}|^2 = 0 \& \mathbf{x} \cdot \dot{\mathbf{x}} = 0\}$: $L = \frac{m}{2} |\dot{\mathbf{x}}|^2 mg \,\hat{\mathbf{e}}_3 \cdot \mathbf{x} + \mu(1 |\mathbf{x}|^2)$
- 5. Charged particle in a magnetic field, $(\mathbf{x}, \dot{\mathbf{x}}) \in T\mathbb{R}^2$: $L = \frac{m}{2} |\dot{\mathbf{x}}|^2 + \frac{e}{c} \dot{\mathbf{x}} \cdot \mathbf{A}(\mathbf{x}) \implies \ddot{\mathbf{x}} = \frac{e}{mc} \dot{\mathbf{x}} \times \mathbf{B}$ with $\mathbf{B} = \operatorname{curl} \mathbf{A}$
- 6. Kepler problem, $(r, \dot{r}, \theta, \dot{\theta}) \in T\mathbb{R}_+ \times TS^1$: $L = \frac{m}{2} \left(\dot{r}^2 + r^2 \dot{\theta}^2 \right) + \frac{GMm}{r} \implies \ddot{r} = -\frac{GM}{r^2} + \frac{J^2}{r^3}$ with $J = r^2 \dot{\theta} = const$
- 7. Free motion on a sphere, $(\mathbf{x}, \dot{\mathbf{x}}) \in T\mathbb{R}^3$, constrained to $S^2 = \{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x}| = 1\}$: $L = \frac{1}{2}|\dot{\mathbf{x}}|^2 + \mu(1 |\mathbf{x}|^2)$
- 8. Spherical pendulum, $(\mathbf{x}, \dot{\mathbf{x}}) \in T\mathbb{R}^3$, constrained to $S^2 = \{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x}| = 1\}$: $L = \frac{m}{2}|\dot{\mathbf{x}}|^2 mg\,\hat{\mathbf{e}}_3 \cdot \mathbf{x} + \mu(1 |\mathbf{x}|^2)$
- 9. Rotating rigid body, $\widehat{\Omega} = O^{-1}\dot{O} \in T(SO(3) \simeq \mathfrak{so}(3) \quad \ell = \frac{1}{2}\Omega \cdot I\Omega \quad \text{with} \quad \Omega \times = \widehat{\Omega}, \quad \text{that is,} \quad -\epsilon_{ijk}\Omega_k = \widehat{\Omega}_{ijk}\Omega_k = \widehat{\Omega$
- 10. Heavy top $(\widehat{\Omega} = O^{-1}\dot{O}, \Gamma = O^{-1}\hat{\mathbf{e}}_3) \in T(SO(3) \otimes \mathbb{R}^3) \simeq \mathfrak{so}(3) \otimes \mathbb{R}^3$: $\ell = \frac{1}{2} \mathbf{\Omega} \cdot I\mathbf{\Omega} mg\mathbf{\Gamma} \cdot \boldsymbol{\chi}$ with $\mathbf{\Omega} \times = \widehat{\Omega}$

3.2 Lie group symmetries and Noether's theorem

• Introduction of Lie group symmetries:

- A **group** is a set of elements with an associative binary product that has a unique inverse and identity element.
- A *Lie group* G is a group whose transformations depends smoothly on a set of parameters in $\mathbb{R}^{\dim(G)}$. A Lie group is also a smooth manifold, so it is an ideal arena for geometric mechanics, e.g., rigid body motion on SO(3).
- **Noether's theorem:** Suppose $q(t, \epsilon) = q_{\epsilon}(t)$ is a group of transformations of q(t) that depends smoothly on a set of parameters ϵ . Its linearisation is computed from a Taylor series as

$$q(t) \to q_{\epsilon}(t) = q(t) + \epsilon \frac{dq(t,\epsilon)}{d\epsilon} \Big|_{\epsilon=0} + O(\epsilon^2) = q(t) + \epsilon \delta q(t) + O(\epsilon^2),$$

where the linear term

$$\delta q(t) := \frac{dq(t,\epsilon)}{d\epsilon} \bigg|_{\epsilon=0} \quad \text{is called the } infinitesimal transformation}$$

Suppose also that the Lagrangian $L(q, \dot{q})$ in Hamilton's principle $\delta S = 0$ with $S = \int_a^b L(q, \dot{q}) dt$ is *invariant* under these infinitesimal transformations, so that $\delta S = 0$ as a consequence of this invariance. Then the endpoint term above $\langle \frac{\partial L}{\partial \dot{q}}, \delta q \rangle \Big|_a^b$ is a *constant of the motion*. That is, the quantity $\langle \frac{\partial L}{\partial \dot{q}}, \delta q \rangle$ is a constant, whenever q(t) is a solution of the EL equations for this invariant Lagrangian. This proves the following.

Theorem 2 (Noether, 1918).

To each Lie symmetry of the Lagrangian there corresponds a conservation law.

Example: Ignorable coordinates: For $L(q, \dot{q}, \dot{\varphi})$ invariant under $\varphi \to \varphi + \epsilon$, $\delta \varphi = \epsilon$, we have $\frac{d}{dt} \left\langle \frac{\partial L}{\partial \dot{\varphi}}, \epsilon \right\rangle = \left\langle \frac{\partial L}{\partial \varphi}, \epsilon \right\rangle = 0$.

3.3 Noether theorem with gauge symmetry

• **Exercise:** Suppose the Lagrangian is not invariant under the infinitesimal transformation δq , but instead changes by a total time derivative,

$$L(q,\dot{q}) \to L(q,\dot{q}) + \epsilon \frac{d}{dt} \Lambda(q,\dot{q}) \quad \text{under} \quad q(t) \to q'(t) = q(t) + \epsilon \delta q(t) \,.$$

What is the change in the statement of Noether's theorem for this case?

• Answer:

$$\begin{split} \delta S &= \delta \int_{a}^{b} L(q,\dot{q}) \, dt = \int_{a}^{b} \delta L(q,\dot{q}) \, dt + \int_{a}^{b} \frac{d}{dt} \Lambda(q,\dot{q}) \, dt \\ &= \int_{a}^{b} \left\langle \frac{\partial L}{\partial \dot{q}}, \delta \dot{q} \right\rangle + \left\langle \frac{\partial L}{\partial q}, \delta q \right\rangle dt + \Lambda(q,\dot{q}) \Big|_{a}^{b} \\ &= \int_{a}^{b} \left\langle \underbrace{-\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} + \frac{\partial L}{\partial q}}_{\text{EL equation}}, \delta q \right\rangle dt + \underbrace{\left[\left\langle \frac{\partial L}{\partial \dot{q}}, \delta q \right\rangle + \Lambda(q,\dot{q}) \right]_{a}^{b}}_{\text{Total endpoint term}} \end{split}$$

That is, the sum $\langle \frac{\partial L}{\partial \dot{q}}, \delta q \rangle + \Lambda(q, \dot{q})$ is a constant, whenever q(t) is a solution of the EL equations for Lagrangian L. In this case, the infinitesimal transformation δq is called a **gauge symmetry**, since it does not alter the EL equations of motion, even though it does alter the Lagrangian and thus change the form of the associated Noether conservation law.

Example: This sort of gauge transformation arises, for example, in Maxwell's equations for electromagnetism. For example, the magnetic field $\mathbf{B} = \operatorname{curl} \mathbf{A}$ is invariant under the gauge transformation $\mathbf{A} \to \mathbf{A} + \nabla \phi$, since $\operatorname{curl} \nabla \phi = 0$.

Of course, in Maxwell's equations, we have space-time dependence $\phi : \mathbb{R}^{3,1} \to \mathbb{R}$, but the same principle applies, since Hamilton's principle applies on manifolds in any coordinate system.

3.4 HP for geodesics (covariant derivatives)

• **Geodesics:** When $L = KE = \frac{1}{2}g_q(\dot{q}, \dot{q}) =: \frac{1}{2}||\dot{q}||^2$, the solution q(t) of the EL equations that passes from point q(a) to q(b) is called the *geodesic path* with respect to the metric $g_q : TM \times TM \to \mathbb{R}$. The geodesic represents the path of shortest distance $q(a) \to q(b)$ measured by

$$ds := \sqrt{dq^a g_{ab}(q) dq^b} = \sqrt{g_q(\dot{q}, \dot{q})} dt$$

- **Exercise:** Compute the EL equations for a geodesic with respect to the metric $g_q : TM \times TM \to \mathbb{R}$. That is, compute the EL equations for L = KE.
- **Answer:** The KE Lagrangian is

$$L(q, \dot{q}) = \frac{1}{2} \dot{q}^{b} g_{bc}(q) \dot{q}^{c}.$$

Its partial derivatives are given by

$$\frac{\partial L}{\partial \dot{q}^a} = g_{ac}(q) \dot{q}^c \quad \text{and} \quad \frac{\partial L}{\partial q^a} = \frac{1}{2} \frac{\partial g_{bc}(q)}{\partial q^a} \dot{q}^b \dot{q}^c \,.$$

Consequently, its Euler–Lagrange equations are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^a} - \frac{\partial L}{\partial q^a} = g_{ae}(q)\ddot{q}^e + \frac{\partial g_{ae}(q)}{\partial q^b}\dot{q}^b\dot{q}^e - \frac{1}{2}\frac{\partial g_{be}(q)}{\partial q^a}\dot{q}^b\dot{q}^e = 0.$$

Symmetrising the coefficient of the middle term and contracting with co-metric g^{ca} satisfying $g^{ca}g_{ae} = \delta_e^c$ yields

$$\ddot{q}^c + \Gamma^c_{be}(q)\dot{q}^b\dot{q}^e = 0, \qquad (1)$$

with

$$\Gamma_{be}^{c}(q) = \frac{1}{2}g^{ca} \left[\frac{\partial g_{ae}(q)}{\partial q^{b}} + \frac{\partial g_{ab}(q)}{\partial q^{e}} - \frac{\partial g_{be}(q)}{\partial q^{a}} \right],\tag{2}$$

in which the Γ_{be}^{c} are called the *Christoffel symbols* for the Riemannian metric g_{ab} .

These Euler–Lagrange equations are the *geodesic equations* of a free particle moving in a Riemannian space. They are often written as

$$\ddot{q} + \nabla_{\dot{q}}\dot{q} = 0,$$

in terms of the *covariant derivative* $\nabla_{\dot{q}}$.

The covariant derivative also arises when U(1) gauge symmetry is locally broken, but that will be deferred to another lecture.

• **Alternative viewpoints, IVP vs BVP:** In mechanics the point q(b) is determined at time t = b from the solution q(t) to the initial value problem (IVP) for EL equations with q and \dot{q} specified at the initial time, e.g., at t = a.

It is also possible to re-phrase this as a boundary value problem (BVP) in time, by specifying endpoint positions q(a) and q(b) instead of the initial values of q and \dot{q} . Variational BVPs (sometimes called optimal control problems) are not treated in this course.

3.5 Example - The isoperimetric problem (what Lagrange wrote to Euler about).

This problem is to find the curve between two points (x_1, y_1) and (x_2, y_2) , of specified length, that maximises the area integral $\int_{x_1}^{x_2} y(x) dx$.

In this example the length of the curve is

$$L[y] = \int_{x_1}^{x_2} \sqrt{1 + {y'}^2} dx,$$

which takes the specified value l = const. The area is

$$A[y] = \iint dx \wedge dy = \int_{x_1}^{x_2} y(x) dx.$$

We look for extrema of the modified functional

$$S[y] = \int_{x_1}^{x_2} y dx - \lambda \int_{x_1}^{x_2} (\sqrt{1 + y'^2} dx - l),$$

where λ is a scalar constant (Lagrange multiplier), to be determined. The Euler-Lagrange equation is

$$\lambda \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{y'}{\sqrt{1+y'^2}} \right) + 1 = 0.$$
(3)

Hence, a first integration yields $\frac{y'}{\sqrt{1+y'^2}} = -(x-x_0)/\lambda$, giving the parametric solution, after solving for y'^2 ,

$$x = x_0 \pm \lambda \sin(\theta), \qquad y = y_0 \pm \lambda \cos(\theta),$$
(4)

so $(x - x_0)^2 + (y - y_0)^2 = \lambda^2$ and the extremum is the arc of a circle of radius λ .

The variational problem satisfied by a soap bubble is analogous to the isoperimetric problem. For the soap bubble, the surface area is extremised, holding the volume integral constant. The Lagrange multiplier is the pressure, p.

• **Legendre transform:** $LT: (q, \dot{q}) \in TM \to (q, p) \in T^*M$ defines momentum p as the fibre derivative of L, namely

$$p := \frac{\partial L(q, \dot{q})}{\partial \dot{q}} \in T^*M \qquad (fibre \ derivative).$$

The LT is *invertible* for $\dot{q} = f(q, p)$, provided the Hessian $\partial^2 L(q, \dot{q})/\partial \dot{q} \partial \dot{q}$ has nonzero determinant. Note, dim $T^*M = 2n$.

• In terms of the LT, the **Hamiltonian** $H: T^*M \to \mathbb{R}$ is defined by

$$H(q,p) = \left\langle p, \dot{q} \right\rangle - L(q, \dot{q})$$

in which the expression $\langle p, \dot{q} \rangle$ in this calculation identifies a *pairing* $\langle \cdot, \cdot \rangle : T^*M \times TM \to \mathbb{R}$. Taking the differential of this definition yields

$$dH = \left\langle H_p, dp \right\rangle + \left\langle H_q, dq \right\rangle$$
$$= \left\langle dp, \dot{q} \right\rangle + \left\langle p - L_{\dot{q}}, d\dot{q} \right\rangle - \left\langle L_q, dq \right\rangle$$

from which Hamilton's principle $\delta S = 0$ for $S = \int_a^b \langle p, \dot{q} \rangle - H(q, p) dt$ produces Hamilton's canonical equations on phase space T^*M ,

$$H_p = \dot{q}$$
 and $H_q = -L_q = -\dot{p}$.

• Hamilton's principle $\delta S = 0$ for $S = \int_a^b \langle p, \dot{q} \rangle - H(q, p) dt$ produces Hamilton's canonical equations on phase space T^*M ,

$$H_p = \dot{q}$$
 and $H_q = -L_q = -\dot{p}$.

Exercise: Verify the previous statement. That is, compute the results of the **Phase-space form of Hamilton's principle** on T^*M , given by $\delta S = 0$ with $S = \int_a^b \langle p, \dot{q} \rangle - H(q, p) dt$.

• **Answer:** One computes

$$\begin{split} \delta S &= \delta \int_{a}^{b} \langle p, \dot{q} \rangle - H(q, p) \, dt = \int_{a}^{b} \delta \langle p, \dot{q} \rangle - \delta H(q, p) \, dt \\ &= \int_{a}^{b} \left\langle \delta p, \dot{q} - H_{p} \right\rangle - \left\langle \dot{p} + H_{q}, \delta q \right\rangle dt + \underbrace{\left\langle p, \delta q \right\rangle}_{a}^{b} \\ & \text{Endpoint term} \end{split}$$

Remark: We will return to the endpoint term in formulating Noether's theorem on phase space, that is, on T^*M .

4 Hamilton's equations

4.1 Legendre transform in simple mechanical systems

• Legendre transform: $H(q, p) = \langle p, \dot{q} \rangle - L(q, \dot{q}) = T(p) + V(q) = KE + PE.$

For example,

- 1. Planar isotropic oscillator, $(\mathbf{x}, \mathbf{p}) \in T^* \mathbb{R}^2$: $H = \frac{1}{2m} |\mathbf{p}|^2 + \frac{k}{2} |\mathbf{x}|^2$
- 2. Planar anisotropic oscillator, $(\mathbf{x}, \mathbf{p}) \in T^* \mathbb{R}^2$: $H = \frac{1}{2m} |\mathbf{p}|^2 + \frac{k_1}{2} x_1^2 + \frac{k_2}{2} x_2^2$
- 3. Planar pendulum in polar coordinates, $(\theta, p_{\theta}) \in T^*S^1$: $H = \frac{1}{2mR^2}p_{\theta}^2 + mgR(1 \cos\theta)$
- 4. Planar pendulum, $(\mathbf{x}, \mathbf{p}) \in T^* \mathbb{R}^2$, constrained to $S^1 = \{\mathbf{x} \in \mathbb{R}^2 : 1 |\mathbf{x}|^2 = 0\}$: $H = \frac{1}{2m} |\mathbf{p}|^2 + mg \, \hat{\mathbf{e}}_2 \cdot \mathbf{x} \mu(1 |\mathbf{x}|^2)$
- 5. Charged particle in a magnetic field, $(\mathbf{x}, \mathbf{p}) \in T^* \mathbb{R}^2$: $H = \frac{1}{2m} |\mathbf{p} \frac{e}{c} \mathbf{A}(\mathbf{x})|^2$ $\mathbf{p} := \partial L / \partial \dot{\mathbf{q}} = m \dot{\mathbf{x}} + \frac{e}{c} \mathbf{A}(\mathbf{x}) \in T^* M$

6. Kepler problem, $(r, p_r, \theta, p_\theta) \in T^* \mathbb{R}_+ \times T^* S^1$: $H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} - \frac{GMm}{r}$ with $p_\theta = r^2 \dot{\theta} = const$

- 7. Free motion on a sphere, $(\mathbf{x}, \mathbf{p}) \in T^* \mathbb{R}^3$, constrained to $S^2 = \{\mathbf{x} \in \mathbb{R}^3 : 1 |\mathbf{x}|^2 = 0\}$: $H = \frac{1}{2m} |\mathbf{p}|^2 \mu(1 |\mathbf{x}|^2)$
- 8. Spherical pendulum, $(\mathbf{x}, \mathbf{p}) \in T^* \mathbb{R}^3$, constrained to $S^2 = \{\mathbf{x} \in \mathbb{R}^3 : 1 |\mathbf{x}|^2 = 0\}$: $H = \frac{1}{2m} |\mathbf{p}|^2 + mg \, \hat{\mathbf{e}}_3 \cdot \mathbf{x} \mu(1 |\mathbf{x}|^2)$
- 9. Rotating rigid body, $\Pi \in T^*(SO(3) \simeq \mathfrak{so}(3)^* \simeq \mathbb{R}^3$ $H = \frac{1}{2}\Pi \cdot I^{-1}\Pi$ with $\Pi = \frac{\partial \ell}{\partial \Omega} = I\Omega$
- 10. Heavy top $(\Pi, \Gamma = \in T^*(SO(3) \otimes \mathbb{R}^3) \simeq \mathfrak{so}(3)^* \otimes \mathbb{R}^3 \simeq \mathbb{R}^3 \otimes \mathbb{R}^3$: $H = \frac{1}{2}\Pi \cdot I^{-1}\Pi + mg\Gamma \cdot \chi$

4.2 Canonical Poisson bracket

• The Hamiltonian dynamics of a phase-space function is given by

$$\frac{d}{dt}F(q,p) = \frac{\partial F}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial F}{\partial p}\frac{\partial H}{\partial q} := \{F, H\}$$

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The operation $\{F, H\}$ is called the *canonical Poisson bracket* of F with H on the phase space T^*M . The canonical Poisson bracket operation $\{\cdot, \cdot\}$ is a map among smooth real functions $\mathcal{F}(T^*M) : T^*M \to \mathbb{R}$

$$\{\cdot, \cdot\}: \mathcal{F}(T^*M) \times \mathcal{F}(T^*M) \to \mathcal{F}(T^*M), \qquad (5)$$

so that Hamiltonian dynamics on phase space T^*M is given by $\dot{F} = \{F, H\}$ for any $F \in \mathcal{F}(T^*M)$.

Definition 3 (Poisson bracket). A *Poisson bracket operation* $\{\cdot, \cdot\}$ *is defined by its properties listed below:*

- It is bilinear.
- It is skew-symmetric, $\{F, H\} = -\{H, F\}$.
- It satisfies the **Leibniz rule** (product rule),

$$\{FG, H\} = \{F, H\}G + F\{G, H\},\$$

for the product of any two functions F and G on M.

- It satisfies the Jacobi identity,

$$\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0,$$
(6)

for any three functions F, G and H on M.

Remark. The Leibniz rule associates Poisson brackets with differential operators on smooth functions $F \in \mathcal{F}(T^*M)$. The differential operator or **Hamiltonian vector field** generated by the canonical Poisson bracket with F is

$$X_F := \{ \cdot, F \} = \frac{\partial F}{\partial p} \frac{\partial}{\partial q} - \frac{\partial F}{\partial q} \frac{\partial}{\partial p}$$

- *Exercise:* What is *Noether's theorem for Hamilton's principle in phase-space, on* T^*M ?
- **Answer:** For an infinitesimal transformation $(\delta q, \delta p)$ that induces $\delta L = \delta (\langle p, \dot{q} \rangle H(q, p))$ we have

$$\delta S = \delta \int_{a}^{b} \left\langle p, \dot{q} \right\rangle - H(q, p) \, dt = \int_{a}^{b} \delta \left\langle p, \dot{q} \right\rangle - \delta H(q, p) = \int_{a}^{b} \left\langle \delta p, \dot{q} - H_{p} \right\rangle - \left\langle \dot{p} + H_{q}, \delta q \right\rangle dt + \underbrace{\left\langle p, \delta q \right\rangle}_{\text{Endpoint}}^{b}$$

4.3 Cotangent lift and Noether's theorem on the Hamiltonian side

• Suppose the variations due to the infinitesimal transformations on M take the form $\delta q = \xi_M(q)$. Then the corresponding Hamiltonian for these infinitesimal transformations is

$$J^{\xi} := \left\langle p, \xi_M(q) \right\rangle \quad \text{so that} \quad \delta q = \frac{\partial J^{\xi}}{\partial p} = \xi_M(q) \quad \text{and} \quad \delta p = -\frac{\partial J^{\xi}}{\partial q} = -\xi'_M(q)^T p$$

The last expression is called the *cotangent lift* to T_q^*M of the infinitesimal transformation $q \to q_{\epsilon} = q + \epsilon \xi_M(q)$ on M. The cotangent lift specifies the infinitesimal transformation of $p \in T_q^*M$, given the infinitesimal transformation of $q \in M$.

$$q \to q_{\epsilon} = q + \epsilon \xi_M(q) \text{ on } M \Longrightarrow (q, p) \to (q_{\epsilon}, p_{\epsilon}) = (q + \epsilon \xi_M(q), p - \epsilon \xi'_M(q)^T p) \text{ on } T_q M.$$

The time derivative of $J^{\xi}(q, p)$ is given by

$$\frac{d}{dt}J^{\xi}(q,p) = \frac{\partial J^{\xi}}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial J^{\xi}}{\partial p}\frac{\partial H}{\partial q} = -\frac{\partial H}{\partial p}\delta p - \frac{\partial H}{\partial q}\delta q = -\delta H.$$

In the last step we recalled the (gauge) symmetry of H. This calculation proves the following.

Corollary 4. Noether's theorem for gauge symmetry on the Hamiltonian side implies conservation of J^{ξ} .

The differential operator or **Hamiltonian vector field** generated by the canonical Poisson bracket with J^{ξ} is

$$X_{J^{\xi}} := \{ \cdot, J^{\xi} \} = \frac{\partial J^{\xi}}{\partial p} \frac{\partial}{\partial q} - \frac{\partial J^{\xi}}{\partial q} \frac{\partial}{\partial p} = \xi_M(q) \frac{\partial}{\partial q} - \xi'(q)^T p \frac{\partial}{\partial p} = \delta q \frac{\partial}{\partial q} + \delta p \frac{\partial}{\partial p}.$$

4.4 Example: Angular momentum

Let $G \times M \to M$ with G = SO(3) and $M = \mathbb{R}^3$. That is, $SO(3) \times \mathbb{R}^3 \to \mathbb{R}^3$. Let $q(\epsilon) = O(\epsilon)q(0)$ with $O \in SO(3)$, so that $O^T O = Id$ and $q \in \mathbb{R}^3$. Then the infinitesimal transformation is

$$\delta q := q'(\epsilon)\big|_{\epsilon=0} = \left[O'(\epsilon)q(0)\right]_{\epsilon=0} = \left[O'(\epsilon)O^{-1}(\epsilon)q(\epsilon)\right]_{\epsilon=0} := \widehat{\xi}q = \xi \times q.$$

The Hamiltonian

$$J^{\xi}(q,p) = q \times p \cdot \xi = p \cdot \xi_M(q) = p \cdot \xi \times q$$

generates infinitesimal SO(3) rotations around the vector $\xi \in \mathfrak{so}(3) \simeq \mathbb{R}^3$, as we compute

$$\delta q = \left\{ q, J^{\xi}(q, p) \right\} = \xi \times q(t), \quad \delta p = \left\{ p, J^{\xi}(q, p) \right\} = \xi \times p(t),$$

using the canonical Poisson bracket $\{\cdot, \cdot\}$. Thus, the *cotangent lift* of an infinitesimal rotation of q given by $\xi_M(q) = \xi \times q$ is an infinitesimal rotation of p given by $-\xi'_M(q)^T p = \xi \times p$. These equations imply the following variation for $J(q, p) = q \times p \in \mathfrak{so}(3)^* \simeq \mathbb{R}^3$

$$\delta J = \xi \times J(t)$$
 for $\xi \in \mathfrak{so}(3) \simeq \mathbb{R}^3$ and $J \in \mathfrak{so}(3)^* \simeq \mathbb{R}^3$,

as obtained by using the product rule and the Jacobi identity for the cross product of vectors in \mathbb{R}^3 .

The quantity $J(q, p) = q \times p \in \mathfrak{so}(3)^* \simeq \mathbb{R}^3$ is called the *angular momentum*.

The map $J(q, p) = q \times p : T_q^* M \to \mathfrak{so}(3)^* \simeq \mathbb{R}^3$ is the **cotangent lift momentum map** for the action of the Lie group of spatial rotations G = SO(3) on the manifold $M = \mathbb{R}^3$.

4.5 An angular momentum map that generalises the notion of Poisson brackets for SO(3).

• **Exercise:** Show for vectors $\xi, \eta \in \mathbb{R}^3$ that for angular momentum

$$\left\{J^{\xi}, J^{\eta}\right\} = J^{\xi \times \eta}.$$

• **Answer:** The proof follows by a direct calculation using Jacobi's identity for vector cross products:

$$\left\{J^{\xi}, J^{\eta}\right\} = \left\{J \cdot \xi, J \cdot \eta\right\} = \left\{q \times p \cdot \xi, q \times p \cdot \eta\right\}_{can} = (q \times p) \cdot (\xi \times \eta) = J \cdot (\xi \times \eta) = J^{\xi \times \eta}.$$

Hence, for functions of the angular momentum map J we have

$$\left\{J_k, J_l\right\} = \epsilon_{kl}{}^m J_m \quad \text{and} \quad \left\{F(J), H(J)\right\} = J \cdot \frac{\partial F}{\partial J} \times \frac{\partial H}{\partial J} \quad \text{so that} \quad \frac{dJ}{dt} = \left\{J, H(J)\right\} = -J \times \frac{\partial H}{\partial J}$$

Thus, the angular momentum map $J(q, p) : T^* \mathbb{R}^3 \to \mathbb{R}^3$ is Poisson, which means that $\{F \circ J, H \circ J\} = \{F, H\} \circ J$.

Upon denoting $\mathbf{x} \in \mathbb{R}^3$ the Poisson bracket becomes $\{F, H\} = \nabla C \cdot \nabla F \times \nabla H$ with motion equation $\dot{\mathbf{x}} = -\nabla C \times \nabla H$ where $C(\mathbf{x}) = \frac{1}{2}|\mathbf{x}|^2$. This means the motion takes place on *spheres* along intersections of level sets of C and H.

4.6 An angular momentum map that generalises from SO(3) to other Lie groups.

For any Lie algebra \mathfrak{g} , the cotangent lift momentum map satisfies

$$\left\{J^{\xi},\,J^{\eta}\right\}=J^{[\xi,\,\eta]},$$

where $[\xi, \eta] = -[\eta, \xi]$ is the Lie bracket between $\xi, \eta \in \mathfrak{g}$, which we also denote as $[\xi, \eta] =: \pm \mathrm{ad}_{\xi} \eta$. (We'll discuss the sign later.)

The corresponding $\boxed{Lie-Poisson \ bracket}$ is

$$\left\{F(J), H(J)\right\} = \mp \left\langle J, \left[\frac{\partial H}{\partial J}, \frac{\partial F}{\partial J}\right]\right\rangle_{\mathfrak{g}^* \times \mathfrak{g}} = \mp \left\langle J, \operatorname{ad}_{\partial H/\partial J}\frac{\partial F}{\partial J}\right\rangle_{\mathfrak{g}^* \times \mathfrak{g}} = \mp \left\langle \operatorname{ad}_{\partial H/\partial J}^* J, \frac{\partial F}{\partial J}\right\rangle_{\mathfrak{g}^* \times \mathfrak{g}}$$

Consequently, for Lie-Poisson systems, the dynamics of the cotangent lift momentum map is governed by

$$\frac{dJ}{dt} = \left\{ J, H(J) \right\} = \mp \operatorname{ad}_{\partial H/\partial J}^* J.$$

This generalises the angular momentum map Exercise for SO(3) to arbitrary Lie groups and their Lie algebras.

The proof follows by a direct calculation using the Lie-Poisson bracket:

$$\left\{J^{\xi}, J^{\eta}\right\} = \left\{\left\langle J, \xi\right\rangle, \left\langle J, \eta\right\rangle\right\} = \left\langle J, [\xi, \eta]\right\rangle = J^{[\xi, \eta]}$$

where we have used $\xi = \xi^j e_j$, $\eta = \eta^k e_k$ and $[e_j, e_k] = c_{jk}{}^i e_i$ to compute

$$[\xi,\eta] = [\xi^j e_j, \,\eta^k e_k] = \xi^j [e_j, \,e_k] \eta^k = \xi^j \eta^k c_{jk}{}^i e_i = [\xi,\eta]^i e_i \,.$$

Hence, for functions of the $\boxed{momentum map} J$ we now have the result that

$$\left\{J_k, J_l\right\} = c_{kl}{}^m J_m \quad \text{and} \quad \left\{F(J), H(J)\right\} = \mp \left\langle J, \operatorname{ad}_{\partial H/\partial J} \frac{\partial F}{\partial J}\right\rangle_{\mathfrak{g}^* \times \mathfrak{g}} \text{ so } \left[\frac{dJ/dt}{dt} = \left\{J, H(J)\right\} = \mp \operatorname{ad}_{\partial H/\partial J}^* J_{\mathcal{G}} \right\}$$

Thus, the momentum map $J(q,p): T^*M \to \mathfrak{g}^*$ is Poisson, which means that $\{F \circ J, H \circ J\} = \{F, H\} \circ J$.

The Lagrangian counterpart of Lie–Poisson theory is **Euler–Poincaré** theory, from Poincaré [1901] that we will study next.

5 Example: Motion under gravity of a particle on a sphere



Figure 7: This is a sketch of the spherical pendulum, described as motion under gravity of a particle on a sphere.

The Lagrangian for this type of motion would be the difference of kinetic minus potential energy for a particle of unit mass,

$$L(\mathbf{x}, \dot{\mathbf{x}}) = \frac{1}{2} |\dot{\mathbf{x}}|^2 - g \hat{\mathbf{e}}_3 \cdot \mathbf{x} \text{ for } (\mathbf{x}, \dot{\mathbf{x}}) \in T \mathbb{R}^3.$$

The motion on the sphere comprises *rotation*, which may be written as the action of the rotation group on a vector in \mathbb{R}^3 , by setting

$$\mathbf{x}(t) = O(t)\mathbf{x}_0, \quad \dot{\mathbf{x}}(t) = \dot{O}(t)\mathbf{x}_0 \quad \text{for} \quad (O, \dot{O}) \in TSO(3),$$
(7)

where $\mathbf{x}_0 = \mathbf{x}(0)$ is the initial position of the particle. Relations (7) replace motion on the sphere S^2 by motion on the group SO(3), whose action $SO(3) \times \mathbb{R}^3 \to \mathbb{R}^3$ on vectors in \mathbb{R}^3 leaves their Euclidean lengths invariant and, thus, preserves the sphere,

$$|\mathbf{x}(t)|^2 = \operatorname{tr}\left[\left(O(t)\mathbf{x}_0\right)^T \left(O(t)\mathbf{x}_0\right)\right] = \operatorname{tr}\left(\mathbf{x}_0^T O^T O \mathbf{x}_0\right) = |\mathbf{x}_0|^2 \quad \text{since} \quad O^T O = O^{-1} O = \operatorname{Id}.$$

That is, the SO(3) rotations (7) of vectors in \mathbb{R}^3 are summoned for this problem, because they map the sphere into itself.¹

The kinetic and potential energies of the particle on the sphere may be written on the group SO(3) by using the transformation (7). In these terms, the kinetic energy is given by

$$\begin{aligned} |\dot{\mathbf{x}}(t)|^2 &= \operatorname{tr} \left[\left(\dot{O}(t) \mathbf{x}_0 \right)^T \left(\dot{O}(t) \mathbf{x}_0 \right) \right] \\ &= \operatorname{tr} \left(\mathbf{x}_0^T \dot{O}^T \dot{O} \mathbf{x}_0 \right) \\ &= \operatorname{tr} \left(\mathbf{x}_0^T \dot{O}^T O O^{-1} \dot{O} \mathbf{x}_0 \right) \\ &= \operatorname{tr} \left(\mathbf{x}_0^T (O^T \dot{O})^T (O^{-1} \dot{O}) \mathbf{x}_0 \right) \\ &= \operatorname{tr} \left(\mathbf{x}_0^T \widehat{\Omega}^T \widehat{\Omega} \mathbf{x}_0 \right) \quad \text{on defining} \quad \widehat{\Omega} := O^{-1} \dot{O} \quad \text{and using} \quad O^T = O^{-1} \\ &= |\widehat{\Omega} \mathbf{x}_0|^2 \\ &= |\mathbf{\Omega} \times \mathbf{x}_0|^2 \quad \text{by applying the} \boxed{hat map} \quad \widehat{\Omega} = \mathbf{\Omega} \times , \end{aligned}$$

and the potential energy is given by

$$g\hat{\mathbf{e}}_{3} \cdot \mathbf{x} = g \operatorname{tr} (\hat{\mathbf{e}}_{3}^{T} \mathbf{x})$$

= $g \operatorname{tr} (\hat{\mathbf{e}}_{3}^{T} O \mathbf{x}_{0})$
= $g \operatorname{tr} (O^{T} \hat{\mathbf{e}}_{3})^{T} \mathbf{x}_{0}$
= $g \operatorname{tr} (\mathbf{\Gamma}^{T} \mathbf{x}_{0})$ with $\mathbf{\Gamma} := O^{-1} \hat{\mathbf{e}}_{3}$
= $g \mathbf{\Gamma} \cdot \mathbf{x}_{0}$

¹ *Exercises* What about other such motions?

⁽¹⁾ Consider particle motion under gravity on a triaxial ellipsoid, or on a hyperboloid. What Lie groups are summoned in those cases?

⁽²⁾ What if the particle had an electrical charge and the motion under gravity was taking place in a constant vertical magnetic field?

We now substitute the transformed (constrained) Lagrangian into Hamilton's principle,

$$0 = \delta S = \delta \int_{a}^{b} \left(\frac{1}{2}|\widehat{\Omega}\mathbf{x}_{0}|^{2} - g\,\mathbf{\Gamma}\cdot\mathbf{x}_{0}\right) dt = \delta \int_{a}^{b} \left(\frac{1}{2}|\mathbf{\Omega}\times\mathbf{x}_{0}|^{2} - g\,\mathbf{\Gamma}\cdot\mathbf{x}_{0}\right) dt$$

A computation with $\widehat{\Omega} := O^{-1}\dot{O}$, $\widehat{\Xi} := O^{-1}\delta O$ and $\Gamma := O^{-1}\hat{\mathbf{e}}_3$ yields the *variational identities*

$$\delta \widehat{\Omega} = \frac{d\widehat{\Xi}}{dt} + [\widehat{\Omega}, \widehat{\Xi}] \text{ or } \delta \Omega = \dot{\Xi} + \Omega \times \Xi \text{ and } \delta \Gamma := -\widehat{\Xi}\Gamma = -\Xi \times \Gamma$$

Then we find

$$0 = \delta S = \int_{a}^{b} \left(\mathbf{\Omega} \times \mathbf{x}_{0} \cdot \delta \mathbf{\Omega} \times \mathbf{x}_{0} - g \, \delta \mathbf{\Gamma} \cdot \mathbf{x}_{0} \right) dt$$
$$= \int_{a}^{b} \underbrace{\mathbf{x}_{0} \times (\mathbf{\Omega} \times \mathbf{x}_{0})}_{=: \mathbf{\Pi}} \cdot \delta \mathbf{\Omega} - g \, \mathbf{x}_{0} \cdot \delta \mathbf{\Gamma} \, dt$$
$$= \int_{a}^{b} \mathbf{\Pi} \cdot \left(\dot{\mathbf{\Xi}} + \mathbf{\Omega} \times \mathbf{\Xi} \right) + g \, \mathbf{x}_{0} \cdot \mathbf{\Xi} \times \mathbf{\Gamma} \, dt$$
$$= \int_{a}^{b} \left(- \dot{\mathbf{\Pi}} - \mathbf{\Omega} \times \mathbf{\Pi} + g \, \mathbf{\Gamma} \times \mathbf{x}_{0} \right) \cdot \mathbf{\Xi} \, dt + \left[\mathbf{\Pi} \cdot \mathbf{\Xi} \right]_{a}^{b}$$

This yields the Eulr-Poincaré equation for the *body angular momentum* Π ,

$$\dot{\mathbf{\Pi}} + \mathbf{\Omega} \times \mathbf{\Pi} = -g\,\mathbf{\Gamma} \times \mathbf{x}_0 \quad \text{with} \quad \mathbf{\Pi} := \mathbf{x}_0 \times (\mathbf{\Omega} \times \mathbf{x}_0) = \mathbf{\Omega} |\mathbf{x}_0|^2 - \mathbf{x}_0 (\mathbf{x}_0 \cdot \mathbf{\Omega}) =: \mathbb{I}(\mathbf{x}_0)\,\mathbf{\Omega}\,, \tag{8}$$

where I is called the moment of inertia of the particle on the sphere. Finally, from its definition, $\Gamma := O^{-1}(t)\hat{\mathbf{e}}_3$ satisfies

$$\dot{\boldsymbol{\Gamma}} := -\,\widehat{\boldsymbol{\Omega}}\boldsymbol{\Gamma} = -\,\boldsymbol{\Omega} \times \boldsymbol{\Gamma}\,. \tag{9}$$

This system conserves the quantities

$$E := \frac{1}{2} \mathbf{\Pi} \cdot \mathbb{I}^{-1}(\mathbf{x}_0) \mathbf{\Pi} + g \, \mathbf{\Gamma} \cdot \mathbf{x}_0, \quad C_1 := \mathbf{\Gamma} \cdot \mathbf{\Pi} = O^{-1} \hat{\mathbf{e}}_3 \cdot \mathbf{\Pi} = \hat{\mathbf{e}}_3 \cdot O(t) \mathbf{\Pi} \quad \text{and} \quad C_2 := |\mathbf{\Gamma}|^2 = 1.$$

We could now Legendre transform and find the Hamiltonian structure, too.

But first we need a theorem about how motion on a sphere relates to rigid body motion.

Theorem 5. When expressed on the Lie group SO(3), spherical motion maps isomorphically to the motion of a rigid body \mathcal{B} , by integration of Hamilton's principle for a single particle mass element over the mass density distribution $\rho(\mathbf{x}_0)$ of the entire body.

Proof. In rigid body motion, every particle in a body \mathcal{B} is undergoing a rotation and the relative positions of any two particle mass elements are fixed in the frame of the body's motion. Therefore, it is not unexpected that the motion of the rigid body could be expressed entirely on the rotation group.

The explicit isomorphism between motion of a particle on a sphere and rigid body motion results from applying the linear integral operator

$$\int_{\mathcal{B}} \rho(\mathbf{x}_0)(\,\cdot\,)\,d^3\mathbf{x}_0$$

to the transformed (constrained) Lagrangian in Hamilton's principle,

$$\ell\left(\mathbf{\Omega},\mathbf{\Gamma}\right) = \int_{\mathcal{B}} \rho(\mathbf{x}_0) \left(\frac{1}{2} |\mathbf{\Omega} \times \mathbf{x}_0|^2 - g \,\mathbf{\Gamma} \cdot \mathbf{x}_0\right) d^3 \mathbf{x}_0 = \frac{1}{2} \mathbf{\Omega} \cdot \mathbb{I} \mathbf{\Omega} - g \,\mathbf{\Gamma} \cdot \boldsymbol{\chi}$$

where, as before, \mathbf{x}_0 is the initial position of any given particle and one introduces the following definitions

$$\begin{split} M &= \int_{\mathcal{B}} \rho(\mathbf{x}_0) \, d^3 \mathbf{x}_0 \quad (\text{mass}) \,, \\ \boldsymbol{\chi} &= M^{-1} \int_{\mathcal{B}} \rho(\mathbf{x}_0) \mathbf{x}_0 \, d^3 \mathbf{x}_0 \quad (\text{centre of mass}) \,, \\ \mathbb{I} &= \int_{\mathcal{B}} \rho(\mathbf{x}_0) \left(|\mathbf{x}_0|^2 \, \text{Id} - \mathbf{x}_0 \otimes \mathbf{x}_0 \right) d^3 \mathbf{x}_0 \quad (\text{moment of inertia}) \,. \end{split}$$

5.1 Lie–Poisson brackets for the spherical pendulum

The Hamiltonian

$$H(\mathbf{\Pi}, \mathbf{\Gamma}) := \mathbf{\Pi} \cdot \mathbf{\Omega} - \ell(\mathbf{\Omega}, \mathbf{\Gamma}) = \frac{1}{2} \mathbf{\Pi} \cdot \mathbb{I}^{-1} \mathbf{\Pi} + g \, \mathbf{\Gamma} \cdot \mathbf{x}_0 \tag{10}$$

recovers equations (8) and (22) for the spherical pendulum in Poisson bracket form, as

$$\dot{\mathbf{\Pi}} = \{\mathbf{\Pi}, H\} = \mathbf{\Pi} \times \mathbb{I}^{-1}\mathbf{\Pi} + \mathbf{\Gamma} \times g \mathbf{x}_{0} = \mathbf{\Pi} \times \frac{\partial H}{\partial \mathbf{\Pi}} + \mathbf{\Gamma} \times \frac{\partial H}{\partial \mathbf{\Gamma}},$$

$$\dot{\mathbf{\Gamma}} = \{\mathbf{\Gamma}, H\} = \mathbf{\Gamma} \times \mathbb{I}^{-1}\mathbf{\Pi} = \mathbf{\Gamma} \times \frac{\partial h}{\partial \mathbf{\Pi}}.$$
(11)

In matrix form this is

$$\begin{bmatrix} \dot{\mathbf{\Pi}} \\ \dot{\mathbf{\Gamma}} \end{bmatrix} = \begin{bmatrix} \mathbf{\Pi} \times & \mathbf{\Gamma} \times \\ \mathbf{\Gamma} \times & 0 \end{bmatrix} \begin{bmatrix} \mathbb{I}^{-1} \mathbf{\Pi} \\ g \mathbf{x}_0 \end{bmatrix} \implies \begin{bmatrix} \dot{\mathbf{\Pi}} \\ \dot{\mathbf{\Gamma}} \end{bmatrix} = \begin{bmatrix} \mathbf{\Pi} \times & \mathbf{\Gamma} \times \\ \mathbf{\Gamma} \times & 0 \end{bmatrix} \begin{bmatrix} \partial H / \partial \mathbf{\Pi} \\ \partial H / \partial \mathbf{\Gamma} \end{bmatrix}$$
(12)

The proposed *Poisson bracket for the spherical-pendulum* is

$$\frac{d}{dt}F(\mathbf{\Pi},\mathbf{\Gamma}) = \begin{bmatrix} \frac{\partial F}{\partial \mathbf{\Pi}} \end{bmatrix}^T \begin{bmatrix} \mathbf{\Pi} \times & \mathbf{\Gamma} \times \\ \mathbf{\Gamma} \times & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial \mathbf{\Pi}} \\ \frac{\partial H}{\partial \mathbf{\Gamma}} \end{bmatrix} = \{F, H\}(\mathbf{\Pi},\mathbf{\Gamma}) = -\mathbf{\Pi} \cdot \frac{\partial F}{\partial \mathbf{\Pi}} \times \frac{\partial H}{\partial \mathbf{\Pi}} - \mathbf{\Gamma} \cdot \left(\frac{\partial F}{\partial \mathbf{\Pi}} \times \frac{\partial H}{\partial \mathbf{\Gamma}} - \frac{\partial H}{\partial \mathbf{\Pi}} \times \frac{\partial F}{\partial \mathbf{\Gamma}} \right).$$
(13)

For now, we simply assert that equation (13) defines a proper Poisson bracket. Later developments of the course will associate such brackets to the invariance properties of the Lagrangian in Hamilton's principle. This leads to the Euler-Poincaré theory which follows from [Po1901].

For now, we only point out that the Lie algebra of the special Euclidean group SE(3) in three dimensions is $\mathfrak{se}(3) = \mathbb{R}^3 \times \mathbb{R}^3$, which possesses the Lie bracket

$$[(\boldsymbol{\xi}, \mathbf{u}), (\boldsymbol{\eta}, \mathbf{v})] = (\boldsymbol{\xi} \times \boldsymbol{\eta}, \boldsymbol{\xi} \times \mathbf{v} - \boldsymbol{\eta} \times \mathbf{u}).$$
(14)

If we identify the dual space $\mathfrak{se}(3)^* \simeq \mathbb{R}^3 \times \mathbb{R}^3$ with pairs (Π, Γ) and Lie algebra elements with pairs $(\boldsymbol{\xi}, \mathbf{u}) = (\partial F / \partial \Pi, \partial F / \partial \Gamma)$ and $(\boldsymbol{\eta}, \mathbf{v}) = (\partial H / \partial \Pi, \partial H / \partial \Gamma)$; then we may write the Poisson bracket (13) as

$$\{F, H\}(\Pi, \Gamma) = -\left\langle (\Pi, \Gamma), \left[\left(\frac{\partial F}{\partial \Pi}, \frac{\partial F}{\partial \Gamma} \right), \left(\frac{\partial H}{\partial \Pi}, \frac{\partial H}{\partial \Gamma} \right) \right] \right\rangle = -\Pi \cdot \left(\frac{\partial F}{\partial \Pi} \times \frac{\partial H}{\partial \Pi} \right) - \Gamma \cdot \left(\frac{\partial F}{\partial \Pi} \times \frac{\partial H}{\partial \Gamma} - \frac{\partial H}{\partial \Pi} \times \frac{\partial F}{\partial \Gamma} \right).$$
(15)

Being dual to a Lie algebra means being a linear functional of a set whose bracket satisfies the Jacobi identity. From this duality, the Poisson bracket (15) will also satisfy the Jacobi identity.

5.2 The *elastic* spherical pendulum

The Lagrangian for this type of motion would be the difference of kinetic minus potential energy for a particle of unit mass,

$$L(\mathbf{x}, \dot{\mathbf{x}}) = \frac{m}{2} |\dot{\mathbf{x}}|^2 - mg \,\hat{\mathbf{e}}_3 \cdot \mathbf{x} - \frac{k}{2} \left(|\mathbf{x}| - |\mathbf{x}_0| \right)^2 \quad \text{for} \quad (\mathbf{x}, \dot{\mathbf{x}}) \in T\mathbb{R}^3,$$
(16)

for constants m, k and g. The motion comprises *rotation and stretching*, which may be written as the action of the scale-rotation group on a vector in \mathbb{R}^3 , by setting

$$\mathbf{x}(t) = R(t)O(t)\mathbf{x}_0, \quad \text{so that} \quad \dot{\mathbf{x}}(t) = \dot{R}(t)O(t)\mathbf{x}_0 + R(t)\dot{O}(t)\mathbf{x}_0 \quad \text{for} \quad (O, \dot{O}) \in TSO(3) \quad \text{and} \quad (R, \dot{R}) \in T\mathbb{R},$$
(17)

where $\mathbf{x}_0 = \mathbf{x}(0)$ is the initial position of the particle, for R(0) = 1. Relations (17) encode the motion as taking place on the scale-rotation group $SO(3) \times \mathbb{R}$, whose action $(SO(3) \times \mathbb{R}) \times \mathbb{R}^3 \to \mathbb{R}^3$ on vectors in \mathbb{R}^3 rotates them and scales their Euclidean lengths,

$$|\mathbf{x}(t)|^{2} = \operatorname{tr}\left[\left(R(t)O(t)\mathbf{x}_{0}\right)^{T}\left(R(t)O(t)\mathbf{x}_{0}\right)\right] = R^{2}(t)\operatorname{tr}\left(\mathbf{x}_{0}^{T}O^{T}O\mathbf{x}_{0}\right) = R^{2}(t)|\mathbf{x}_{0}|^{2} \quad \text{since} \quad O^{T}O = O^{-1}O = \operatorname{Id}.$$

That is, this problem summons the scale-rotation group $SO(3) \times \mathbb{R}$ acting on vectors in \mathbb{R}^3 .

As for the spherical pendulum, we define

$$\widehat{\Omega} := O^{-1}\dot{O} = \mathbf{\Omega} \times, \quad \mathbf{\Gamma} := O^{-1}\widehat{\mathbf{e}}_3 \quad \text{and note that} \quad \dot{\mathbf{\Gamma}} := -\widehat{\Omega}\mathbf{\Gamma} = -\mathbf{\Omega} \times \mathbf{\Gamma}.$$

Consequently, our Lagrangian (16) for the spring-pendulum, or swinging spring, can be written as

$$\ell(\mathbf{\Omega}, \mathbf{\Gamma}, R, \dot{R}) = \frac{m}{2} \dot{R}^2 |\mathbf{x}_0|^2 + \frac{m}{2} R^2 |\widehat{\mathbf{\Omega}} \mathbf{x}_0|^2 - mg R \mathbf{\Gamma} \cdot \mathbf{x}_0 - \frac{k}{2} (|\mathbf{x}| - |\mathbf{x}_0|)^2$$

= $\frac{m}{2} \dot{R}^2 |\mathbf{x}_0|^2 + \frac{m}{2} R^2 |\mathbf{\Omega} \times \mathbf{x}_0|^2 - mg R \mathbf{\Gamma} \cdot \mathbf{x}_0 - \frac{k}{2} (R-1)^2 |\mathbf{x}_0|^2$
= $\frac{m}{2} \dot{R}^2 |\mathbf{x}_0|^2 + \frac{m}{2} R^2 \mathbf{\Omega} \cdot \mathbb{I}(\mathbf{x}_0) \mathbf{\Omega} - mg R \mathbf{\Gamma} \cdot \mathbf{x}_0 - \frac{k}{2} (R-1)^2 |\mathbf{x}_0|^2$

with $\mathbb{I}(\mathbf{x}_0) = |\mathbf{x}_0|^2 \operatorname{Id} - \mathbf{x}_0 \otimes \mathbf{x}_0.$

As before, a computation with $\widehat{\Omega} := O^{-1}\dot{O} = \mathbf{\Omega} \times$, $\widehat{\Xi} := O^{-1}\delta O = \mathbf{\Xi} \times$ and $\mathbf{\Gamma} := O^{-1}\hat{\mathbf{e}}_3$ yields the *variational identities*

$$\delta\widehat{\Omega} = \frac{d\widehat{\Xi}}{dt} + [\widehat{\Omega}, \widehat{\Xi}] \quad \text{or} \quad \delta\Omega = \dot{\Xi} + \Omega \times \Xi \quad \text{and} \quad \delta\Gamma := -\widehat{\Xi}\Gamma = -\Xi \times \Gamma.$$
(18)

Then we find by following the calculation for the spherical pendulum, modulo terms involving R that

$$\frac{d}{dt}\frac{\delta\ell}{\delta\Omega} = \frac{\delta\ell}{\delta\Omega} \times \Omega + \frac{\delta\ell}{\delta\Gamma} \times \Gamma \quad \text{and} \quad \frac{d}{dt}\frac{\delta\ell}{\delta\dot{R}} = \frac{\delta\ell}{\delta R} \quad \text{for} \quad \frac{\delta\ell}{\delta\Omega} =: \Pi = mR^2 \mathbb{I}\Omega \quad \text{and} \quad \frac{\delta\ell}{\delta\dot{R}} = m\dot{R}|\mathbf{x}_0|^2 \,. \tag{19}$$

Namely, this Hamilton's principle yields three types of equations :

1. the **Euler-Poincaré equation** for the body angular momentum $\Pi(t)$,

$$\dot{\mathbf{\Pi}} + \mathbf{\Omega} \times \mathbf{\Pi} = mg \, R \, \mathbf{x}_0 \times \mathbf{\Gamma} \quad \text{with} \quad \mathbf{\Pi} := \frac{\delta \ell}{\delta \mathbf{\Omega}} =: m R^2(t) \mathbb{I}(\mathbf{x}_0) \, \mathbf{\Omega} \,, \tag{20}$$

with gravitational torque scaled by the factor R(t),

2. the **Euler-Lagrange equation** for the scale factor of the pendulum length, R(t),

$$|\mathbf{x}_0|^2 \ddot{R} = R |\mathbf{\Omega} \times \mathbf{x}_0|^2 - g \,\mathbf{\Gamma} \cdot \mathbf{x}_0 - \frac{k}{m} \left(R - 1 \right) |\mathbf{x}_0|^2 \,, \tag{21}$$

with the effects of centrifugal, gravitational and spring restoring forces, and

3. the *auxiliary equation for* Γ

$$\dot{\mathbf{\Gamma}} = -\mathbf{\Omega} \times \mathbf{\Gamma} \,, \tag{22}$$

obtained from its definition.

The system (20)-(22) conserves the total energy and the vertical component of spatial angular momentum. Namely, it conserves the quantities

$$E := \frac{m}{2} \dot{R}^2 |\mathbf{x}_0|^2 + \frac{m}{2} \mathbf{\Pi} \cdot \mathbb{I}^{-1}(\mathbf{x}_0) \mathbf{\Pi} + mg \, \mathbf{\Gamma} \cdot \mathbf{x}_0 + \frac{k}{2} (R-1)^2 |\mathbf{x}_0|^2 ,$$

$$C_1 := \mathbf{\Gamma} \cdot \mathbf{\Pi} = O^{-1} \hat{\mathbf{e}}_3 \cdot \mathbf{\Pi} = \hat{\mathbf{e}}_3 \cdot O(t) \mathbf{\Pi} \quad \text{and} \quad C_2 := |\mathbf{\Gamma}|^2 = 1 .$$
(23)

6 Spherical pendulum as a constrained system

6.1 Formulation

A spherical pendulum of unit length swings from a fixed point of support under the constant acceleration of gravity g (Figure 8). This motion is equivalent to a particle of unit mass moving on the surface of the unit sphere S^2 under the influence of the gravitational (linear) potential V(z) with $z = \hat{\mathbf{e}}_3 \cdot \mathbf{x}$. The only forces acting on the mass are the reaction from the sphere and gravity. This system is treated as an enhanced coursework example by using spherical polar coordinates and the traditional methods of Newton, Lagrange and Hamilton. The present section treats this problem more geometrically.

In this section, the equations of motion for the spherical pendulum will be derived according to the approaches of Lagrange and Hamilton on the tangent bundle TS^2 of $S^2 \in \mathbb{R}^3$:

$$TS^{2} = \left\{ (\mathbf{x}, \dot{\mathbf{x}}) \in T\mathbb{R}^{3} \simeq \mathbb{R}^{6} \mid 1 - |\mathbf{x}|^{2} = 0, \, \mathbf{x} \cdot \dot{\mathbf{x}} = 0 \right\}.$$

$$\tag{24}$$

After the Legendre transformation to the Hamiltonian side, the canonical equations will be transformed to quadratic variables that are invariant under S^1 rotations about the vertical axis. This is the **quotient map** for the spherical pendulum.

Then the **Nambu bracket** in \mathbb{R}^3 will be found in these S^1 quadratic invariant variables and the equations will be reduced to the **orbit manifold**, which is the zero level set of a distinguished function called the **Casimir function** for this bracket. On the intersections of the Hamiltonian with the orbit manifold, the reduced equations for the spherical pendulum will simplify to the equations of a quadratically nonlinear oscillator.

The solution for the motion of the spherical pendulum will be finished by finding expressions for its geometrical and dynamical phases.

The constrained Lagrangian We begin with the Lagrangian $L(\mathbf{x}, \dot{\mathbf{x}}) : T\mathbb{R}^3 \to \mathbb{R}$ given by

$$L(\mathbf{x}, \dot{\mathbf{x}}) = \frac{1}{2} |\dot{\mathbf{x}}|^2 - g \hat{\mathbf{e}}_3 \cdot \mathbf{x} - \frac{1}{2} \mu (1 - |\mathbf{x}|^2),$$
(25)

in which the Lagrange multiplier μ constrains the motion to remain on the sphere S^2 by enforcing $(1 - |\mathbf{x}|^2) = 0$ when it is varied in Hamilton's principle. The corresponding Euler-Lagrange equation is

$$\ddot{\mathbf{x}} = -g\hat{\mathbf{e}}_3 + \mu\mathbf{x}\,.\tag{26}$$



Figure 8: Spherical pendulum moving under gravity on TS^2 in \mathbb{R}^3 .

Equation (26) preserves both of the TS^2 relations $1 - |\mathbf{x}|^2 = 0$ and $\mathbf{x} \cdot \dot{\mathbf{x}} = 0$, provided the Lagrange multiplier is given by

$$\mu = g\hat{\mathbf{e}}_3 \cdot \mathbf{x} - |\dot{\mathbf{x}}|^2 \,. \tag{27}$$

Remark 6. In Newtonian mechanics, the motion equation obtained by substituting (27) into (26) may be interpreted as

$$\ddot{\mathbf{x}} = \mathbf{F} \cdot (\mathrm{Id} - \mathbf{x} \otimes \mathbf{x}) - |\dot{\mathbf{x}}|^2 \mathbf{x},$$

where $\mathbf{F} = -g\hat{\mathbf{e}}_3$ is the force exerted by gravity on the particle,

$$\mathbf{T} = \mathbf{F} \cdot (\mathrm{Id} - \mathbf{x} \otimes \mathbf{x})$$

is its component tangential to the sphere and, finally, $-|\dot{\mathbf{x}}|^2 \mathbf{x}$ is the centripetal force for the motion to remain on the sphere.

 S^1 symmetry and Noether's theorem The Lagrangian in (25) is invariant under S^1 rotations about the vertical axis, whose infinitesimal generator is $\delta \mathbf{x} = \hat{\mathbf{e}}_3 \times \mathbf{x}$. Noether's theorem tell us that each smooth symmetry of the Lagrangian in which an action principle implies a conservation law for its Euler-Lagrange equations. It implies in this case that Equation (26) conserves

$$J_3(\mathbf{x}, \dot{\mathbf{x}}) = \dot{\mathbf{x}} \cdot \delta \mathbf{x} = \mathbf{x} \times \dot{\mathbf{x}} \cdot \hat{\mathbf{e}}_3, \qquad (28)$$

which is the angular momentum about the vertical axis.

Legendre transform and canonical equations The fibre derivative of the Lagrangian L in (25) is

$$\mathbf{y} = \frac{\partial L}{\partial \dot{\mathbf{x}}} = \dot{\mathbf{x}} \,. \tag{29}$$

The variable \mathbf{y} will be the momentum canonically conjugate to the radial position \mathbf{x} , after the Legendre transform to the corresponding Hamiltonian,

$$H(\mathbf{x}, \mathbf{y}) = \frac{1}{2} |\mathbf{y}|^2 + g \hat{\mathbf{e}}_3 \cdot \mathbf{x} + \frac{1}{2} (g \hat{\mathbf{e}}_3 \cdot \mathbf{x} - |\mathbf{y}|^2) (1 - |\mathbf{x}|^2), \qquad (30)$$

whose canonical equations on $(1 - |\mathbf{x}|^2) = 0$ are

$$\dot{\mathbf{x}} = \mathbf{y} \quad \text{and} \quad \dot{\mathbf{y}} = -g\hat{\mathbf{e}}_3 + (g\hat{\mathbf{e}}_3 \cdot \mathbf{x} - |\mathbf{y}|^2)\mathbf{x}.$$
 (31)

This Hamiltonian system on $T^*\mathbb{R}^3$ admits TS^2 as an invariant manifold, provided the initial conditions satisfy the defining relations for TS^2 in (24). On TS^2 , Equations (31) conserve the energy

$$E(\mathbf{x}, \mathbf{y}) = \frac{1}{2} |\mathbf{y}|^2 + g \hat{\mathbf{e}}_3 \cdot \mathbf{x}$$
(32)

and the vertical angular momentum

$$J_3(\mathbf{x},\mathbf{y}) = \mathbf{x} \times \mathbf{y} \cdot \mathbf{\hat{e}}_3$$

Under the (\mathbf{x}, \mathbf{y}) canonical Poisson bracket, the angular momentum component J_3 generates the Hamiltonian vector field

$$X_{J_3} = \{ \cdot, J_3 \} = \frac{\partial J_3}{\partial \mathbf{y}} \cdot \frac{\partial}{\partial \mathbf{x}} - \frac{\partial J_3}{\partial \mathbf{x}} \cdot \frac{\partial}{\partial \mathbf{y}}$$

= $\hat{\mathbf{e}}_3 \times \mathbf{x} \cdot \frac{\partial}{\partial \mathbf{x}} + \hat{\mathbf{e}}_3 \times \mathbf{y} \cdot \frac{\partial}{\partial \mathbf{y}},$ (33)

for infinitesimal rotations about the vertical axis $\hat{\mathbf{e}}_3$. Because of the S^1 symmetry of the Hamiltonian in (30) under these rotations, we have the conservation law,

$$J_3 = \{J_3, H\} = X_{J_3}H = 0.$$

6.2 Lie symmetry reduction

Algebra of invariants To take advantage of the S^1 symmetry of the spherical pendulum, we transform to S^1 -invariant quantities. A convenient choice of basis for the algebra of polynomials in (\mathbf{x}, \mathbf{y}) that are S^1 -invariant under rotations about the third axis is chosen as

Quotient map The transformation defined by

$$\pi: (\mathbf{x}, \mathbf{y}) \to \{\sigma_j(\mathbf{x}, \mathbf{y}), \ j = 1, \dots, 6\}$$
(34)

is the *quotient map* $T\mathbb{R}^3 \to \mathbb{R}^6$ for the spherical pendulum. Each of the fibres of the quotient map π is an S^1 orbit generated by the Hamiltonian vector field X_{J_3} in (33).

The six S^1 invariants that define the quotient map in (34) for the spherical pendulum satisfy the cubic algebraic relation

$$\sigma_5^2 + \sigma_6^2 = \sigma_4(\sigma_3 - \sigma_2^2).$$
(35)

They also satisfy the positivity conditions

$$\sigma_4 \ge 0, \quad \sigma_3 \ge \sigma_2^2. \tag{36}$$

In these variables, the defining relations (24) for TS^2 become

$$\sigma_4 + \sigma_1^2 = 1$$
 and $\sigma_5 + \sigma_1 \sigma_2 = 0$. (37)

Perhaps not unexpectedly, since TS^2 is invariant under the S^1 rotations, it is also expressible in terms of S^1 invariants. The three relations in Equations (35)–(37) will define the orbit manifold for the spherical pendulum in \mathbb{R}^6 .

Reduced space and orbit manifold in \mathbb{R}^3 On TS^2 , the variables $\sigma_j(\mathbf{x}, \mathbf{y})$, $j = 1, \ldots, 6$ satisfying (37) allow the elimination of σ_4 and σ_5 to satisfy the algebraic relation

$$\sigma_1^2 \sigma_2^2 + \sigma_6^2 = (\sigma_3 - \sigma_2^2)(1 - \sigma_1^2)$$

which on expansion simplifies to

$$\sigma_2^2 + \sigma_6^2 = \sigma_3 (1 - \sigma_1^2), \qquad (38)$$

where $\sigma_3 \ge 0$ and $(1 - \sigma_1^2) \ge 0$. Restoring $\sigma_6 = J_3$, we may write the previous equation as

$$C(\sigma_1, \sigma_2, \sigma_3; J_3^2) = \sigma_3(1 - \sigma_1^2) - \sigma_2^2 - J_3^2 = 0.$$
(39)

,

This is the **orbit manifold** for the spherical pendulum in \mathbb{R}^3 . The motion takes place on the following family of surfaces depending on $(\sigma_1, \sigma_2, \sigma_3) \in \mathbb{R}^3$ and parameterised by the conserved value of J_3^2 ,

$$\sigma_3 = \frac{\sigma_2^2 + J_3^2}{1 - \sigma_1^2} \,. \tag{40}$$

The orbit manifold for the spherical pendulum is a graph of σ_3 over $(\sigma_1, \sigma_2) \in \mathbb{R}^2$, provided $1 - \sigma_1^2 \neq 0$. The two solutions of $1 - \sigma_1^2 = 0$ correspond to the north and south poles of the sphere. In the case $J_3^2 = 0$, the spherical pendulum is restricted to the *planar* pendulum.



Figure 9: The dynamics of the spherical pendulum in the space of S^1 invariants $(\sigma_1, \sigma_2, \sigma_3)$ is recovered by taking the union in \mathbb{R}^3 of the intersections of level sets of two families of surfaces. These surfaces are the roughly cylindrical level sets of angular momentum about the vertical axis given in (40) and the (planar) level sets of the Hamiltonian in (41). (Only one member of each family is shown in the figure here, although the curves show a few of the other intersections.) On each *planar* level set of the Hamiltonian, the dynamics reduces to that of a quadratically nonlinear oscillator for the vertical coordinate (σ_1) given in Equation (47).
Reduced Poisson bracket in \mathbb{R}^3 When evaluated on TS^2 , the Hamiltonian for the spherical pendulum is expressed in these S^1 -invariant variables by the linear relation

$$H = \frac{1}{2}\sigma_3 + g\sigma_1, \tag{41}$$

whose level surfaces are planes in \mathbb{R}^3 . The motion in \mathbb{R}^3 takes place on the intersections of these Hamiltonian planes with the level sets of J_3^2 given by C = 0 in Equation (39). Consequently, in \mathbb{R}^3 -vector form, the motion is governed by the cross-product formula

$$\dot{\boldsymbol{\sigma}} = \frac{\partial C}{\partial \boldsymbol{\sigma}} \times \frac{\partial H}{\partial \boldsymbol{\sigma}}.$$
(42)

In components, this evolution is expressed as

$$\dot{\sigma}_i = \{\sigma_i, H\} = \epsilon_{ijk} \frac{\partial C}{\partial \sigma_j} \frac{\partial H}{\partial \sigma_k} \quad \text{with} \quad i, j, k = 1, 2, 3.$$
(43)

The motion may be expressed in Hamiltonian form by introducing the following bracket operation, defined for a function F of the S^1 -invariant vector $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3) \in \mathbb{R}^3$,

$$\{F,H\} = -\frac{\partial C}{\partial \boldsymbol{\sigma}} \cdot \frac{\partial F}{\partial \boldsymbol{\sigma}} \times \frac{\partial H}{\partial \boldsymbol{\sigma}} = -\epsilon_{ijk} \frac{\partial C}{\partial \sigma_i} \frac{\partial F}{\partial \sigma_i} \frac{\partial H}{\partial \sigma_k}.$$
(44)

This is an example of the Nambu \mathbb{R}^3 bracket. The proof that this bracket satisfies the defining relations to be a Poisson bracket was assigned in a homework. In the present case, the distinguished function $C(\sigma_1, \sigma_2, \sigma_3; J_3^2)$ in (39) defines a level set of the squared vertical angular momentum J_3^2 in \mathbb{R}^3 given by C = 0. The distinguished function C is a **Casimir function** for the Nambu bracket in \mathbb{R}^3 . That is, the Nambu bracket in (44) with C obeys $\{C, H\} = 0$ for any Hamiltonian $H(\sigma_1, \sigma_2, \sigma_3) : \mathbb{R}^3 \to \mathbb{R}$. Consequently, the motion governed by this \mathbb{R}^3 bracket takes place on level sets of J_3^2 given by C = 0.

Poisson map Introducing the Nambu bracket in (44) ensures that the quotient map for the spherical pendulum $\pi : T\mathbb{R}^3 \to \mathbb{R}^6$ in (34) is a **Poisson map**. That is, the subspace obtained by using the relations (37) to restrict to the invariant manifold TS^2 produces a set of Poisson brackets $\{\sigma_i, \sigma_j\}$ for i, j = 1, 2, 3 that close amongst themselves. Namely,

$$\{\sigma_i, \sigma_j\} = \epsilon_{ijk} \frac{\partial C}{\partial \sigma_k}, \tag{45}$$

with C given in (39). These brackets may be expressed in tabular form, as

$\{\cdot,\cdot\}$	σ_1	σ_2	σ_3
σ_1	0	$1 - \sigma_1^2$	$2\sigma_2$
σ_2	$-1 + \sigma_1^2$	0	$-2\sigma_1\sigma_3$
σ_3	$-2\sigma_2$	$2\sigma_1\sigma_3$	0

In addition, $\{\sigma_i, \sigma_6\} = 0$ for i = 1, 2, 3, since $\sigma_6 = J_3$ and the $\{\sigma_i | i = 1, 2, 3\}$ are all S¹-invariant under X_{J_3} in (33).

Reduced motion: Restriction in \mathbb{R}^3 to Hamiltonian planes The individual components of the equations of motion may be obtained from (43) as

$$\dot{\sigma}_1 = -\sigma_2, \qquad \dot{\sigma}_2 = \sigma_1 \sigma_3 + g(1 - \sigma_1^2), \qquad \dot{\sigma}_3 = 2g\sigma_2.$$
 (46)

Substituting $\sigma_3 = 2(H - g\sigma_1)$ from Equation (41) and setting the acceleration of gravity to be unity g = 1 yields

$$\ddot{\sigma}_1 = 3\sigma_1^2 - 2H\sigma_1 - 1 \tag{47}$$

which has equilibria at $\sigma_1^{\pm} = \frac{1}{3}(H \pm \sqrt{H^2 + 3})$ and conserves the energy integral

$$\frac{1}{2}\dot{\sigma}_1^2 + V(\sigma_1) = E \tag{48}$$

with the potential $V(\sigma_1)$ parameterised by H in (41) and given by

$$V(\sigma_1) = -\sigma_1^3 + H\sigma_1^2 + \sigma_1.$$
(49)

Equation (48) is an energy equation for a particle of unit mass, with position σ_1 and energy E, moving in a cubic potential field $V(\sigma_1)$. For H = 0, its equilibria in the $(\sigma_1, \dot{\sigma}_1)$ phase plane are at $(\sigma_1, \dot{\sigma}_1) = (\pm \sqrt{3}/3, 0)$, as sketched in Figure 10.

Each curve in the lower panel of Figure 10 represents the intersection in the reduced phase space with S^1 -invariant coordinates $(\sigma_1, \sigma_2, \sigma_3) \in \mathbb{R}^3$ of one of the Hamiltonian planes (41) with a level set of J_3^2 given by C = 0 in Equation (39). The critical points of the potential are relative equilibria, corresponding to S^1 -periodic solutions. The case H = 0 includes the homoclinic trajectory, for which the level set E = 0 in (48) starts and ends with zero velocity at the north pole of the unit sphere. A discussion of the properties of motion in a cubic potential and the details of how to compute its homoclinic trajectory was assigned as a homework.



Figure 10: The upper panel shows a sketch of the cubic potential $V(\sigma_1)$ in Equation (49) for the case H = 0. For H = 0, the potential has three zeros located at $\sigma_1 = 0, \pm 1$ and two critical points (relative equilibria) at $\sigma_1 = -\sqrt{3}/3$ (centre) and $\sigma_1 = +\sqrt{3}/3$ (saddle). The lower panel shows a sketch of its fish-shaped saddle-centre configuration in the $(\sigma_1, \dot{\sigma}_1)$ phase plane, comprising several level sets of $E(\sigma_1, \dot{\sigma}_1)$ from Equation (48) for H = 0.

6.3 Geometric phase for the spherical pendulum

We write the Nambu bracket (44) for the spherical pendulum as a differential form in \mathbb{R}^3 ,

$$\{F,H\}\,d^3\sigma = dC \wedge dF \wedge dH\,,\tag{50}$$

with oriented volume element $d^3\sigma = d\sigma_1 \wedge d\sigma_2 \wedge d\sigma_3$. Hence, on a level set of H we have the canonical Poisson bracket

$$\{f,h\}d\sigma_1 \wedge d\sigma_2 = df \wedge dh = \left(\frac{\partial f}{\partial \sigma_1}\frac{\partial h}{\partial \sigma_2} - \frac{\partial f}{\partial \sigma_2}\frac{\partial h}{\partial \sigma_1}\right)d\sigma_1 \wedge d\sigma_2 \tag{51}$$

and we recover Equation (47) in canonical form with Hamiltonian

$$h(\sigma_1, \sigma_2) = -\left(\frac{1}{2}\sigma_2^2 - \sigma_1^3 + H\sigma_1^2 + \sigma_1\right) = -\left(\frac{1}{2}\sigma_2^2 + V(\sigma_1)\right),\tag{52}$$

which, not unexpectedly, is also the conserved energy integral in (48) for motion on level sets of H.

For the S^1 reduction considered in the present case, the canonical one-form is

$$p_i dq_i = \sigma_2 \, d\sigma_1 + H d\psi \,, \tag{53}$$

where σ_1 and σ_2 are the symplectic coordinates for the level surface of H on which the reduced motion takes place and $\psi \in S^1$ is canonically conjugate to H.

Our goal is to finish the solution for the spherical pendulum motion by reconstructing the phase $\psi \in S^1$ from the symmetry-reduced motion in $(\sigma_1, \sigma_2, \sigma_3) \in \mathbb{R}^3$ on a level set of H. Rearranging Equation (53) gives

$$Hd\psi = -\sigma_2 \, d\sigma_1 + p_i dq_i \,. \tag{54}$$

Thus, the phase change around a closed periodic orbit on a level set of H in the $(\sigma_1, \sigma_2, \psi, H)$ phase space decomposes into the sum of the following two parts:

$$\oint H \, d\psi = H \, \Delta \psi = \underbrace{-\oint \sigma_2 \, d\sigma_1}_{\text{geometric}} + \underbrace{\oint p_i dq_i}_{\text{dynamic}}.$$
(55)

On writing this decomposition of the phase as

$$\Delta \psi = \Delta \psi_{geom} + \Delta \psi_{dyn} \,, \tag{56}$$

one sees from (46) that

$$H\Delta\psi_{geom} = \oint \sigma_2^2 \, dt = \iint d\sigma_1 \wedge d\sigma_2 \tag{57}$$

is the area enclosed by the periodic orbit on a level set of H. Thus the name **geometric phase** for $\Delta \psi_{geom}$, because this part of the phase equals the geometric area of the periodic orbit. The rest of the phase is given by

$$H\Delta\psi_{dyn} = \oint p_i \, dq_i = \int_0^T (\sigma_2 \dot{\sigma}_1 + H\dot{\psi}) \, dt \,.$$
(58)

Hence, from the canonical equations $\dot{\sigma}_1 = \partial h / \partial \sigma_2$ and $\dot{\psi} = \partial h / \partial H$ with Hamiltonian h in (52), we have

$$\Delta \psi_{dyn} = \frac{1}{H} \int_{0}^{T} \left(\sigma_{2} \frac{\partial h}{\partial \sigma_{2}} + H \frac{\partial h}{\partial H} \right) dt$$

$$= \frac{2T}{H} \left(h + \langle V(\sigma_{1}) \rangle - \frac{1}{2} H \langle \sigma_{1}^{2} \rangle \right)$$

$$= \frac{2T}{H} \left(h + \langle V(\sigma_{1}) \rangle \right) - T \langle \sigma_{1}^{2} \rangle, \qquad (59)$$

where T is the period of the orbit around which the integration is performed and the angle brackets $\langle \cdot \rangle$ denote time average.

The second summand $\Delta \psi_{dyn}$ in (56) depends on the Hamiltonian h = E, the orbital period T, the value of the level set H and the time averages of the potential energy and σ_1^2 over the orbit. Thus, $\Delta \psi_{dyn}$ deserves the name **dynamic phase**, since it depends on several aspects of the dynamics along the orbit, not just its area.

This finishes the solution for the periodic motion of the spherical pendulum up to quadratures for the phase. The remaining homoclinic trajectory is determined as in Section ??.

DD Holm

7 Mechanics on Lie groups

7.1 This is a topic invented by H. Poincaré in 1901.



MEMOIRES ET COMMUNICATIONS

DES MEMBRES ET DES CORRESPONDANTS DE L'ACADÉMIE.

MÉCANIQUE RATIONNELLE. — Sur une forme nouvelle des équations de la Mécanique. Note de M. H. POINCARÉ.

« Ayant eu l'occasion de m'occuper du mouvement de rotation d'un corps solide creux, dont la cavité est remplie de liquide, j'ai été conduit à mettre les équations générales de la Mécanique sous une forme que je crois nouvelle et qu'il peut être intéressant de faire connaître.

» Supposons qu'il y ait *n* degrés de liberté et désignons par $x_1, x_2, ..., x_n$ les variables qui définissent l'état du système. Soient T et U l'énergie cinétique et l'énergie potentielle.

» Envisageons un groupe transitif continu quelconque. Soit X_i(f) une substitution infinitésimale quelconque de ce groupe, telle que

$$X_i(f) = X_i^4 \frac{df}{dx_i} + X_i^2 \frac{df}{dx_i} + \ldots + X_i^n \frac{df}{dx_n},$$

C. R., 1901, 1^{er} Semestre. (T. CXXXII, N° 7.) (48)

Figure 11: This is the first page of Poincaré's short paper, C.R. Acad. Sci. 132 (1901) 369-371 [Po1901].

Keywords for mechanics on Lie groups:

group	$\operatorname{conjugation}$ map	structure constants
Lie group, G	Lie algebra bracket,	reduced Lagrangian
identity element, e	$[\cdot,\cdot]:\mathfrak{g} imes\mathfrak{g} o\mathfrak{g}$	dual Lie algebra, \mathfrak{g}^*
Lie algebra, \mathfrak{g}	Jacobi identity	dual basis, $e^k \in \mathfrak{g}^*$
tangent vectors	basis vectors, $e_k \in \mathfrak{g}$	pairing, $\mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$

- Recall that a group is a set of elements with an associative binary product that has a unique inverse and identity element.
- A Lie group G is a group that depends smoothly on a set of parameters in $\mathbb{R}^{\dim(G)}$.

A Lie group is also a smooth manifold, so it is an interesting arena for geometric mechanics.

- Choose the manifold M for mechanics as discussed above to be the Lie group G and denote the *identity element* as the point e. The identity element e satisfies eg = g = ge for all $g \in G$, where the group product denoted by concatenation.
- The Lie algebra \mathfrak{g} of the Lie group G is defined as the space of tangent vectors $\mathfrak{g} \cong T_e G$ at the identity e of the group.

The Lie algebra has a *bracket* operation $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, which it inherits from linearisation at the identity e of the *conjugation* map $h \cdot g = hgh^{-1}$ for $g, h \in G$. For this, one begins with the conjugation map $h(t) \cdot g(s) = h(t)g(s)h(t)^{-1}$ for curves $g(s), h(t) \in G$, with g(0) = e = h(0). One linearises at the identity, first in s to get the operation $\operatorname{Ad} : G \times \mathfrak{g} \to \mathfrak{g}$ and then in t to get the operation ad $: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, which yields the Lie bracket. The bracket operation is antisymmetric [a, b] = -[b, a] and satisfies the Jacobi condition for $a, b, c \in \mathfrak{g}$,

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$$

The bracket operation among the basis vectors $e_k \in \mathfrak{g}$ with $k = 1, 2, \ldots, \dim(\mathfrak{g})$ defines the Lie algebra by its structure constants c_{ij}^{k} in (summing over repeated indices)

$$[e_i, e_j] = c_{ij}^k e_k$$

The requirement of skew-symmetry and the Jacobi condition put constraints on the structure constants. These constraints are

– skew-symmetry

$$c_{ji}^k = -c_{ij}^k \,, \tag{60}$$

- Jacobi identity

$$c_{ij}^k c_{lk}^m + c_{li}^k c_{jk}^m + c_{jl}^k c_{ik}^m = 0. ag{61}$$

Conversely, any set of constants c_{ij}^k that satisfy relations (60)–(61) defines a Lie algebra \mathfrak{g} .

Exercise:

Prove that the Jacobi condition requires the relation (61).

Hint: the Jacobi condition involves summing three terms of the form

$$[\mathbf{e}_l, [\mathbf{e}_i, \mathbf{e}_j]] = c_{ij}^k [\mathbf{e}_l, \mathbf{e}_k] = c_{ij}^k c_{lk}^m \mathbf{e}_m.$$

7.2 Understanding H. Poincaré's contribution [Po1901].

To understand [Po1901], let's introduce two more definitions.

- 1. Define a *reduced Lagrangian* $l: \mathfrak{g} \to \mathbb{R}$ and an associated variational principle $\delta S = 0$ with $S = \int_a^b l(\xi) dt$ where $\xi = \xi^k e_k \in \mathfrak{g}$ has components ξ^k in the set of basis vectors e_k .
- 2. Define the *dual Lie algebra* \mathfrak{g}^* by using the fibre derivative of the Lagrangian $l:\mathfrak{g}\to\mathbb{R}$ to define a pairing as

$$\mu := \frac{\partial l(\xi)}{\partial \xi} \in \mathfrak{g}^*, \quad \text{written in components as} \quad \mu_i := \frac{\partial l(\xi)}{\partial \xi^i}, \quad \text{with a basis} \quad \mu = \mu_j e^j, \quad \text{and pairing} \quad \langle e^j, e_i \rangle = \delta_i^j.$$

In particular, the relation $dl = \langle \mu, d\xi \rangle$ defines a natural pairing $\langle \cdot, \cdot \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$.

The natural **dual basis** for \mathfrak{g}^* satisfies $\langle e^j, e_k \rangle = \delta_k^j$ in this pairing and an element $\mu \in \mathfrak{g}^*$ has components in this dual basis given by $\mu = \mu_k e^k$, again with with $k = 1, 2, \ldots, \dim(\mathfrak{g})$.

• Exercise:

(a) Show that Hamilton's principle $\delta S = 0$ with $S = \int_a^b l(\xi) dt$ implies the Euler-Poincaré (EP) equations:

$$\frac{d}{dt}\mu_i = -c_{ij}{}^k \xi^j \mu_k , \quad \text{with} \quad \mu_k = \frac{\partial l(\xi)}{\partial \xi^k} ,$$

for variations given by $\delta \xi = \dot{\eta} + [\xi, \eta]$ with $\xi, \eta \in \mathfrak{g}$.

Note: $[e_j, e_k] = c_{jk}{}^i e_i$, so

$$[\xi,\eta] = [\xi^{j}e_{j}, \eta^{k}e_{k}] = \xi^{j}[e_{j}, e_{k}]\eta^{k} = \xi^{j}\eta^{k}c_{jk}{}^{i}e_{i} = [\xi,\eta]^{i}e_{i}$$

• Answer:

$$\delta S = \delta \int_{a}^{b} l(\xi) dt = \int_{a}^{b} \left\langle \frac{\partial l}{\partial \xi}, \, \delta \xi \right\rangle dt = \int_{a}^{b} \left\langle \frac{\partial l}{\partial \xi}, \, \dot{\eta} + [\xi, \eta] \right\rangle dt$$
$$= \int_{a}^{b} \left\langle \frac{\partial l}{\partial \xi^{n}} e^{n}, \, \dot{\eta}^{i} e_{i} + \xi^{j} \eta^{k} c_{jk}{}^{i} e_{i} \right\rangle dt \quad \text{since} \quad \left\langle e^{n}, \, e_{i} \right\rangle = \delta_{i}^{r}$$
$$= \int_{a}^{b} \left(-\frac{d}{dt} \frac{\partial l}{\partial \xi^{i}} + \frac{\partial l}{\partial \xi^{k}} \xi^{j} c_{ji}{}^{k} \right) \eta^{i} dt + \left[\frac{\partial l}{\partial \xi^{i}} \eta^{i} \right]_{a}^{b}$$

where, in the last step, we integrated by parts and relabelled indices. Hence, when η^i vanishes at the endpoints in time, but is otherwise arbitrary, we recover the EP equations as

$$\frac{d}{dt}\frac{\partial l}{\partial\xi^i} + \frac{\partial l}{\partial\xi^k}\xi^j c_{ij}{}^k = 0\,,$$

where we have used the antisymmetry of the structure constant $c_{ij}{}^k = -c_{ji}{}^k$.

These are the equations introduced by Poincaré in [Po1901], which we now write as $\frac{d}{dt}\frac{\partial l}{\partial\xi} - \mathrm{ad}_{\xi}^*\frac{\partial l}{\partial\xi} = 0$. Here the notation ad^* is defined by $\langle -\mathrm{ad}_{\xi}^*\frac{\partial l}{\partial\xi}, \eta \rangle := \frac{\partial l}{\partial\xi^k}\xi^j c_{ij}{}^k\eta^i = \frac{\partial l}{\partial\xi^k}[e_i\eta^i, e_j\xi^j]^k =: \langle \frac{\partial l}{\partial\xi}, -\mathrm{ad}_{\xi}\eta \rangle.$

- *Exercise:* Write Noether's theorem for the Euler-Poincaré theory.
- **Answer:** To each continuous symmetry group G of the Lagrangian $l(\xi)$, the quantity $(\frac{\partial l}{\partial \xi^i} \eta^i)$ is conserved, where $\eta^i e_i \in \mathfrak{g}$ is the infinitesimal transformation of the action of the group $G \times \mathfrak{g} \to \mathfrak{g}$.
- **Exercise:** The Lie algebra $\mathfrak{so}(3)$ of the Lie group SO(3) of rotations in three dimensions has structure constants $c_{ij}{}^k = \epsilon_{ij}{}^k$, where $\epsilon_{ij}{}^k$ with $i, j, k \in \{1, 2, 3\}$ is totally antisymmetric under pairwise permutations of its indices, with $\epsilon_{12}{}^3 = 1$, $\epsilon_{21}{}^3 = -1$, etc. Identify the Lie bracket [a, b] of two elements $a = a^i e_i, b = b^j e_j \in \mathfrak{so}(3)$ with the cross product $\mathbf{a} \times \mathbf{b}$ of two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ according to 2

$$[a,b] = [a^i e_i, b^j e_j] = a^i b^j \epsilon_{ij}{}^k e_k = (\mathbf{a} \times \mathbf{b})^k e_k.$$

(a) Show that in this case the EP equation

$$\dot{\mu}_i = -\epsilon_{ij}{}^k \xi^j \mu_k$$

is equivalent to the vector equation for $\boldsymbol{\xi}, \boldsymbol{\mu} \in \mathbb{R}^3$

$$\dot{oldsymbol{\mu}}=-oldsymbol{\xi} imesoldsymbol{\mu}$$
 .

(b) Show that when the Lagrangian is given by the quadratic

$$l(\boldsymbol{\xi}) = \frac{1}{2} \|\boldsymbol{\xi}\|_{K}^{2} = \frac{1}{2} \boldsymbol{\xi} \cdot K \boldsymbol{\xi} = \frac{1}{2} \xi^{i} K_{ij} \xi^{j}$$

for a symmetric constant Riemannian metric $K^T = K$, then Euler's equations for a rotating rigid body are recovered.

That is, Euler's equations for rigid body motion are contained in Poincaré's equations for motion on Lie groups! And Poincaré's equations generalise Euler's equations for rigid body motion from \mathbb{R}^3 to motion on Lie groups!

(c) Identify the functional dependence of μ on ξ and give the physical meanings of the symbols ξ , μ and K in Euler's rigid body equations.

² (a') Show that this formula implies the Jacobi identity for the cross product of vectors in \mathbb{R}^3 . This is no surprise because, that familiar cross product relation for vectors may be proven directly by using the antisymmetric tensor ϵ_{ij}^k .

8 Motion on SO(n): the rigid body

8.1 Manakov's formulation of the SO(n) rigid body

Proposition 7 (Manakov [Ma1976]). Euler's equations for a rigid body on SO(n) take the matrix commutator form,

$$\frac{dM}{dt} = [M, \Omega] \quad with \quad M = \mathbb{A}\Omega + \Omega\mathbb{A}, \qquad (62)$$

where the $n \times n$ matrices M, Ω are skew-symmetric (forgoing superfluous hats) and \mathbb{A} is symmetric.

Proof. Manakov's commutator form of the SO(n) rigid-body Equations (62) follows as the Euler-Lagrange equations for Hamilton's principle $\delta S = 0$ with $S = \int l \, dt$ for the Lagrangian

$$l = \frac{1}{2} \operatorname{tr}(\Omega^T \mathbb{A} \Omega) = -\frac{1}{2} \operatorname{tr}(\Omega \mathbb{A} \Omega) \, ,$$

where $\Omega = O^{-1}\dot{O} \in so(n)$ and the $n \times n$ matrix \mathbb{A} is symmetric. Taking matrix variations in Hamilton's principle yields

$$\delta S = -\frac{1}{2} \int_{a}^{b} \operatorname{tr} \left(\delta \Omega \left(\mathbb{A} \Omega + \Omega \mathbb{A} \right) \right) dt = -\frac{1}{2} \int_{a}^{b} \operatorname{tr} \left(\delta \Omega M \right) dt \,,$$

after cyclically permuting the order of matrix multiplication under the trace and substituting $M := \mathbb{A}\Omega + \Omega\mathbb{A}$.

Using the variational formula

$$\delta\Omega = \delta(O^{-1}\dot{O}) = \Xi + [\Omega, \Xi], \quad \text{with} \quad \Xi = (O^{-1}\delta O)$$
(63)

for $\delta\Omega$ now leads to

$$\delta S = -\frac{1}{2} \int_{a}^{b} \operatorname{tr} \left((\Xi + \Omega \Xi - \Xi \Omega) M \right) dt$$

Integrating by parts and permuting under the trace then yields the equation

$$\delta S = \frac{1}{2} \int_{a}^{b} \operatorname{tr} \left(\Xi \left(\dot{M} + \Omega M - M \Omega \right) \right) dt \,.$$

Finally, invoking stationarity for arbitrary Ξ implies the commutator form (62).

8.2 Matrix Euler–Poincaré equations

Manakov's commutator form of the rigid-body equations in (62) recalls much earlier work by Poincaré [Po1901], who also noticed that the matrix commutator form of Euler's rigid-body equations suggests an additional mathematical structure going back to Lie's theory of groups of transformations depending continuously on parameters. In particular, Poincaré [Po1901] remarked that the commutator form of Euler's rigid-body equations would make sense for any Lie algebra, not just for so(3). The proof of Manakov's commutator form (62) by Hamilton's principle is essentially the same as Poincaré's proof in [Po1901].

Theorem 8 (Matrix Euler–Poincaré equations). The Euler–Lagrange equations for Hamilton's principle $\delta S = 0$ with $S = \int l(\Omega) dt$ may be expressed in matrix commutator form,

$$\frac{dM}{dt} = [M, \Omega] \quad with \quad M = \frac{\delta l}{\delta \Omega}, \tag{64}$$

for any Lagrangian $l(\Omega)$, where $\Omega = g^{-1}\dot{g} \in \mathfrak{g}$ and \mathfrak{g} is the matrix Lie algebra of any matrix Lie group G.

Proof. The proof here is the same as the proof of Manakov's commutator formula via Hamilton's principle, modulo replacing $O^{-1}\dot{O} \in so(n)$ with $g^{-1}\dot{g} \in \mathfrak{g}$.

Remark 9.

Poincaré's observation leading to the matrix Euler–Poincaré Equation (64) was reported in two pages with no references [Po1901]. The proof above shows that the matrix Euler–Poincaré equations possess a natural variational principle. Note that if $\Omega = g^{-1}\dot{g} \in \mathfrak{g}$, then $M = \delta l/\delta \Omega \in \mathfrak{g}^*$, where the dual is defined in terms of the matrix trace pairing.

Exercise. Retrace the proof of the variational principle for the Euler–Poincaré equation, replacing the left-invariant quantity $g^{-1}\dot{g}$ with the right-invariant quantity $\dot{g}g^{-1}$.

8.3 An isospectral eigenvalue problem for the SO(n) rigid body

The solution of the SO(n) rigid-body dynamics

$$\frac{dM}{dt} = [M, \Omega] \quad \text{with} \quad M = \mathbb{A}\Omega + \Omega\mathbb{A},$$

for the evolution of the $n \times n$ skew-symmetric matrices M, Ω , with constant symmetric \mathbb{A} , is given by a similarity transformation (later to be identified as coadjoint motion),

$$M(t) = O(t)^{-1} M(0) O(t) =: \operatorname{Ad}_{O(t)}^* M(0),$$

with $O(t) \in SO(n)$ and $\Omega := O^{-1}\dot{O}(t)$. Consequently, the evolution of M(t) is *isospectral*. This means that

• The initial eigenvalues of the matrix M(0) are preserved by the motion; that is, $d\lambda/dt = 0$ in

$$M(t)\psi(t) = \lambda\psi(t) ,$$

provided its eigenvectors $\psi \in \mathbb{R}^n$ evolve according to

$$\psi(t) = O(t)^{-1}\psi(0) \,.$$

The proof of this statement follows from the corresponding property of similarity transformations.

• Its matrix invariants are preserved:

$$\frac{d}{dt} \operatorname{tr}(M - \lambda \operatorname{Id})^{K} = 0 \,,$$

for every non-negative integer power K.

This is clear because the invariants of the matrix M may be expressed in terms of its eigenvalues; but these are invariant under a similarity transformation.

Proposition 10. Isospectrality allows the quadratic rigid-body dynamics (62) on SO(n) to be rephrased as a system of two coupled linear equations: the eigenvalue problem for M and an evolution equation for its eigenvectors ψ , as follows:

 $M\psi = \lambda\psi$ and $\dot{\psi} = -\Omega\psi$, with $\Omega = O^{-1}\dot{O}(t)$.

Proof. Applying isospectrality in the time derivative of the first equation yields

$$(\dot{M} + [\Omega, M])\psi + (M - \lambda \mathrm{Id})(\dot{\psi} + \Omega\psi) = 0.$$

Now substitute the second equation to recover (62).

8.4 Manakov's integration of the SO(n) rigid body

Manakov [Ma1976] observed that Equations (62) may be "deformed" into

$$\frac{d}{dt}(M + \lambda A) = \left[(M + \lambda A), (\Omega + \lambda B)\right],\tag{65}$$

where A, B are also $n \times n$ matrices and λ is a scalar constant parameter. For these deformed rigid-body equations on SO(n) to hold for any value of λ , the coefficient of each power must vanish.

• The coefficient of λ^2 is

0 = [A, B].

Therefore, A and B must commute. For this, let them be constant and diagonal:

$$A_{ij} = \operatorname{diag}(a_i)\delta_{ij}, \quad B_{ij} = \operatorname{diag}(b_i)\delta_{ij} \quad (\text{no sum}).$$

• The coefficient of λ is

$$0 = \frac{dA}{dt} = [A, \Omega] + [M, B].$$

Therefore, by antisymmetry of M and Ω ,

$$(a_i - a_j)\Omega_{ij} = (b_i - b_j)M_{ij},$$

which implies that

$$\Omega_{ij} = \frac{b_i - b_j}{a_i - a_j} M_{ij} \quad \text{(no sum)}.$$

Hence, angular velocity Ω is a linear function of angular momentum, M.

• Finally, the coefficient of λ^0 recovers the Euler equation

$$\frac{dM}{dt} = [M, \Omega],$$

but now with the restriction that the moments of inertia are of the form

$$\Omega_{ij} = \frac{b_i - b_j}{a_i - a_j} M_{ij} \quad \text{(no sum)}.$$

This relation turns out to possess only five free parameters for n = 4.

Under these conditions, Manakov's deformation of the SO(n) rigid-body equation into the commutator form (65) implies for every non-negative integer power K that

$$\frac{d}{dt}(M + \lambda A)^{K} = \left[(M + \lambda A)^{K}, (\Omega + \lambda B) \right].$$

Since the commutator is antisymmetric, its trace vanishes and K conservation laws emerge, as

$$\frac{d}{dt}\operatorname{tr}(M+\lambda A)^{K}=0\,,$$

after commuting the trace operation with the time derivative. Consequently,

$$\operatorname{tr}(M + \lambda A)^K = \operatorname{constant},$$

for each power of λ . That is, all the coefficients of each power of λ are constant in time for the SO(n) rigid body. Manakov [?] proved that these constants of motion are sufficient to completely determine the solution for n = 4.

Remark 11.

This result generalises considerably. For example, Manakov's method determines the solution for all the algebraically solvable rigid bodies on SO(n). The moments of inertia of these bodies possess only 2n-3 parameters. (Recall that in Manakov's case for SO(4) the moment of inertia possesses only five parameters.)

Exercise. Try computing the constants of motion $tr(M + \lambda A)^K$ for the values K = 2, 3, 4.

Hint: Keep in mind that M is a skew-symmetric matrix, $M^T = -M$, so the trace of the product of any diagonal matrix times an odd power of M vanishes.

Answer. The traces of the powers trace $(M + \lambda A)^n$ are given by

$$\begin{split} \boxed{n=2} &: \operatorname{tr} M^2 + 2\lambda \operatorname{tr}(AM) + \lambda^2 \operatorname{tr} A^2, \\ \boxed{n=3} &: \operatorname{tr} M^3 + 3\lambda \operatorname{tr}(AM^2) + 3\lambda^2 \operatorname{tr} A^2 M + \lambda^3 \operatorname{tr} A^3, \\ \boxed{n=4} &: \operatorname{tr} M^4 + 4\lambda \operatorname{tr}(AM^3) \\ &+ \lambda^2 (2\operatorname{tr} A^2 M^2 + 4\operatorname{tr} AMAM) \\ &+ \lambda^3 \operatorname{tr} A^3 M + \lambda^4 \operatorname{tr} A^4. \end{split}$$

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The number of conserved quantities for n = 2, 3, 4 are, respectively, one $(C_2 = \operatorname{tr} M^2)$, one $(C_3 = \operatorname{tr} AM^2)$ and two $(C_4 = \operatorname{tr} M^4$ and $I_4 = 2\operatorname{tr} A^2M^2 + 4\operatorname{tr} AMAM)$.

Exercise. How do the Euler equations look on $so(4)^*$ as a matrix equation? Is there an analogue of the hat map for so(4)? Hint: The Lie algebra so(4) is locally isomorphic to $so(3) \times so(3)$.

9 Transformation Theory

motion	linearisation	differential, d
motion equation	infinitesimal transformation	differential k -form
vector field	pull-back	wedge product, \wedge
diffeomorphism	push-forward	Lie derivative, \pounds_Q
flow	Jacobian matrix	product rule
fixed point	directional derivative	fluid dynamics
equilibrium	commutator	other flows

9.1 Motions, pull-backs, push-forwards, commutators & differentials

• A *motion* is defined as a smooth curve $q(t) \in M$ parameterised by $t \in \mathbb{R}$ that solves the *motion equation*, which is a system of differential equations

$$\dot{q}(t) = \frac{dq}{dt} = f(q) \in TM, \qquad (66)$$

or in components

$$\dot{q}^{i}(t) = \frac{dq^{i}}{dt} = f^{i}(q) \quad i = 1, 2, \dots, n,$$
(67)

• The map $f: q \in M \to f(q) \in T_qM$ is a vector field.

According to standard theorems about differential equations that are not proven in this course, the solution, or integral curve, q(t) exists, provided f is sufficiently smooth, which will always be assumed to hold.

• Vector fields can also be defined as *differential operators* that act on functions, as

$$\frac{d}{dt}G(q) = \dot{q}^{i}(t)\frac{\partial G}{\partial q^{i}} = f^{i}(q)\frac{\partial G}{\partial q^{i}} \quad i = 1, 2, \dots, n, \quad \text{(sum on repeated indices)}$$
(68)

for any smooth function $G(q): M \to \mathbb{R}$.

• To indicate the dependence of the solution of its initial condition $q(0) = q_0$, we write the motion as a smooth transformation

 $q(t) = \phi_t(q_0) \,.$

Because the vector field f is independent of time t, for any fixed value of t we may regard ϕ_t as mapping from M into itself that satisfies the composition law

$$\phi_t \circ \phi_s = \phi_{t+s}$$

and

 $\phi_0 = \mathrm{Id}$.

Setting s = -t shows that ϕ_t has a smooth inverse. A smooth mapping that has a smooth inverse is called a *diffeomorphism*. Geometric mechanics deals with diffeomorphisms.

• The smooth mapping $\phi_t : \mathbb{R} \times M \to M$ that determines the solution $\phi_t \circ q_0 = q(t) \in M$ of the motion equation (66) with initial condition $q(0) = q_0$ is called the **flow** of the vector field Q.

A point $q^* \in M$ at which $f(q^*) = 0$ is called a *fixed point* of the flow ϕ_t , or an *equilibrium*.

Vice versa, the vector field f is called the *infinitesimal transformation* of the mapping ϕ_t , since

$$\left. \frac{d}{dt} \right|_{t=0} (\phi_t \circ q_0) = f(q) \,.$$

That is, f(q) is the *linearisation* of the flow map ϕ_t at the point $q \in M$.

More generally, the *directional derivative* of the function h along the vector field f is given by the action of a differential operator, as

$$\frac{d}{dt}\Big|_{t=0}h\circ\phi_t = \left[\frac{\partial h}{\partial\phi_t}\frac{d}{dt}(\phi_t\circ q_0)\right]_{t=0} = \frac{\partial h}{\partial q^i}\dot{q}^i = \frac{\partial h}{\partial q^i}f^i(q) =: Qh.$$

• Under a smooth change of variables q = c(r) the vector field Q in the expression Qh transforms as

$$Q = f^{i}(q)\frac{\partial}{\partial q^{i}} \quad \mapsto \quad R = g^{j}(r)\frac{\partial}{\partial r^{j}} \quad \text{with} \quad g^{j}(r)\frac{\partial c^{i}}{\partial r^{j}} = f^{i}(c(r)) \quad \text{or} \quad g = c_{r}^{-1}f \circ c \,, \tag{69}$$

where c_r is the **Jacobian matrix** of the transformation. That is, since h(q) is a function of q,

 $(Qh) \circ c = R(h \circ c) \,.$

We express the transformation between the vector fields as $R = c^*Q$ and write this relation as

$$(Qh) \circ c =: c^* Q(h \circ c) \,. \tag{70}$$

The expression c^*Q is called the **pull-back** of the vector field Q by the map c. Two vector fields are equivalent under a map c, if one is the pull-back of the other, and fixed points are mapped into fixed points.

The inverse of the pull-back is called the *push-forward*. It is the pull-back by the inverse map.

• The *commutator*

$$QR - RQ =: [Q, R]$$

of two vector fields Q and R defines another vector field. Indeed, if

$$Q = f^i(q) \frac{\partial}{\partial q^i}$$
 and $R = g^j(q) \frac{\partial}{\partial q^j}$

then

$$\left[Q, R\right] = \left(f^{i}(q)\frac{\partial g^{j}(q)}{\partial q^{i}} - g^{i}(q)\frac{\partial f^{j}(q)}{\partial q^{i}}\right)\frac{\partial}{\partial q^{j}}$$

because the second-order derivative terms cancel. By the pull-back relation (70) we have

$$c^*[Q, R] = [c^*Q, c^*R]$$
(71)

under a change of variables defined by a smooth map, c. This means the definition of the vector field commutator is independent of the choice of coordinates. As we shall see, the **tangent** to the relation $c_t^*[Q, R] = [c_t^*Q, c_t^*R]$ at the identity t = 0 is the **Jacobi** condition for the vector fields to form an algebra.

• The *differential* of a smooth function $f: M \to M$ is defined as

$$df = \frac{\partial f}{\partial q^i} dq^i$$

• Under a smooth change of variables $s = \phi \circ q = \phi(q)$ the differential of the composition of functions $d(f \circ \phi)$ transforms according to the chain rule as

$$df = \frac{\partial f}{\partial q^i} dq^i, \quad d(f \circ \phi) = \frac{\partial f}{\partial \phi^j(q)} \frac{\partial \phi^j}{\partial q^i} dq^i = \frac{\partial f}{\partial s^j} ds^j \implies d(f \circ \phi) = (df) \circ \phi$$
(72)

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That is, the differential d commutes with the pull-back ϕ^* of a smooth transformation ϕ ,

$$d(\phi^* f) = \phi^* df \,. \tag{73}$$

In a moment, this pull-back formula will give us the rule for transforming differential forms of any order.

9.2 Wedge products

• Differential k-forms on an n-dimensional manifold are defined in terms of the differential d and the antisymmetric wedge product (\wedge) satisfying

$$dq^{i} \wedge dq^{j} = -dq^{j} \wedge dq^{i}, \quad \text{for} \quad i, j = 1, 2, \dots, n$$

$$\tag{74}$$

By using wedge product, any k-form $\alpha \in \Lambda^k$ on M may be written locally at a point $q \in M$ in the differential basis dq^j as

$$\alpha_m = \alpha_{i_1 \dots i_k}(m) dq^{i_1} \wedge \dots \wedge dq^{i_k} \in \Lambda^k \,, \quad i_1 < i_2 < \dots < i_k \,, \tag{75}$$

where the sum over repeated indices is ordered, so that it must be taken over all i_j satisfying $i_1 < i_2 < \cdots < i_k$. Roughly speaking differential forms Λ^k are objects that can be integrated. As we shall see, vector fields also act on differential forms in interesting ways.

• Pull-backs of other differential forms may be built up from their basis elements, the dq^{i_k} . By equation (73),

Theorem 12 (Pull-back of a wedge product). The pull-back of a wedge product of two differential forms is the wedge product of their pull-backs:

$$\phi_t^*(\alpha \wedge \beta) = \phi_t^* \alpha \wedge \phi_t^* \beta \,. \tag{76}$$

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9.3 Lie derivatives

Definition 13 (Lie derivative of a differential k-form). The Lie derivative of a differential k-form Λ^k by a vector field $Q \in \mathfrak{X}$ is defined by linearising its flow ϕ_t around the identity t = 0,

$$\pounds_Q \Lambda^k = \frac{d}{dt} \bigg|_{t=0} \phi_t^* \Lambda^k \quad maps \quad \pounds_Q \Lambda^k \mapsto \Lambda^k \,.$$

Hence, by equation (76), the Lie derivative satisfies the product rule for the wedge product.

Corollary 14 (Product rule for the Lie derivative of a wedge product).

$$\pounds_Q(\alpha \wedge \beta) = \pounds_Q \alpha \wedge \beta + \alpha \wedge \pounds_Q \beta \,. \tag{77}$$

• Pullbacks of vector fields lead to Lie derivative expressions, too.

Definition 15 (Lie derivative of a vector field). The Lie derivative of a vector field $Y \in \mathfrak{X}$ by another vector field $X \in \mathfrak{X}$ is defined by linearising the flow ϕ_t of X around the identity t = 0,

$$\pounds_X Y = \frac{d}{dt} \Big|_{t=0} \phi_t^* Y \quad maps \quad \pounds_X \in \mathfrak{X} \mapsto \mathfrak{X}.$$

Theorem 16. The Lie derivative $\pounds_X Y$ of a vector field Y by a vector field X satisfies

$$\pounds_X Y = \frac{d}{dt} \bigg|_{t=0} \phi_t^* Y = [X, Y],$$
(78)

where [X, Y] = XY - YX is the commutator of the vector fields X and Y.

Proof. Denote the vector fields in components as

$$X = X^{i}(q) \frac{\partial}{\partial q^{i}} = \frac{d}{dt} \Big|_{t=0} \phi_{t}^{*} \text{ and } Y = Y^{j}(q) \frac{\partial}{\partial q^{j}}.$$

Then, by the pull-back relation (70) a direct computation yields, on using the matrix identity $dM^{-1} = -M^{-1}dMM^{-1}$,

$$\begin{aligned} \pounds_X Y &= \frac{d}{dt} \bigg|_{t=0} \phi_t^* Y = \frac{d}{dt} \bigg|_{t=0} \left(Y^j(\phi_t q) \frac{\partial}{\partial (\phi_t q)^j} \right) \\ &= \frac{d}{dt} \bigg|_{t=0} \left(Y^j(\phi_t q) \left[\frac{\partial (\phi_t q)}{\partial q}^{-1} \right]_j^k \frac{\partial}{\partial q^k} \right) \\ &= \left(X^j \frac{\partial Y^k}{\partial q^j} - Y^j \frac{\partial X^k}{\partial q^j} \right) \frac{\partial}{\partial q^k} \\ &= [X, Y] \,. \end{aligned}$$

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Corollary 17. The Lie derivative of the relation (71) for the pull-back of the commutator $c_t^*[Y, Z] = [c_t^*Y, c_t^*Z]$ yields the **Jacobi** condition for the vector fields to form an algebra.

Proof. By the product rule and the definition of the Lie bracket (78) we have

$$\frac{d}{dt}\Big|_{t=0}\phi_t^*[Y,Z] = \left[X, \left[Y,Z\right]\right] = \left[\left[X,Y\right],Z\right] + \left[Y,\left[X,Z\right]\right] = \frac{d}{dt}\Big|_{t=0}\left[\phi_t^*Y,\phi_t^*Z\right]$$

This is the *Jacobi identity* for vector fields.

Use the hat map and the relation $R_t(\mathbf{x} \times \mathbf{y}) = R_t \mathbf{x} \times R_t \mathbf{y}$ to show that the same argument gives the Jacobi identity for the cross product of vectors in \mathbb{R}^3 , when ϕ_t^* is a rotation.

9.4 Contraction

$$\partial_q \, \sqcup \, dq = 1 = \partial_p \, \sqcup \, dp \,, \quad and \quad \partial_q \, \sqcup \, dp = 0 = \partial_p \, \sqcup \, dq \,, \tag{79}$$

so that differential forms are linear functions of vector fields. A Hamiltonian vector field,

$$X_H = \dot{q}\frac{\partial}{\partial q} + \dot{p}\frac{\partial}{\partial p} = H_p\partial_q - H_q\partial_p = \{\cdot, H\}, \qquad (80)$$

satisfies the intriguing linear functional relations with the basis elements in phase space,

$$X_H \, \lrcorner \, dq = H_p \quad and \quad X_H \, \lrcorner \, dp = -H_q \,. \tag{81}$$

Definition 19 (Contraction rules with higher forms). The rule for contraction or substitution of a vector field into a differential form is to sum the substitutions of X_H over the permutations of the factors in the differential form that bring the corresponding dual basis element into its leftmost position. For example, substitution of the Hamiltonian vector field X_H into the symplectic form $\omega = dq \wedge dp$ yields

$$X_H \sqcup \omega = X_H \sqcup (dq \land dp) = (X_H \sqcup dq) dp - (X_H \sqcup dp) dq.$$

In this example, $X_H \sqcup dq = H_p$ and $X_H \sqcup dp = -H_q$, so

$$X_H \sqcup \omega = H_p dp + H_q dq = dH \,,$$

which follows from the duality relations (79).

This calculation has proved the following.

Theorem 20 (Hamiltonian vector field). The Hamiltonian vector field $X_H = \{\cdot, H\}$ satisfies $X_H \sqcup \omega = dH \quad with \quad \omega = dq \land dp.$ (82)

Remark 21.

The purely geometric nature of relation (82) argues for it to be taken as the definition of a Hamiltonian vector field.

Lemma 22. $d^2 = 0$ for smooth phase-space functions.

Proof. For any smooth phase-space function H(q, p), one computes

$$dH = H_q dq + H_p dp$$

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and taking the second exterior derivative yields

$$d^{2}H = H_{qp} dp \wedge dq + H_{pq} dq \wedge dp$$

= $(H_{pq} - H_{qp}) dq \wedge dp = 0.$

Relation (82) also implies the following.

Corollary 23 (Poincaré's theorem). The flow of X_H preserves the exact two-form ω for any Hamiltonian H. Proof. Preservation of ω may be verified first by a formal calculation using (82). Along

$$X_H = (dq/dt, dp/dt) = (\dot{q}, \dot{p}) = (H_p, -H_q)$$

for a solution of Hamilton's equations, we have

$$\begin{aligned} \pounds_{X_H} \omega &= \pounds_{X_H} (dq \wedge dp) \\ &= \left. \frac{d}{dt} \right|_{t=0} g_t^* (dq \wedge dp) \\ &= \left. \frac{d}{dt} \right|_{t=0} (g_t^* dq \wedge g_t^* dp) \\ &= \left. d\dot{q} \wedge dp + dq \wedge d\dot{p} \right| \\ &= \left. dH_p \wedge dp - dq \wedge dH_q \right| \\ &= \left. d(H_p \, dp + H_q \, dq) \right| \\ &= \left. d(X_H \sqcup \omega) \\ &= \left. d(dH) = 0 \right. \end{aligned}$$

The first two steps use the product rule for Lie derivatives of differential forms

$$\begin{aligned}
\pounds_{X_H}(dq \wedge dp) &= \frac{d}{dt}\Big|_{t=0} g_t^*(dq \wedge dp) = \frac{d}{dt}\Big|_{t=0} (g_t^* dq \wedge g_t^* dp) \\
&= \left[\frac{d}{dt} g_t^* dq \wedge g_t^* dp + g_t^* dq \wedge \frac{d}{dt} g_t^* dp\right]_{t=0} \\
&= \pounds_{X_H} dq \wedge dp + dq \wedge \pounds_{X_H} dp
\end{aligned}$$
(83)

and the third-to-the-last and last steps use the property of the exterior derivative d that $d^2 = 0$ for continuous forms. The latter is due to the equality of cross derivatives $H_{pq} = H_{qp}$ and antisymmetry of the wedge product $dq \wedge dp = -dp \wedge dq$.

Definition 24 (Symplectic flow). A flow is symplectic if it preserves the phase-space area or symplectic two-form, $\omega = dq \wedge dp$.

According to this definition, Corollary 23 may be simply re-stated as

Corollary 25 (Poincaré's theorem). The flow of a Hamiltonian vector field is symplectic.

Definition 26 (Canonical transformations). A smooth invertible map g of the phase space T^*M is called a canonical transformation if it preserves the canonical symplectic form ω on T^*M , i.e., $g^*\omega = \omega$, where $g^*\omega$ denotes the pull-back of ω under the map g.

Remark 27 (Criterion for a canonical transformation).

Suppose in original coordinates (p,q) the symplectic form is expressed as $\omega = dq \wedge dp$. A transformation $g: T^*M \mapsto T^*M$ written as (Q, P) = (Q(p,q), P(p,q)) is canonical if the direct computation shows that $dQ \wedge dP = g^*(dq \wedge dp) = c dq \wedge dp$, up to a constant factor c. (Such a constant factor c is unimportant, since it may be absorbed into the units of time in Hamilton's canonical equations.)

Remark 28.

By Corollary 25 (Poincaré's Theorem), the Hamiltonian phase flow g_t is a one-parameter group of canonical transformations.

Theorem 29 (Preservation of Hamiltonian form). Canonical transformations preserve the Hamiltonian form.

Proof. The coordinate-free relation $X_H \perp \omega = dH$ with $\omega = dq \wedge dp$ keeps its form if

$$dQ \wedge dP = g^*(dq \wedge dp) = c \, dq \wedge dp \,,$$

up to the constant factor c. Hence, Hamilton's equations re-emerge in canonical form in the new coordinates, up to a rescaling by c which may be absorbed into the units of time.

9.5 Summary of differential-form operations

Besides the wedge product, three basic operations are commonly applied to differential forms. These are contraction, exterior derivative and Lie derivative.

$$X \sqcup \Lambda^k \mapsto \Lambda^{k-1} \,.$$

• Exterior derivative *d* raises the degree:

$$d\Lambda^k \mapsto \Lambda^{k+1}$$

• Lie derivative \pounds_X by vector field X preserves the degree:

$$\pounds_X \Lambda^k \mapsto \Lambda^k$$
, where $\pounds_X \Lambda^k = \frac{d}{dt} \bigg|_{t=0} \phi_t^* \Lambda^k$,

in which ϕ_t is the flow of the vector field X. In analogy with fluids one may write $\pounds_X \Lambda^k = \frac{d}{dt} \Lambda^k$ along $\frac{dx}{dt} = X$.

• Lie derivative \pounds_X satisfies *Cartan's formula*: (The proof is a direct calculation.)

$$\pounds_X \alpha = X \, \sqcup \, d\alpha + d(X \, \sqcup \, \alpha) \quad \text{for} \quad \alpha = \alpha_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k} \in \Lambda^k \, .$$

Remark 30.

Note also that the Lie derivative commutes with the exterior derivative. That is,

$$d(\pounds_X \alpha) = \pounds_X d\alpha$$
, for $\alpha \in \Lambda^k(M)$ and $X \in \mathfrak{X}(M)$.

9.6 Examples of contraction, or interior product

Definition 31 (Contraction, or interior product). Let $\alpha \in \Lambda^k$ be a k-form on a manifold M,

$$\alpha = \alpha_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k} \in \Lambda^k, \quad with \quad i_1 < i_2 < \dots < i_k,$$

and let $X = X^j \partial_j$ be a vector field. The contraction or interior product $X \perp \alpha$ of a vector field X with a k-form α is defined by

$$X \sqcup \alpha = X^{j} \alpha_{ji_2\dots i_k} dx^{i_2} \wedge \dots \wedge dx^{i_k} \,. \tag{84}$$

Note that

$$X \sqcup (Y \sqcup \alpha) = X^{l} Y^{m} \alpha_{m l i_{3} \dots i_{k}} dx^{i_{3}} \wedge \dots \wedge dx^{i_{l}}$$
$$= -Y \sqcup (X \sqcup \alpha),$$

by antisymmetry of $\alpha_{mli_3...i_k}$, particularly in its first two indices.

Remark 32 (Examples of contraction).

1. A mnemonic device for keeping track of signs in contraction or substitution of a vector field into a differential form is to sum the substitutions of $X = X^j \partial_j$ over the permutations that bring the corresponding dual basis element into the leftmost position in the k-form α . For example, in two dimensions, contraction of the vector field $X = X^j \partial_j = X^1 \partial_1 + X^2 \partial_2$ into the two-form $\alpha = \alpha_{jk} dx^j \wedge dx^k$ with $\alpha_{21} = -\alpha_{12}$ yields

$$X \, \sqcup \, \alpha = X^{j} \alpha_{ji_{2}} dx^{i_{2}} = X^{1} \alpha_{12} dx^{2} + X^{2} \alpha_{21} dx^{1} \, .$$

Likewise, in three dimensions, contraction of the vector field $X = X^1 \partial_1 + X^2 \partial_2 + X^3 \partial_3$ into the three-form $\alpha = \alpha_{123} dx^1 \wedge dx^2 \wedge dx^3$ with $\alpha_{213} = -\alpha_{123}$, etc. yields

 $X \sqcup \alpha = X^{1} \alpha_{123} dx^{2} \wedge dx^{3} + \text{cyclic permutations}$ $= X^{j} \alpha_{ij_{2}i_{3}} dx^{i_{2}} \wedge dx^{i_{3}} \quad \text{with} \quad i_{2} < i_{3}.$

2. The rule for contraction of a vector field with a differential form develops from the relation

$$\partial_j \, \sqcup \, dx^k = \delta_j^k \, ,$$

in the coordinate basis $e_j = \partial_j := \partial/\partial x^j$ and its dual basis $e^k = dx^k$. Contraction of a vector field with a one-form yields the dot product, or inner product, between a covariant vector and a contravariant vector is given by

$$X^j \partial_j \sqcup v_k dx^k = v_k \delta^k_j X^j = v_j X^j \,,$$

or, in vector notation,

$$X \, \sqcup \, \mathbf{v} \cdot d\mathbf{x} = \mathbf{v} \cdot \mathbf{X}$$

This is the *dot product of vectors* **v** and **X**.

3. By the linearity of its definition (84), contraction of a vector field X with a differential k-form α satisfies

$$(hX) \, \sqcup \, \alpha = h(X \, \sqcup \, \alpha) = X \, \sqcup \, h\alpha \, .$$

Our previous calculations for two-forms and three-forms provide the following additional expressions for contraction of a vector field with a differential form, which may be written in vector notation as:

$$X \sqcup \mathbf{B} \cdot d\mathbf{S} = -\mathbf{X} \times \mathbf{B} \cdot d\mathbf{x},$$

$$X \sqcup d^{3}x = \mathbf{X} \cdot d\mathbf{S},$$

$$d(X \sqcup d^{3}x) = d(\mathbf{X} \cdot d\mathbf{S}) = (\operatorname{div} \mathbf{X}) d^{3}x.$$

Remark 33 (Physical examples of contraction).

The first of these contraction relations represents the Lorentz, or Coriolis force, when \mathbf{X} is particle velocity and \mathbf{B} is either magnetic field, or rotation rate, respectively. The second contraction relation is the flux of the vector \mathbf{X} through a surface element. The third is the exterior derivative of the second, thereby yielding the divergence of the vector \mathbf{X} .

Exercise. Show that

$$X \perp (X \perp \mathbf{B} \cdot d\mathbf{S}) = 0$$

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and

$$(X \perp \mathbf{B} \cdot d\mathbf{S}) \wedge \mathbf{B} \cdot d\mathbf{S} = 0,$$

for any vector field X and two-form $\mathbf{B} \cdot d\mathbf{S}$.

Proposition 34 (Contracting through wedge product). Let α be a k-form and β be a one-form on a manifold M and let $X = X^j \partial_j$ be a vector field. Then the contraction of X through the wedge product $\alpha \wedge \beta$ satisfies

$$X \sqcup (\alpha \land \beta) = (X \sqcup \alpha) \land \beta + (-1)^k \alpha \land (X \sqcup \beta).$$
(85)

Proof. The proof is a straightforward calculation using the definition of contraction. The exponent k in the factor $(-1)^k$ counts the number of exchanges needed to get the one-form β to the left most position through the k-form α .

Proposition 35. [Contraction is natural under pull-back] That is,

$$\phi^*(X(m) \sqcup \alpha) = X(\phi(m)) \sqcup \phi^* \alpha = \phi^* X \sqcup \phi^* \alpha \,. \tag{86}$$

Proof. Direct verification using the relation between pull-back of forms and push-forward of vector fields. Note the implication, $\pounds_X(Y \sqcup \alpha) = [X, Y] \sqcup \alpha + Y \sqcup (\pounds_X \alpha).$

Definition 36 (Alternative notations for contraction). Besides the hook notation with \Box , one also finds in the literature the following two alternative notations for contraction of a vector field X with k-form $\alpha \in \Lambda^k$ on a manifold M:

$$X \sqcup \alpha = i_X \alpha = \alpha(X, \underbrace{\cdot, \cdot, \ldots, \cdot}_{k-1 \ slots}) \in \Lambda^{k-1}.$$

$$(87)$$

In the last alternative, one leaves a dot (\cdot) in each remaining slot of the form that results after contraction. For example, contraction of the Hamiltonian vector field $X_H = \{\cdot, H\}$ with the symplectic two-form $\omega \in \Lambda^2$ produces the one-form

$$X_H \sqcup \omega = \omega(X_H, \cdot) = -\omega(\cdot, X_H) = dH.$$

 \star

In this alternative notation, the proof of formula (86) in Proposition 35 may be written, as follows.

Proof. Since forms are multilinear maps to the real numbers, one may define the pull back of a k-form, α , by

$$\phi^* \alpha(X_1, X_2, ...) := \alpha(\phi_* X, \phi_* X_2, ...).$$

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Therefore, we are able to use the following proof.

$$\phi^* X \sqcup \phi^* \alpha(X_1, X_2, ...) = \phi^* \alpha(\phi^* X, X_1, X_2, ...)$$

= $\alpha(\phi_* \phi^* X, \phi_* X_1, \phi_* X_2, ...)$
= $\alpha(X, \phi_* X_1, \phi_* X_2, ...)$
= $(X \sqcup \alpha)(\phi_* X_1, \phi_* X_2, ...)$
= $\phi^*(X \sqcup \alpha)(X_1, X_2, ...)$

Now, if we allow X_1, X_2, \ldots to be arbitrary, then formula (86) in Proposition 35 follows.

Proposition 37 (Hamiltonian vector field definitions). The two definitions of Hamiltonian vector field X_H

$$dH = X_H \, \sqcup \, \omega \quad and \quad X_H = \{ \, \cdot \,, \, H \}$$

are equivalent.

Proof. The symplectic Poisson bracket satisfies $\{F, H\} = \omega(X_F, X_H)$, because

$$\omega(X_F, X_H) := X_H \, \sqcup \, X_F \, \sqcup \, \omega = X_H \, \sqcup \, dF = -X_F \, \sqcup \, dH = \{F, H\} \, .$$

Remark 38.

The relation $\{F, H\} = \omega(X_F, X_H)$ means that the Hamiltonian vector field defined via the symplectic form coincides exactly with the Hamiltonian vector field defined using the Poisson bracket.

9.7 Exercises in exterior calculus operations

Vector notation for differential basis elements One denotes differential basis elements dx^i and $dS_i = \frac{1}{2}\epsilon_{ijk}dx^j \wedge dx^k$, for i, j, k = 1, 2, 3 in vector notation as

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$$d\mathbf{x} := (dx^{1}, dx^{2}, dx^{3}),$$

$$d\mathbf{S} = (dS_{1}, dS_{2}, dS_{3})$$

$$:= (dx^{2} \wedge dx^{3}, dx^{3} \wedge dx^{1}, dx^{1} \wedge dx^{2}),$$

$$dS_{i} := \frac{1}{2} \epsilon_{ijk} dx^{j} \wedge dx^{k},$$

$$d^{3}x = dVol := dx^{1} \wedge dx^{2} \wedge dx^{3}$$

$$= \frac{1}{6} \epsilon_{ijk} dx^{i} \wedge dx^{j} \wedge dx^{k}.$$

Exercise. (Vector calculus operations) Show that contraction of the vector field $X = X^j \partial_j =: \mathbf{X} \cdot \nabla$ with the differential basis elements recovers the following familiar operations among vectors:

$$X \sqcup d\mathbf{x} = \mathbf{X},$$

$$X \sqcup d\mathbf{S} = \mathbf{X} \times d\mathbf{x},$$

(or, $X \sqcup dS_i = \epsilon_{ijk} X^j dx^k$)

$$Y \sqcup X \sqcup d\mathbf{S} = \mathbf{X} \times \mathbf{Y},$$

$$X \sqcup d^3 x = \mathbf{X} \cdot d\mathbf{S} = X^k dS_k,$$

$$Y \sqcup X \sqcup d^3 x = \mathbf{X} \times \mathbf{Y} \cdot d\mathbf{x} = \epsilon_{ijk} X^i Y^j dx^k,$$

$$Z \sqcup Y \sqcup X \sqcup d^3 x = \mathbf{X} \times \mathbf{Y} \cdot \mathbf{Z}.$$

Exercise. (Exterior derivatives in vector notation)

that the exterior derivative and wedge product satisfy the following relations in components and in three-dimensional vector

★ Show

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notation:

$$\begin{split} df &= f_{,j} \, dx^j =: \nabla f \cdot d\mathbf{x} \,, \\ 0 &= d^2 f &= f_{,jk} \, dx^k \wedge dx^j \,, \\ df \wedge dg &= f_{,j} \, dx^j \wedge g_{,k} \, dx^k \\ &=: (\nabla f \times \nabla g) \cdot d\mathbf{S} \,, \\ df \wedge dg \wedge dh &= f_{,j} \, dx^j \wedge g_{,k} \, dx^k \wedge h_{,l} \, dx^l \\ &=: (\nabla f \cdot \nabla g \times \nabla h) \, d^3 x \,. \end{split}$$

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Exercise. (Vector calculus formulas) Show that the exterior derivative yields the following vector calculus formulas:

$$df = \nabla f \cdot d\mathbf{x},$$

$$d(\mathbf{v} \cdot d\mathbf{x}) = (\operatorname{curl} \mathbf{v}) \cdot d\mathbf{S},$$

$$d(\mathbf{A} \cdot d\mathbf{S}) = (\operatorname{div} \mathbf{A}) d^{3}x.$$

The compatibility condition $d^2 = 0$ is written for these forms as

$$0 = d^2 f = d(\nabla f \cdot d\mathbf{x}) = (\operatorname{curl}\operatorname{grad} f) \cdot d\mathbf{S},$$

$$0 = d^2(\mathbf{v} \cdot d\mathbf{x}) = d((\operatorname{curl} \mathbf{v}) \cdot d\mathbf{S}) = (\operatorname{div}\operatorname{curl} \mathbf{v}) d^3 x.$$

The product rule is written for these forms as

$$d(f(\mathbf{A} \cdot d\mathbf{x})) = df \wedge \mathbf{A} \cdot d\mathbf{x} + f \operatorname{curl} \mathbf{A} \cdot d\mathbf{S}$$
$$= (\nabla f \times \mathbf{A} + f \operatorname{curl} \mathbf{A}) \cdot d\mathbf{S}$$
$$= \operatorname{curl} (f\mathbf{A}) \cdot d\mathbf{S},$$

$$d((\mathbf{A} \cdot d\mathbf{x}) \wedge (\mathbf{B} \cdot d\mathbf{x})) = (\operatorname{curl} \mathbf{A}) \cdot d\mathbf{S} \wedge \mathbf{B} \cdot d\mathbf{x} - \mathbf{A} \cdot d\mathbf{x} \wedge (\operatorname{curl} \mathbf{B}) \cdot d\mathbf{S}$$
$$= (\mathbf{B} \cdot \operatorname{curl} \mathbf{A} - \mathbf{A} \cdot \operatorname{curl} \mathbf{B})d^{3}x$$
$$= d((\mathbf{A} \times \mathbf{B}) \cdot d\mathbf{S})$$
$$= \operatorname{div}(\mathbf{A} \times \mathbf{B})d^{3}x .$$

These calculations yield familiar formulas from vector calculus for quantities $\operatorname{curl}(\operatorname{grad})$, $\operatorname{div}(\operatorname{curl})$, $\operatorname{curl}(f\mathbf{A})$ and $\operatorname{div}(\mathbf{A} \times \mathbf{B})$.

9.8 Integral calculus formulas

Exercise. (Integral calculus formulas) Show that the Stokes' theorem for the vector calculus formulas yields the following familiar results in \mathbb{R}^3 :

• The *fundamental theorem of calculus*, upon integrating df along a curve in \mathbb{R}^3 starting at point a and ending at point b:

$$\int_{a}^{b} df = \int_{a}^{b} \nabla f \cdot d\mathbf{x} = f(b) - f(a)$$

• The *classical Stokes theorem*, for a compact surface S with boundary ∂S :

$$\int_{S} (\operatorname{curl} \mathbf{v}) \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{v} \cdot d\mathbf{x} \,.$$

(For a planar surface $S \in \mathbb{R}^2$, this is *Green's theorem*.)

• The *Gauss divergence theorem*, for a compact spatial domain D with boundary ∂D :

$$\int_{D} (\operatorname{div} \mathbf{A}) \, d^{3}x = \oint_{\partial D} \mathbf{A} \cdot d\mathbf{S} \, . \qquad \bigstar$$

These exercises illustrate the following,

Theorem 39 (Stokes' theorem). Suppose M is a compact oriented k-dimensional manifold with boundary ∂M and α is a smooth (k-1)-form on M. Then

$$\int_M d\alpha = \oint_{\partial M} \alpha \,.$$

9.9 Summary and an exercise

Summary

The pull-back ϕ_t^* of a smooth flow ϕ_t generated by a smooth vector field X on a smooth manifold M commutes with the exterior derivative d, wedge product \wedge and contraction \square .

That is, for k-forms $\alpha, \beta \in \Lambda^k(M)$, and $m \in M$, the pull-back ϕ_t^* satisfies

$$d(\phi_t^*\alpha) = \phi_t^* d\alpha ,$$

$$\phi_t^*(\alpha \wedge \beta) = \phi_t^* \alpha \wedge \phi_t^* \beta ,$$

$$\phi_t^*(X \sqcup \alpha) = \phi_t^* X \sqcup \phi_t^* \alpha$$

In addition, the Lie derivative $\pounds_X \alpha$ of a k-form $\alpha \in \Lambda^k(M)$ by the vector field X tangent to the flow ϕ_t on M is defined either dynamically or geometrically (by Cartan's formula) as

$$\pounds_X \alpha = \left. \frac{d}{dt} \right|_{t=0} (\phi_t^* \alpha) = X \, \sqcup \, d\alpha + d(X \, \sqcup \, \alpha), \tag{88}$$

in which the last is Cartan's geometric formula in (88) for the Lie derivative.

Exercise.

- (a) Verify the formula $[X, Y] \sqcup \alpha = \pounds_X(Y \sqcup \alpha) Y \sqcup (\pounds_X \alpha)$.
- (b) Use (b) to verify $\pounds_{[X,Y]}\alpha = \pounds_X \pounds_Y \alpha \pounds_Y \pounds_X \alpha$.
- (c) Use (c) to verify the Jacobi identity.
- (d) Use (c) to verify that the divergence-free vector fields are closed under commutation.
- (e) For a top-form α show that

$$[X, Y] \,\lrcorner\, \alpha = d\big(X \,\lrcorner\, (Y \,\lrcorner\, \alpha)\big) \,. \tag{89}$$

(f) Write the equivalent of equation (89) as a formula in vector calculus.

Answer.

(a) The required formula follows immediately from the product rule in (83) for the dynamical definition of the Lie derivative. Since pull-back commutes with contraction, insertion of a vector field into a k-form transforms under the flow ϕ_t of a smooth vector field Y as

$$\phi_t^*(Y \, \lrcorner \, \alpha) = \phi_t^* Y \, \lrcorner \, \phi_t^* \alpha$$

A direct computation using the dynamical definition of the Lie derivative $\pounds_Y \alpha = \frac{d}{dt}|_{t=0}(\phi_t^*\alpha)$, then yields

$$\frac{d}{dt}\Big|_{t=0}\phi_t^*(Y \,\lrcorner\, \alpha) = \left(\frac{d}{dt}\Big|_{t=0}\phi_t^*Y\right) \,\lrcorner\, \alpha \\ + Y \,\lrcorner\, \left(\frac{d}{dt}\Big|_{t=0}\phi_t^*\alpha\right)$$

Hence, we recognise that the desired formula is the *product rule* met earlier in equation (83):

$$\pounds_X(Y \,\lrcorner\, \alpha) = (\pounds_X Y) \,\lrcorner\, \alpha + Y \,\lrcorner\, (\pounds_X \alpha) \,.$$

 \star

(b) Insert $\pounds_X Y = [X, Y]$ into the product rule formula in part (b). Then

$$[X, Y] \, \lrcorner \, \alpha = \pounds_X(Y \, \lrcorner \, \alpha) - Y \, \lrcorner \, (\pounds_X \alpha).$$

Now use Cartan's formula in (88)

$$\pounds_X \alpha = \frac{d}{dt} \Big|_{t=0} (\phi_t^* \alpha) = X \, \sqcup \, d\alpha + d(X \, \sqcup \, \alpha),$$

to compute the required result, as

$$\begin{aligned} \pounds_{[X,Y]} \alpha &= d([X,Y] \,\lrcorner\, \alpha) + [X,Y] \,\lrcorner\, d\alpha \\ &= d\big(\pounds_X(Y \,\lrcorner\, \alpha) - Y \,\lrcorner\, (\pounds_X \alpha)\big) \\ &+ \pounds_X(Y \,\lrcorner\, d\alpha) - Y \,\lrcorner\, (\pounds_X d\alpha) \\ &= \pounds_X d(Y \,\lrcorner\, \alpha) - d(Y \,\lrcorner\, (\pounds_X \alpha) \\ &+ \pounds_X(Y \,\lrcorner\, d\alpha) - Y \,\lrcorner\, d(\pounds_X \alpha) \\ &= \pounds_X(\pounds_Y \alpha) - \pounds_Y(\pounds_X \alpha) \,. \end{aligned}$$

Can you think of an alternative proof based on the dynamical definition of the Lie derivative?

- (c) Applying part (b), $(\pounds_{[X,Y]}\alpha = \pounds_X \pounds_Y \alpha \pounds_Y \pounds_X \alpha)$ to $\alpha = d^3x$ proves that $\pounds_{[X,Y]} d^3x = 0$; since both $\pounds_Y d^3x = 0 = \pounds_X d^3x$, because, e.g., $\pounds_Y d^3x = (\operatorname{div} Y) d^3x$.
- (d) As a consequence of part (b),

$$\pounds_{[Z,[X,Y]]} \alpha = \pounds_Z (\pounds_X \pounds_Y - \pounds_Y \pounds_X) \alpha - (\pounds_X \pounds_Y - \pounds_Y \pounds_X) \pounds_Z \alpha = \pounds_Z \pounds_X \pounds_Y \alpha - \pounds_Z \pounds_Y \pounds_X \alpha - \pounds_X \pounds_Y \pounds_Z \alpha + \pounds_Y \pounds_X \pounds_Z \alpha ,$$

and summing over cyclic permutations verifies that

$$\pounds_{[Z,[X,Y]]} \alpha + \pounds_{[X,[Y,Z]]} \alpha + \pounds_{[Y,[Z,X]]} \alpha = 0.$$

This is the Jacobi identity for the Lie derivative.
(e) Substituting the relation $\pounds_X Y = [X, Y]$ into the product rule above in part (b) and rearranging yields

$$[X, Y] \,\lrcorner\, \alpha = \pounds_X(Y \,\lrcorner\, \alpha) - Y \,\lrcorner\, (\pounds_X \alpha) \,, \tag{90}$$

as required, for an arbitrary k-form α .

From formula (90), we have

$$[X, Y] \sqcup \alpha = \pounds_X (Y \sqcup \alpha) - Y \sqcup (\pounds_X \alpha)$$

= $d(X \sqcup (Y \sqcup \alpha) + X \sqcup d(Y \sqcup \alpha)) - Y \sqcup (\pounds_X \alpha)$
= $d(X \sqcup (Y \sqcup \alpha)) + X \sqcup (\pounds_Y \alpha - Y \sqcup d\alpha) - Y \sqcup (\pounds_X \alpha)$
= $d(X \sqcup (Y \sqcup \alpha)) + X \sqcup (\pounds_Y \alpha) - Y \sqcup (\pounds_X \alpha)$
 $[X, Y] \sqcup \alpha = d(X \sqcup (Y \sqcup \alpha)) + (\operatorname{div} \mathbf{Y}) X \sqcup \alpha - (\operatorname{div} \mathbf{X}) Y \sqcup \alpha.$ (91)

The last two steps to obtain (91) follow, because $d\alpha = 0$ and $\pounds_X \alpha = (\operatorname{div} \mathbf{X}) \alpha$ for a top-form α . For divergence-free vectors \mathbf{X} and \mathbf{Y} , the last result takes the elegant form,

$$[X, Y] \,\lrcorner\, \alpha = d \big(X \,\lrcorner\, (Y \,\lrcorner\, \alpha) \big), \tag{92}$$

when div $\mathbf{X} = 0 = \operatorname{div} \mathbf{Y}$.

(f) The vector calculus formula to which equation (91) is equivalent may be found by writing its left and right sides in a coordinate basis, as

$$[X, Y] \sqcup \alpha = (\mathbf{X} \cdot \nabla \mathbf{Y} - \mathbf{Y} \cdot \nabla \mathbf{X}) \cdot d\mathbf{S}$$
$$d(X \sqcup (Y \sqcup \alpha)) + X \sqcup (\pounds_Y \alpha) - Y \sqcup (\pounds_X \alpha) = -\operatorname{curl} (\mathbf{X} \times \mathbf{Y}) \cdot d\mathbf{S} + (\operatorname{div} \mathbf{Y}) \mathbf{X} \cdot d\mathbf{S} - (\operatorname{div} \mathbf{X}) \mathbf{Y} \cdot d\mathbf{S}$$

Thus, equation (91) is equivalent to the vector calculus identity

$$\left(\mathbf{X} \cdot \nabla \mathbf{Y} - \mathbf{Y} \cdot \nabla \mathbf{X}\right) = -\operatorname{curl}\left(\mathbf{X} \times \mathbf{Y}\right) + \left(\operatorname{div} \mathbf{Y}\right)\mathbf{X} - \left(\operatorname{div} \mathbf{X}\right)\mathbf{Y}.$$

This is the fundamental identity of fluid mechanics when $\mathbf{X} = \mathbf{u}$ and $\mathbf{Y} = \boldsymbol{\omega}$. That is,

 $-\operatorname{curl}\left(\mathbf{u}\times\boldsymbol{\omega}\right) = \mathbf{u}\cdot\nabla\boldsymbol{\omega} + \left(\operatorname{div}\mathbf{u}\right)\boldsymbol{\omega} - \boldsymbol{\omega}\cdot\nabla\mathbf{u} - \left(\operatorname{div}\boldsymbol{\omega}\right)\mathbf{u}.$

10 Geometric formulations of ideal fluid dynamics

10.1 Euler's fluid equations

Euler's equations for the incompressible motion of an ideal flow of a fluid of unit density and velocity \mathbf{u} satisfying div $\mathbf{u} = 0$ in a rotating frame with Coriolis parameter curl $\mathbf{R} = 2\mathbf{\Omega}$ are given in the form of Newton's law of force by

$$\underbrace{\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}}_{\text{acceleration}} = \underbrace{\mathbf{u} \times 2\Omega}_{\text{Coriolis}} - \underbrace{\nabla p}_{\text{pressure}}$$
(93)

Requiring preservation of the divergence-free (volume-preserving) constraint $\nabla \cdot \mathbf{u} = 0$ results in a Poisson equation for pressure p, which may be written in several equivalent forms,

$$-\Delta p = \operatorname{div} (\mathbf{u} \cdot \nabla \mathbf{u} - \mathbf{u} \times 2\mathbf{\Omega})$$

= $u_{i,j} u_{j,i} - \operatorname{div} (\mathbf{u} \times 2\mathbf{\Omega})$
= $\operatorname{tr} S^2 - \frac{1}{2} |\operatorname{curl} \mathbf{u}|^2 - \operatorname{div} (\mathbf{u} \times 2\mathbf{\Omega}),$ (94)

where $S = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$ is the *strain-rate tensor*.

The Newton's law equation for Euler fluid motion in (93) may be rearranged into an alternative form,

$$\partial_t \mathbf{v} - \mathbf{u} \times \boldsymbol{\omega} + \nabla \left(p + \frac{1}{2} |\mathbf{u}|^2 \right) = 0,$$
(95)

where we denote

$$\mathbf{v} \equiv \mathbf{u} + \mathbf{R}, \quad \boldsymbol{\omega} = \operatorname{curl} \mathbf{v} = \operatorname{curl} \mathbf{u} + 2\mathbf{\Omega},$$
(96)

and introduce the Lamb vector,

$$\boldsymbol{\ell} := -\mathbf{u} \times \boldsymbol{\omega} \,, \tag{97}$$

which represents the nonlinearity in Euler's fluid equation (95). The Poisson equation (94) for pressure p may now be expressed in terms of the divergence of the Lamb vector,

$$-\Delta\left(p+\frac{1}{2}|\mathbf{u}|^{2}\right) = \operatorname{div}(-\mathbf{u}\times\operatorname{curl}\mathbf{v}) = \operatorname{div}\boldsymbol{\ell}.$$
(98)

Remark 40 (Boundary conditions).

Because the velocity \mathbf{u} must be tangent to any fixed boundary, the normal component of the motion equation must vanish. This requirement produces a Neumann condition for pressure given by

$$\partial_n \left(p + \frac{1}{2} |\mathbf{u}|^2 \right) + \hat{\mathbf{n}} \cdot \boldsymbol{\ell} = 0, \qquad (99)$$

at a fixed boundary with unit outward normal vector $\hat{\mathbf{n}}$.

Remark 41 (Helmholtz vorticity dynamics).

Taking the curl of the Euler fluid equation (95) yields the Helmholtz vorticity equation

$$\partial_t \,\boldsymbol{\omega} - \operatorname{curl}(\mathbf{u} \times \boldsymbol{\omega}) = 0\,,\tag{100}$$

whose geometrical meaning will emerge in discussing Stokes' Theorem 57 for the vorticity of a rotating fluid.

The rotation terms have now been fully integrated into both the dynamics and the boundary conditions. In this form, the *Kelvin circulation theorem* and the *Stokes vorticity theorem* will emerge naturally together as geometrical statements.

10.2 Kelvin's circulation theorem

Theorem 42 (*Kelvin's circulation theorem*). The Euler equations (93) preserve the circulation integral I(t) defined by

$$I(t) = \oint_{c(\mathbf{u})} \mathbf{v} \cdot d\mathbf{x} \,, \tag{101}$$

where $c(\mathbf{u})$ is a closed circuit moving with the fluid at velocity \mathbf{u} .

Proof. The dynamical definition of the Lie derivative in (88) yields the following for the time rate of change of this circulation integral:

$$\frac{d}{dt} \oint_{c(\mathbf{u})} \mathbf{v} \cdot d\mathbf{x} = \oint_{c(\mathbf{u})} \left(\frac{\partial}{\partial t} + \mathcal{L}_{\mathbf{u}} \right) (\mathbf{v} \cdot d\mathbf{x})
= \oint_{c(\mathbf{u})} \left(\frac{\partial \mathbf{v}}{\partial t} + \frac{\partial \mathbf{v}}{\partial x^{j}} u^{j} + v_{j} \frac{\partial u^{j}}{\partial \mathbf{x}} \right) \cdot d\mathbf{x}
= -\oint_{c(\mathbf{u})} \nabla \left(p + \frac{1}{2} |\mathbf{u}|^{2} - \mathbf{u} \cdot \mathbf{v} \right) \cdot d\mathbf{x}
= -\oint_{c(\mathbf{u})} d \left(p + \frac{1}{2} |\mathbf{u}|^{2} - \mathbf{u} \cdot \mathbf{v} \right) = 0.$$
(102)

The Cartan formula in (88) defines the Lie derivative of the circulation integrand in the equivalent form that we need for the third step and will also use in a moment for Stokes' theorem:

$$\begin{aligned}
\pounds_{\mathbf{u}}(\mathbf{v} \cdot d\mathbf{x}) &= \left(\mathbf{u} \cdot \nabla \mathbf{v} + v_{j} \nabla u^{j}\right) \cdot d\mathbf{x} \\
&= u \, \lrcorner \, d(\mathbf{v} \cdot d\mathbf{x}) + d(u \, \lrcorner \, \mathbf{v} \cdot d\mathbf{x}) \\
&= u \, \lrcorner \, (\operatorname{curl} \mathbf{v} \cdot d\mathbf{S}) + d(\mathbf{u} \cdot \mathbf{v}) \\
&= \left(-\mathbf{u} \times \operatorname{curl} \mathbf{v} + \nabla(\mathbf{u} \cdot \mathbf{v})\right) \cdot d\mathbf{x} \,.
\end{aligned}$$
(103)

This identity recasts Euler's equation into the following geometric form:

$$\begin{pmatrix} \frac{\partial}{\partial t} + \pounds_{\mathbf{u}} \end{pmatrix} (\mathbf{v} \cdot d\mathbf{x}) = (\partial_t \mathbf{v} - \mathbf{u} \times \operatorname{curl} \mathbf{v} + \nabla(\mathbf{u} \cdot \mathbf{v})) \cdot d\mathbf{x}$$

$$= -\nabla \left(p + \frac{1}{2} |\mathbf{u}|^2 - \mathbf{u} \cdot \mathbf{v} \right) \cdot d\mathbf{x}$$

$$= -d \left(p + \frac{1}{2} |\mathbf{u}|^2 - \mathbf{u} \cdot \mathbf{v} \right).$$
(104)

This finishes the last step in the proof (102), because the integral of an exact differential around a closed loop vanishes.

The exterior derivative of the Euler fluid equation in the form (104) yields Stokes' theorem, after using the commutativity of the exterior and Lie derivatives $[d, \mathcal{L}_{\mathbf{u}}] = 0$,

$$d\pounds_{\mathbf{u}}(\mathbf{v} \cdot d\mathbf{x}) = \pounds_{\mathbf{u}} d(\mathbf{v} \cdot d\mathbf{x})$$

$$= \pounds_{\mathbf{u}}(\operatorname{curl} \mathbf{v} \cdot d\mathbf{S})$$

$$= -\operatorname{curl}(\mathbf{u} \times \operatorname{curl} \mathbf{v}) \cdot d\mathbf{S}$$

$$= [\mathbf{u} \cdot \nabla \operatorname{curl} \mathbf{v} + \operatorname{curl} \mathbf{v}(\operatorname{div} \mathbf{u}) - (\operatorname{curl} \mathbf{v}) \cdot \nabla \mathbf{u}] \cdot d\mathbf{S},$$

$$\text{(by div } \mathbf{u} = 0) = [\mathbf{u} \cdot \nabla \operatorname{curl} \mathbf{v} - (\operatorname{curl} \mathbf{v}) \cdot \nabla \mathbf{u}] \cdot d\mathbf{S}$$

$$=: [u, \operatorname{curl} v] \cdot d\mathbf{S},$$
(105)

where $[u, \operatorname{curl} v]$ denotes (minus) the *Jacobi–Lie bracket* of the vector fields u and $\operatorname{curl} v$.

This calculation proves the following.

Theorem 43. Euler's fluid equations (95) imply that

$$\frac{\partial\omega}{\partial t} = -\left[u,\,\omega\right] \tag{106}$$

where $[u, \omega]$ denotes the Jacobi-Lie bracket of the divergenceless vector fields u and $\omega := \operatorname{curl} v$.

The exterior derivative of Euler's equation in its geometric form (104) is equivalent to the curl of its vector form (95). That is,

$$d\left(\frac{\partial}{\partial t} + \pounds_{\mathbf{u}}\right)(\mathbf{v} \cdot d\mathbf{x}) = \left(\frac{\partial}{\partial t} + \pounds_{\mathbf{u}}\right)(\operatorname{curl} \mathbf{v} \cdot d\mathbf{S}) = 0.$$
(107)

Hence from the calculation in (105) and the Helmholtz vorticity equation (107) we have

$$\left(\frac{\partial}{\partial t} + \pounds_{\mathbf{u}}\right) (\operatorname{curl} \mathbf{v} \cdot d\mathbf{S}) = \left(\partial_t \,\boldsymbol{\omega} - \operatorname{curl}(\mathbf{u} \times \boldsymbol{\omega})\right) \cdot d\mathbf{S} = 0\,,\tag{108}$$

in which one denotes $\boldsymbol{\omega} := \operatorname{curl} \mathbf{v}$. This Lie-derivative version of the Helmholtz vorticity equation may be used to prove the following form of Stokes' theorem for the Euler equations in a rotating frame.

Theorem 44. [Kelvin/Stokes' theorem for vorticity of a rotating fluid]

$$\frac{d}{dt} \oint_{c(\mathbf{u})} \mathbf{v} \cdot d\mathbf{x} = \frac{d}{dt} \iint_{S(\mathbf{u})} \operatorname{curl} \mathbf{v} \cdot d\mathbf{S}
= \iint_{S(\mathbf{u})} \left(\frac{\partial}{\partial t} + \mathcal{L}_{\mathbf{u}} \right) (\operatorname{curl} \mathbf{v} \cdot d\mathbf{S})
= \iint_{S(\mathbf{u})} \left(\partial_t \, \boldsymbol{\omega} - \operatorname{curl} \left(\mathbf{u} \times \boldsymbol{\omega} \right) \right) \cdot d\mathbf{S} = 0,$$
(109)

where the surface $S(\mathbf{u})$ is bounded by an arbitrary circuit $\partial S = c(\mathbf{u})$ moving with the fluid.

10.3 Steady solutions: Lamb surfaces

According to Theorem 43, Euler's fluid equations (95) imply that

$$\frac{\partial \omega}{\partial t} = -\left[u,\,\omega\right].\tag{110}$$

Consequently, the vector fields u, ω in *steady* Euler flows, which satisfy $\partial_t \omega = 0$, also satisfy the condition necessary for the Frobenius theorem to hold – namely, that their Jacobi–Lie bracket vanishes. That is, in smooth steady, or equilibrium, solutions of Euler's fluid equations, the flows of the two divergenceless vector fields u and ω commute with each other and lie on a surface in three dimensions.

A sufficient condition for this commutation relation is that the *Lamb vector* $\boldsymbol{\ell} := -\mathbf{u} \times \operatorname{curl} \mathbf{v}$ in (97) satisfies

$$\boldsymbol{\ell} := -\mathbf{u} \times \operatorname{curl} \mathbf{v} = \nabla H(\mathbf{x}), \tag{111}$$

for some smooth function $H(\mathbf{x})$. This condition means that the flows of vector fields u and curl v (which are *steady* flows of the Euler equations) are both confined to the same surface $H(\mathbf{x}) = const$. Such a surface is called a *Lamb surface*.

The vectors of velocity (**u**) and total vorticity (curl **v**) for a steady Euler flow are both perpendicular to the normal vector to the Lamb surface along $\nabla H(\mathbf{x})$. That is, the Lamb surface is invariant under the flows of both vector fields, *viz*

$$\pounds_u H = \mathbf{u} \cdot \nabla H = 0 \quad \text{and} \quad \pounds_{\operatorname{curl} v} H = \operatorname{curl} \mathbf{v} \cdot \nabla H = 0.$$
(112)

The Lamb surface condition (111) has the following coordinate-free representation.

Theorem 45 (*Lamb surface condition*). The Lamb surface condition (111) is equivalent to the following double substitution of vector fields into the volume form,

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$$dH = u \, \sqcup \, \operatorname{curl} v \, \sqcup \, d^{\,3}x \,. \tag{113}$$

Proof. Recall that the contraction of vector fields with forms yields the following useful formula for the surface element:

$$\nabla \, \lrcorner \, d^3 x = d\mathbf{S} \,. \tag{114}$$

Then using results from previous exercises in vector calculus operations one finds by direct computation that

$$u \sqcup \operatorname{curl} v \sqcup d^{3}x = u \sqcup (\operatorname{curl} \mathbf{v} \cdot d\mathbf{S})$$

= $-(\mathbf{u} \times \operatorname{curl} \mathbf{v}) \cdot d\mathbf{x}$
= $\nabla H \cdot d\mathbf{x}$
= dH . (115)

Remark 46.

Formula (115)

$$u \, \sqcup \, (\operatorname{curl} \mathbf{v} \cdot d\mathbf{S}) = dH$$

is to be compared with

 $X_h \, \sqcup \, \omega = dH \,,$

in the definition of a Hamiltonian vector field in Equation (82) of Theorem 20. Likewise, the stationary case of the Helmholtz vorticity equation (107), namely,

$$\pounds_{\mathbf{u}}(\operatorname{curl} \mathbf{v} \cdot d\mathbf{S}) = 0, \qquad (116)$$

is to be compared with the proof of Poincaré's theorem in Corollary 23

$$\pounds_{X_h}\omega = d(X_h \, \bot \, \omega) = d^2 H = 0 \, .$$

Thus, the two-form $\operatorname{curl} \mathbf{v} \cdot d\mathbf{S}$ plays the same role for stationary Euler fluid flows as the symplectic form $dq \wedge dp$ plays for canonical Hamiltonian flows. We seek the corresponding symplectic coordinates.

Definition 47. The Clebsch representation of the one-form $\mathbf{v} \cdot d\mathbf{x}$ is defined by

$$\mathbf{v} \cdot d\mathbf{x} = -\prod d\Xi + d\Psi \,. \tag{117}$$

The functions Ξ , Π and Ψ are called **Clebsch potentials** for the vector \mathbf{v} .³

In terms of the Clebsch representation (117) of the one-form $\mathbf{v} \cdot d\mathbf{x}$, the total vorticity flux curl $\mathbf{v} \cdot d\mathbf{S} = d(\mathbf{v} \cdot d\mathbf{x})$ is the exact two-form,

$$\operatorname{curl} \mathbf{v} \cdot d\mathbf{S} = d\Xi \wedge d\Pi \,. \tag{118}$$

This amounts to writing the flow lines of the vector field of the total vorticity curl v as the intersections of level sets of surfaces $\Xi = const$ and $\Pi = const$. In other words,

$$\operatorname{curl} \mathbf{v} = \nabla \Xi \times \nabla \Pi, \tag{119}$$

with the assumption that these level sets foliate \mathbb{R}^3 . That is, one assumes that any point in \mathbb{R}^3 along the flow of the total vorticity vector field curl v may be assigned to a regular intersection of these level sets. The main result of this assumption is the following theorem.

Theorem 48 (Lamb surfaces are symplectic manifolds). [ArKh1992, ArKh1998] The steady flow of the vector field u satisfying the symmetry relation given by the vanishing of the commutator $[u, \operatorname{curl} v] = 0$ on a three-dimensional manifold $M \in \mathbb{R}^3$ reduces to incompressible flow on a two-dimensional symplectic manifold whose canonically conjugate coordinates (Ξ, Π) are provided by the total vorticity flux

$$\operatorname{curl} v \, \sqcup \, d^3 x = \operatorname{curl} \mathbf{v} \cdot d\mathbf{S} = d\Xi \wedge d\Pi$$
.

The reduced flow is canonically Hamiltonian on this symplectic manifold. Furthermore, the reduced Hamiltonian is precisely the restriction of the invariant H onto the reduced phase space.

Proof. Restricting formula (115) to coordinates on a total vorticity flux surface (118) yields the exterior derivative of the Hamiltonian,

$$dH(\Xi, \Pi) = u \sqcup (\operatorname{curl} \mathbf{v} \cdot d\mathbf{S})$$

$$= u \sqcup (d\Xi \wedge d\Pi)$$

$$= (\mathbf{u} \cdot \nabla\Xi) d\Pi - (\mathbf{u} \cdot \nabla\Pi) d\Xi$$

$$=: \frac{d\Xi}{dT} d\Pi - \frac{d\Pi}{dT} d\Xi$$

$$= \frac{\partial H}{\partial \Pi} d\Pi + \frac{\partial H}{\partial \Xi} d\Xi, \qquad (120)$$

³The Clebsch representation is another example of a cotangent lift momentum map.

where $T \in \mathbb{R}$ is the time parameter along the flow lines of the steady vector field u, which carries the Lagrangian fluid parcels. On identifying corresponding terms, the steady flow of the fluid velocity \mathbf{u} is found to obey the canonical Hamiltonian equations,

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$$(\mathbf{u} \cdot \nabla \Xi) = \pounds_u \Xi \quad =: \quad \frac{d\Xi}{dT} = \frac{\partial H}{\partial \Pi} = \{\Xi, H\},$$
(121)

$$(\mathbf{u} \cdot \nabla \Pi) = \pounds_u \Pi \quad =: \quad \frac{d\Pi}{dT} = -\frac{\partial H}{\partial \Xi} = \left\{ \Pi, H \right\}, \tag{122}$$

where $\{\cdot, \cdot\}$ is the canonical Poisson bracket for the symplectic form $d\Xi \wedge d\Pi$.

Corollary 49. The vorticity flux $d\Xi \wedge d\Pi$ is invariant under the flow of the velocity vector field u.

Proof. By (120), one verifies

$$\ell_u(d\Xi \wedge d\Pi) = d(u \sqcup (d\Xi \wedge d\Pi)) = d^2 H = 0.$$

This is the standard computation in the proof of Poincaré's theorem in Corollary 23 for the preservation of a symplectic form by a canonical transformation. Its interpretation here is that the steady Euler flows preserve the total vorticity flux, curl $\mathbf{v} \cdot d\mathbf{S} = d\Xi \wedge d\Pi$. \Box

10.4 The conserved helicity of ideal incompressible flows

Definition 50 (Helicity). The helicity $\Lambda[\operatorname{curl} \mathbf{v}]$ of a divergence-free vector field $\operatorname{curl} v$ that is tangent to the boundary ∂D of a simply connected domain $D \in \mathbb{R}^3$ is defined as

$$\Lambda[\operatorname{curl} \mathbf{v}] = \int_D \mathbf{v} \cdot \operatorname{curl} \mathbf{v} \, d^3 x \,, \tag{123}$$

where \mathbf{v} is a divergence-free vector-potential for the field curl \mathbf{v} .

Remark 51.

The helicity is unchanged by adding a gradient to the vector \mathbf{v} . Thus, \mathbf{v} is not unique and div $\mathbf{v} = 0$ is not a restriction for simply connected domains in \mathbb{R}^3 , provided curl \mathbf{v} is tangent to the boundary ∂D .

The helicity of a vector field $\operatorname{curl} v$ measures the total linking of its field lines, or their relative winding. (For details and mathematical history, see [ArKh1998].) The idea of helicity goes back to Helmholtz and Kelvin in the 19th century. The principal feature of this concept for fluid dynamics is embodied in the following theorem.

Theorem 52 (*Euler flows preserve helicity*). When homogeneous or periodic boundary conditions are imposed, Euler's equations for an ideal incompressible fluid flow in a rotating frame with Coriolis parameter curl $\mathbf{R} = 2\mathbf{\Omega}$ preserves the helicity

$$\Lambda[\operatorname{curl} \mathbf{v}] = \int_{D} \mathbf{v} \cdot \operatorname{curl} \mathbf{v} \, d^{3}x \,, \tag{124}$$

with $\mathbf{v} = \mathbf{u} + \mathbf{R}$, for which \mathbf{u} is the divergenceless fluid velocity (div $\mathbf{u} = 0$) and curl $\mathbf{v} = \text{curl } \mathbf{u} + 2\mathbf{\Omega}$ is the total vorticity.

Proof. Rewrite the geometric form of the Euler equations (104) for rotating incompressible flow with unit mass density in terms of the circulation one-form $v := \mathbf{v} \cdot d\mathbf{x}$ as

$$\left(\partial_t + \mathcal{L}_u\right)v = -d\left(p + \frac{1}{2}|\mathbf{u}|^2 - \mathbf{u} \cdot \mathbf{v}\right) =: -d\,\boldsymbol{\varpi}\,,\tag{125}$$

and $\pounds_u d^3 x = 0$, where ϖ is an augmented pressure variable,

$$\varpi := p + \frac{1}{2} |\mathbf{u}|^2 - \mathbf{u} \cdot \mathbf{v} \,. \tag{126}$$

The fluid velocity vector field is denoted as $u = \mathbf{u} \cdot \nabla$ with div $\mathbf{u} = 0$. Then the *helicity density*, defined as

$$v \wedge dv = \mathbf{v} \cdot \operatorname{curl} \mathbf{v} \, d^3 x = \lambda \, d^3 x \,, \quad \text{with} \quad \lambda = \mathbf{v} \cdot \operatorname{curl} \mathbf{v} \,,$$

$$\tag{127}$$

obeys the dynamics it inherits from the Euler equations,

$$(\partial_t + \mathcal{L}_u)(v \wedge dv) = -d\varpi \wedge dv - v \wedge d^2 \varpi = -d(\varpi \, dv), \qquad (128)$$

after using $d^2 \varpi = 0$ and $d^2 v = 0$. In vector form, this result may be expressed as a conservation law,

$$\left(\partial_t \lambda + \operatorname{div} \lambda \mathbf{u}\right) d^3 x = -\operatorname{div}(\varpi \operatorname{curl} \mathbf{v}) d^3 x \,. \tag{129}$$

Consequently, the time derivative of the integrated helicity in a domain D obeys

$$\frac{d}{dt}\Lambda[\operatorname{curl} \mathbf{v}] = \int_{D} \partial_{t}\lambda \, d^{3}x = -\int_{D} \operatorname{div}(\lambda \mathbf{u} + \varpi \operatorname{curl} \mathbf{v}) \, d^{3}x$$

$$= -\oint_{\partial D} (\lambda \mathbf{u} + \varpi \operatorname{curl} \mathbf{v}) \cdot d\mathbf{S}, \qquad (130)$$

which vanishes when homogeneous, or periodic, or even Neumann boundary conditions are imposed on the values of \mathbf{u} and curl \mathbf{v} at the boundary ∂D .

Remark 53.

This result means the *helicity integral*

$$\Lambda[\operatorname{curl} \mathbf{v}] = \int_D \lambda \, d^3 x$$

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is conserved in periodic domains, or in all of \mathbb{R}^3 with vanishing boundary conditions at spatial infinity. However, if either the velocity or total vorticity at the boundary possesses a nonzero normal component, then the boundary is a *source* of helicity (that is, it causes winding of field lines of curl v). For a fixed impervious boundary, the normal component of velocity does vanish, but no such condition is imposed on the total vorticity by the physics of fluid flow. Thus, we have the following.

Corollary 54. A flux of total vorticity $\operatorname{curl} \mathbf{v}$ into the domain is a source of helicity.

Exercise. Use Cartan's formula in (88) to compute $\pounds_u(v \wedge dv)$ in Equation (128).

Exercise. Compute the helicity for the one-form $v = \mathbf{v} \cdot d\mathbf{x}$ in the Clebsch representation (117). What does this mean for the linkage of the vortex lines that admit the Clebsch representation?

Theorem 55 (Diffeomorphisms preserve helicity). The helicity $\Lambda[\xi]$ of any divergenceless vector field ξ is preserved under the action on ξ of any volume-preserving diffeomorphism of the manifold M [ArKh1998].

Remark 56 (Helicity is a topological invariant).

The helicity $\Lambda[\xi]$ is a topological invariant, not a dynamical invariant, because its invariance is independent of which diffeomorphism acts on ξ . This means the invariance of helicity is independent of which Hamiltonian flow produces the diffeomorphism. This is the hallmark of a Casimir function. Although it is defined above with the help of a metric, every volume-preserving diffeomorphism carries a divergenceless vector field ξ into another such field with the same helicity. However, independently of any metric properties, the action of diffeomorphisms does not create or destroy linkages of the characteristic curves of divergenceless vector fields.

10.5 Ertel theorem for potential vorticity

Euler–Boussinesq equations The Euler–Boussinesq equations for the incompressible motion of an ideal flow of a stratified fluid and velocity \mathbf{u} satisfying div $\mathbf{u} = 0$ in a rotating frame with Coriolis parameter curl $\mathbf{R} = 2\mathbf{\Omega}$ are given by

$$\underbrace{\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}}_{\text{acceleration}} = \underbrace{-gb\nabla z}_{\text{buoyancy}} + \underbrace{\mathbf{u} \times 2\Omega}_{\text{Coriolis}} - \underbrace{\nabla p}_{\text{pressure}}$$
(131)

 \star

where $-g\nabla z$ is the constant downward acceleration of gravity and b is the buoyancy, which satisfies the *advection relation*,

$$\partial_t b + \mathbf{u} \cdot \nabla b = 0. \tag{132}$$

As for Euler's equations without buoyancy, requiring preservation of the divergence-free (volume-preserving) constraint $\nabla \cdot \mathbf{u} = 0$ results in a Poisson equation for pressure p,

$$-\Delta\left(p+\frac{1}{2}|\mathbf{u}|^{2}\right) = \operatorname{div}(-\mathbf{u}\times\operatorname{curl}\mathbf{v}) + g\partial_{z}b\,,\qquad(133)$$

which satisfies a Neumann boundary condition because the velocity **u** must be tangent to the boundary.

The Newton's law form of the Euler-Boussinesq equations (131) may be rearranged as

$$\partial_t \mathbf{v} - \mathbf{u} \times \operatorname{curl} \mathbf{v} + gb\nabla z + \nabla \left(p + \frac{1}{2} |\mathbf{u}|^2 \right) = 0,$$
(134)

where $\mathbf{v} \equiv \mathbf{u} + \mathbf{R}$ and $\nabla \cdot \mathbf{u} = 0$.

Theorem 57. [The Kelvin/Stokes' theorem for vorticity of a stratified, rotating fluid]

$$\frac{d}{dt} \oint_{c(\mathbf{u})} \mathbf{v} \cdot d\mathbf{x} = \frac{d}{dt} \iint_{S(\mathbf{u})} \operatorname{curl} \mathbf{v} \cdot d\mathbf{S}
= \iint_{S(\mathbf{u})} \left(\frac{\partial}{\partial t} + \pounds_{\mathbf{u}} \right) (\operatorname{curl} \mathbf{v} \cdot d\mathbf{S})
= \iint_{S(\mathbf{u})} \left(\partial_t \,\boldsymbol{\omega} - \operatorname{curl} \left(\mathbf{u} \times \boldsymbol{\omega} \right) \right) \cdot d\mathbf{S}
= \iint_{S(\mathbf{u})} \left(- g \nabla b \times \nabla z \right) \cdot d\mathbf{S},$$
(135)

where the surface $S(\mathbf{u})$ is bounded by an arbitrary circuit $\partial S = c(\mathbf{u})$ moving with the fluid. Thus, non-alignment of the gradient of buoyancy ∇b with the vertical ∇z creates circulation. Compare this result with equation (109) in the absence of stratification.

Geometrically, equation (134) may be written as

$$(\partial_t + \mathcal{L}_u)v + gbdz + d\varpi = 0, \qquad (136)$$

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where ϖ is defined in (125). In addition, the buoyancy satisfies

$$(\partial_t + \mathcal{L}_u)b = 0, \quad \text{with} \quad \mathcal{L}_u d^3 x = 0.$$
(137)

The fluid velocity vector field is denoted as $u = \mathbf{u} \cdot \nabla$ and the circulation one-form as $v = \mathbf{v} \cdot d\mathbf{x}$. The exterior derivatives of the two equations in (136) are written as

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$$\partial_t + \mathcal{L}_u dv = -gdb \wedge dz \quad \text{and} \quad (\partial_t + \mathcal{L}_u)db = 0.$$
 (138)

Consequently, one finds from the product rule for Lie derivatives (83) that

$$(\partial_t + \mathcal{L}_u)(dv \wedge db) = 0 \quad \text{or} \quad \partial_t q + \mathbf{u} \cdot \nabla q = 0,$$
(139)

in which the quantity

$$q = \nabla b \cdot \operatorname{curl} \mathbf{v} \tag{140}$$

is called *potential vorticity* and is abbreviated as PV. The potential vorticity is an important diagnostic for many processes in geophysical fluid dynamics. Conservation of PV on fluid parcels is called *Ertel's theorem*.

Remark 58 (Ertel's theorem for the vorticity vector field).

Writing the vorticity vector field $\omega = \boldsymbol{\omega} \cdot \nabla$, we have

$$(\partial_t + \pounds_u)\omega = \partial_t\omega + [u, \omega] = g\nabla z \times \nabla b \cdot \nabla.$$

Thus, conservation of the potential vorticity may also be proved by the product rule, as

$$(\partial_t + \pounds_u)q = (\partial_t + \pounds_u)(\boldsymbol{\omega} \cdot \nabla b) = (\partial_t + \pounds_u)(\boldsymbol{\omega} b) = ((\partial_t + \pounds_u)\boldsymbol{\omega})b + \boldsymbol{\omega}(\partial_t + \pounds_u)b = 0.$$

Remark 59 (Material derivative formulation).

Denoting

$$\frac{D}{Dt} = \partial_t + \pounds_u \quad \text{and} \quad \omega = \boldsymbol{\omega} \cdot \nabla$$

provides an intuitive expression of the Ertel theorem (139) that helps understand it in terms of the time derivative $\frac{D}{Dt}$ following the flow of the fluid particles. Namely, it suggests writing in vector form

$$\frac{D}{Dt}(\boldsymbol{\omega}\cdot\nabla) = g\nabla z \times \nabla b \cdot \nabla \quad \text{and} \quad \frac{Db}{Dt} = 0\,,$$

so that the product rule for derivatives yields conservation of PV on fluid parcels, as

$$\frac{Dq}{Dt} = \frac{D}{Dt}(\boldsymbol{\omega} \cdot \nabla b) = \left(\frac{D}{Dt}(\boldsymbol{\omega} \cdot \nabla)\right)b + (\boldsymbol{\omega} \cdot \nabla)\frac{Db}{Dt} = g\nabla z \times \nabla b \cdot \nabla b + (\boldsymbol{\omega} \cdot \nabla)\frac{Db}{Dt} = 0.$$

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Remark 60 (The conserved quantities associated with Ertel's theorem).

The constancy of the scalar quantities b and q on fluid parcels implies conservation of the spatially integrated quantity,

$$C_{\Phi} = \int_{D} \Phi(b,q) \, d^{3}x \,, \tag{141}$$

for any smooth function Φ for which the integral exists.

Proof.

$$\frac{d}{dt}C_{\Phi} = \int_{D} \Phi_{b}\partial_{t}b + \Phi_{q}\partial_{t}q \, d^{3}x = -\int_{D} \Phi_{b}\mathbf{u} \cdot \nabla b + \Phi_{q}\mathbf{u} \cdot \nabla q \, d^{3}x$$

$$= -\int_{D} \mathbf{u} \cdot \nabla \Phi(b,q) \, d^{3}x = -\int_{D} \nabla \cdot \left(\mathbf{u} \Phi(b,q)\right) d^{3}x = -\oint_{\partial D} \Phi(b,q) \, \mathbf{u} \cdot \hat{\mathbf{n}} \, dS = 0,$$

when the normal component of the velocity $\mathbf{u} \cdot \hat{\mathbf{n}}$ vanishes at the boundary ∂D .

Remark 61 (Energy conservation).

In addition to C_{Φ} , the Euler-Boussinesq fluid equations (134) also conserve the total energy

$$E = \int_{D} \frac{1}{2} |\mathbf{u}|^2 + gbz \ d^3x \,, \tag{142}$$

which is the sum of the kinetic and potential energies.

We do not develop the Hamiltonian formulation of the three-dimensional stratified rotating fluid equations studied here. However, one may imagine that the conserved quantity C_{Φ} with the arbitrary function Φ would play an important role. For more explanation in the framework of Geometric Mechanics, see [Ho2011GM] and references therein.

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