## 1 M3-4-5A16 Assessed Problems \# 1: Do all three problems

Exercise 1.1 (Quaternions in Cayley-Klein (CK) parameters).
Express all of your answers for this exercise in terms of Cayley-Klein (CK) parameters.
(A) Show that any quaternion may be written as a product of its magnitude and a unit quaternion.
(B) For the unit quaternion $\mathfrak{q}=\left[q_{0}, \mathbf{q}\right]$, compute

$$
\mathfrak{p}=\left[q_{0}, \mathbf{q}\right]\left(\left[q_{0}, \mathbf{q}\right]^{*}\right)^{-1}
$$

and express your answer in Cayley-Klein (CK) parameters.
(C) Write an example of a unit quaternion and take its square root in CK parameters.
(D) De Moivre's theorem for unit complex numbers is

$$
(\cos \theta+i \sin \theta)^{m}=(\cos m \theta+i \sin m \theta)
$$

Prove the analog of this theorem for unit quaternions.
(E) The tilde map.

The pure quaternions provide the isomorphism ( $\cdot)^{\sim}: \mathbb{R}^{3} \mapsto \mathfrak{s u}(2)$ (the tilde map),

$$
\mathbf{n}=\left(n_{1}, n_{2}, n_{3}\right) \in \mathbb{R}^{3} \rightarrow \widetilde{\mathbf{n}}:=\left[\begin{array}{cc}
-i n_{3} & -i n_{1}-n_{2} \\
-i n_{1}+n_{2} & i n_{3}
\end{array}\right]=-i \mathbf{n} \cdot \boldsymbol{\sigma} \in \mathfrak{s u}(2),
$$

in which $\mathbf{n}$ is defined as $\mathbf{n} \cdot \boldsymbol{\sigma}=\sum_{\alpha=1}^{3} n_{\alpha}\left\{\sigma_{\alpha}\right\}_{k l}$. Show that

$$
U\left(-i \sigma_{3}\right) U^{\dagger}=\widetilde{\mathbf{n}}
$$

for a matrix $U \in S U(2)$. Interpret this result in terms of the $S^{2}$ Poincaré sphere $|\mathbf{n}|^{2}=1$.
(F) Express the Hopf fibration in terms of quaternionic multiplication.

Hint: Use the tilde map to represent the Hopf fibration in terms of Pauli matrices as

$$
\begin{equation*}
U\left(-i \mathbf{e}_{3} \cdot \boldsymbol{\sigma}\right) U^{\dagger}=-i \mathbf{n} \cdot \boldsymbol{\sigma} \tag{1.1}
\end{equation*}
$$

in which $U^{\dagger}=U^{-1} \in S U(2)$ and $|\mathbf{n}|^{2}=1$. Then use the Cayley map $\mathfrak{s u}(2) \rightarrow S U(2)$ given by

$$
\begin{equation*}
U=\left(q_{0} \sigma_{0}+i \mathbf{q} \cdot \boldsymbol{\sigma}\right)\left(q_{0} \sigma_{0}-i \mathbf{q} \cdot \boldsymbol{\sigma}\right)^{-1}, \quad \text { for } \quad q_{0}^{2}+|\mathbf{q}|^{2}=1 \tag{1.2}
\end{equation*}
$$

to represent the tilde map.

## Answer.

(A) Any quaternion may be written as a product of its magnitude and a unit quaternion, since any quaternion may be factorized into

$$
\mathfrak{q}=\left[q_{0}, \boldsymbol{q}\right]=|\mathfrak{q}|\left[q_{0} /|\mathfrak{q}|, \boldsymbol{q} /|\mathfrak{q}|\right] \quad \text { where } \quad|\mathfrak{q}|^{2}=q_{0}^{2}+|\boldsymbol{q}|^{2}
$$

and $\left[q_{0} /|\mathfrak{q}|, \boldsymbol{q} /|\mathfrak{q}|\right]$ is a unit quaternion,

$$
\left|\left[q_{0} /|\mathfrak{q}|, \boldsymbol{q} /|\mathfrak{q}|\right]\right|^{2}=\left(q_{0} /|\mathfrak{q}|\right)^{2}+|\boldsymbol{q}|^{2} /|\mathfrak{q}|^{2}=1
$$

(B) One knows that a unit quaternion is isomorphic to an element of the Lie group $S U(2)$

$$
\mathfrak{q}=\left[q_{0}, \boldsymbol{q}\right]=q_{0} \sigma_{0}-i \boldsymbol{q} \cdot \boldsymbol{\sigma} \in S U(2) \quad \text { where } \quad q_{0}^{2}+\boldsymbol{q} \cdot \boldsymbol{q}=1
$$

and that a pure quaternion is an element of the Lie algebra $s u(2)$

$$
\mathfrak{q}=[0, \boldsymbol{q}]=-i \boldsymbol{q} \cdot \boldsymbol{\sigma} \in s u(2)
$$

The ratio

$$
\mathfrak{p}=\left[q_{0}, \mathbf{q}\right]\left(\left[q_{0}, \mathbf{q}\right]^{*}\right)^{-1}
$$

is a version of the Cayley transform for $S O(3)$ which naturally extends to $S U(2)$ by writing it in quaternion form. The ratio may be computed using the formula for the inverse of a quaternion, $\mathfrak{r}^{-1}=\mathfrak{r}^{*} /|\mathfrak{r}|^{2}=\left[r_{0},-\boldsymbol{r}\right] /\left(r_{0}^{2}+\boldsymbol{r} \cdot \boldsymbol{r}\right)$. Thus,

$$
\begin{aligned}
\mathfrak{p} & =\left[q_{0}, \mathbf{q}\right]\left(\left[q_{0}, \mathbf{q}\right]^{*}\right)^{-1}=\left[q_{0}, \mathbf{q}\right]\left[q_{0}, \mathbf{q}\right] /\left(q_{0}^{2}+\boldsymbol{q} \cdot \boldsymbol{q}\right) \\
& =[\cos (\theta / 2), \sin (\theta / 2) \hat{\boldsymbol{n}}]^{2} \quad \text { for } \mathfrak{q}=[\cos (\theta / 2), \sin (\theta / 2) \hat{\boldsymbol{n}}], \\
& =\left[\cos ^{2}(\theta / 2)-\sin ^{2}(\theta / 2), 2 \cos (\theta / 2) \sin (\theta / 2) \hat{\boldsymbol{n}}\right] \\
& =[\cos \theta, \sin \theta \hat{\boldsymbol{n}}]
\end{aligned}
$$

So the Cayley transform of a unit quaternion is simply its square.
(C) According to the previous part, the square root of a quaternion obeys the same law as for polar coordinates. That is, the square root of a unit quaternion is given by

$$
[\cos \theta, \sin \theta \hat{\boldsymbol{n}}]^{1 / 2}= \pm[\cos (\theta / 2), \sin (\theta / 2) \hat{\boldsymbol{n}}] .
$$

Thus, not surprisingly, the unit quaternions are already Cayley.
(D) De Moivre's theorem is

$$
(\cos \theta+i \sin \theta)^{m}=(\cos m \theta+i \sin m \theta)
$$

for complex numbers. The analog of this theorem for unit quaternions is found as above to be

$$
[\cos \theta, \sin \theta \hat{\boldsymbol{n}}]^{m}=[\cos m \theta, \sin m \theta \hat{\boldsymbol{n}}] .
$$

(E) The tilde map.

We may express the Hopf fibration in terms of Pauli matrices as

$$
\left[\begin{array}{cc}
a_{1} & -a_{2}^{*} \\
a_{2} & a_{1}^{*}
\end{array}\right]\left[\begin{array}{cc}
-i & 0 \\
0 & i
\end{array}\right]\left[\begin{array}{cc}
a_{1}^{*} & a_{2}^{*} \\
-a_{2} & a_{1}
\end{array}\right]=\left[\begin{array}{cc}
-i n_{3} & -i n_{1}+n_{2} \\
-i n_{1}-n_{2} & i n_{3}
\end{array}\right] \text { for } \quad\left|a_{1}\right|^{2}+\left|a_{2}\right|^{2}=1 \text {. }
$$

In other words,

$$
\operatorname{Ad}_{U}\left(-i \mathbf{e}_{3} \cdot \boldsymbol{\sigma}\right):=U\left(-i \mathbf{e}_{3} \cdot \boldsymbol{\sigma}\right) U^{\dagger}=-i \mathbf{n} \cdot \boldsymbol{\sigma},
$$

in which $U^{\dagger}=U^{-1} \in S U(2)$ and $|\mathbf{n}|^{2}=1$. This is the tilde map and $\left(n_{1}, n_{2}, n_{3}\right)$ are the components of the Hopf fibration. The interpretation of the tilde map is that Ad-operations of $S U(2)$ on the element in its Lie algebra $\mathfrak{s u}(2)$ corresponding to the unit vector in the $\hat{\mathbf{z}}$ direction can reach any point on the Poincaré sphere.
(F) The Pauli matrix representation of quaternions is

$$
\mathfrak{q}=\left[q_{0}, \boldsymbol{q}\right]=q_{0} \sigma_{0}-i \boldsymbol{q} \cdot \boldsymbol{\sigma}, \text { with } \boldsymbol{q} \cdot \boldsymbol{\sigma}:=\sum_{a=1}^{3} q_{a} \sigma_{a}
$$

and we know that $\mathfrak{q}^{-1}=\mathfrak{q}^{*} /|\mathfrak{q}|^{2}$, which for $|\mathfrak{q}|^{2}=1$ yields $\mathfrak{q}^{-1}=\mathfrak{q}^{*}$. Thus, in the Cayley formula

$$
U=\left(q_{0} \sigma_{0}+i \mathbf{q} \cdot \boldsymbol{\sigma}\right)\left(q_{0} \sigma_{0}-i \mathbf{q} \cdot \boldsymbol{\sigma}\right)^{-1}, \quad \text { for } \quad q_{0}^{2}+|\mathbf{q}|^{2}=1
$$

we have $U=\mathfrak{q}^{*} \mathfrak{q}^{*}$, so the unit quaternions $\mathfrak{q}$ and unit vectors $[0, \hat{\mathbf{z}}]$ and $[0, \boldsymbol{n}]$ satisfy

$$
\mathfrak{q}^{*} \mathfrak{q}^{*}[0, \hat{\mathbf{z}}] \mathfrak{q q}=\left[q_{0},-\boldsymbol{q}\right]\left[q_{0},-\boldsymbol{q}\right][0, \hat{\mathbf{z}}]\left[q_{0}, \boldsymbol{q}\right]\left[q_{0}, \boldsymbol{q}\right]=[0, \boldsymbol{n}]
$$

Thus, conjugating the pure unit quaternion along the $z$-axis $[0, \hat{\mathbf{z}}]$ by the other unit quaternions yields the entire unit two-sphere $S^{2}$. That is, quaternionic conjugation of a single unit quaternion yields the complete set of unit quaternions. This is the quaternionic version of the Hopf fibration.

Exercise 1.2 (Momentum map for unitary transformations).
Definition. The formula determining the momentum map for the cotangent-lifted action of a Lie group $G$ on a smooth manifold $Q$ may be expressed in terms of the pairings $\langle\cdot, \cdot\rangle: \mathfrak{g}^{*} \times \mathfrak{g} \rightarrow \mathbb{R}$ and $\langle\langle\cdot, \cdot\rangle\rangle: T^{*} Q \times T Q \rightarrow \mathbb{R}$ as

$$
\langle J, \xi\rangle=\left\langle\left\langle p, £_{\xi} q\right\rangle\right\rangle
$$

where $(q, p) \in T_{q}^{*} Q$ and $£_{\xi} q$ is the infinitesimal generator of the action of the Lie algebra element $\xi$ on the coordinate $q$.

Consider the matrix Lie group $\mathcal{Q}$ of $n \times n$ Hermitian matrices, so that $Q^{\dagger}=Q$ for $Q \in \mathcal{Q}$. The Poisson (symplectic) manifold is $T^{*} \mathcal{Q}$, whose elements are pairs $(Q, P)$ of Hermitian matrices. The corresponding Poisson bracket is

$$
\{F, H\}=\operatorname{tr}\left(\frac{\partial F}{\partial Q} \frac{\partial H}{\partial P}-\frac{\partial H}{\partial Q} \frac{\partial F}{\partial P}\right)
$$

Let $G$ be the group $U(n)$ of $n \times n$ unitary matrices: $G$ acts on $T^{*} \mathcal{Q}$ through

$$
(Q, P) \mapsto\left(U Q U^{\dagger}, U P U^{\dagger}\right), \quad U U^{\dagger}=I d
$$

(A) What is the linearization of this group action?
(B) What is its momentum map?
(C) Is this momentum map equivariant? Explain why, or why not.
(D) Is the momentum map conserved by the Hamiltonian $H=\frac{1}{2} \operatorname{tr} P^{2}$ ? Prove it.

## Answer.

(A) The linearization of this group action with $U=\exp (t \xi)$, with skew-Hermitian $\xi^{\dagger}=-\xi$ yields the vector field

$$
X_{\xi}=([Q, \xi],[P, \xi])
$$

(B) This is the Hamiltonian vector field for

$$
H_{\xi}=\operatorname{tr}([Q, P] \xi)
$$

thus yielding the momentum map $J(Q, P)=[Q, P]$.
(C) Being defined by a cotangent lift, this momentum map is equivariant.
(D) For $H=\frac{1}{2} \operatorname{tr} P^{2}$,

$$
\{[Q, P], H\}=\operatorname{tr}\left(\frac{\partial[Q, P]}{\partial Q} \frac{\partial H}{\partial P}\right)=\operatorname{tr}\left(P^{2}-P^{2}\right)=0
$$

so the momentum map $J(Q, P)=[Q, P]$ is conserved by this Hamiltonian.

Alternatively, one may simply observe that the map

$$
(Q, P) \mapsto\left(U Q U^{\dagger}, U P U^{\dagger}\right), \quad U U^{\dagger}=I d
$$

preserves $\operatorname{tr}\left(P^{2}\right)$, since it takes

$$
\operatorname{tr}\left(P^{2}\right) \mapsto \operatorname{tr}\left(U P U^{\dagger} U P U^{\dagger}\right)=\operatorname{tr}\left(P^{2}\right)
$$

Exercise 1.3 (Co-quaternions, $S p(2)$, adjoint and coadjoint actions, and all that).
The matrix Lie group $S p(2)$ of symplectic $2 \times 2$ matrices is defined by

$$
S p(2):=\left\{M \in G L(2, \mathbb{R}) \mid M^{T} J M=J\right\} \quad \text { with } \quad J:=\left[\begin{array}{cc}
0 & -1 \\
1 & 0
\end{array}\right]
$$

(A) Show that the unit coquaternions with $|\mathfrak{c}|^{2}=1$ form a representation of the Lie group of $2 \times 2$ symplectic matrices $S p(2, \mathbb{R})$.
(B) Follow the developments in Chapter 3 of GM2 for quaternions far enough to define conjugacy classes for the action of unit coquaternions on vectors in $\mathbb{R}^{3}$.
(C) Compute the Euler-Rodrigues formula for the coquaternions.
(D) Represent the Lie algebra $\mathfrak{s p}(2)$ as $2 \times 2$ matrices and compute their associated flows on the group.
(E) Compute the adjoint and coadjoint actions $\mathrm{AD}, \mathrm{Ad}, \mathrm{ad}, \mathrm{Ad}^{*}$ and $\mathrm{ad}^{*}$ for $S p(2)$.
(F) Explain what corresponds to the quotient map, orbit manifold (image of the quotient map) and Poincaré sphere for the transmission of optical rays by Fermat's principle in an axisymmetric, translation-invariant medium?
Hint: This problem is discussed in the first chapter of [Ho2011GM1]. Use co-quaternions as representations of the symplectic group $S p(2)$, as discussed in the third chapter of [Ho2011GM2].

## Answer.

(A) The unit co-quaternion $\hat{\mathfrak{c}}=[w, x, y, z]$ may be represented as a real $2 \times 2$ matrix by

$$
C=\left[\begin{array}{ll}
w+z & y+x \\
y-x & w-z
\end{array}\right]
$$

where $\hat{\mathfrak{c}}^{*}=[w,-x,-y,-z]$ and

$$
|\hat{\mathfrak{c}}|^{2}=\hat{\mathfrak{c}}^{*} \cdot \hat{\mathfrak{c}}=\frac{1}{2} \operatorname{tr}\left(C^{*} C\right)=w^{2}+x^{2}-y^{2}-z^{2}=\operatorname{det} C=1
$$

(B) As for quaternions, we may define conjugation for coquaternions as

$$
\mathfrak{r}^{\prime}=\hat{\mathfrak{c}} \mathfrak{r} \hat{\mathfrak{c}}^{*}
$$

for the unit coquaternion $\hat{\mathfrak{c}}$. If $\mathfrak{r}^{\prime}$ is a pure coquaternion, the conjugacy class is

$$
\mathcal{C}(\mathfrak{r})=\left\{\mathfrak{r}^{\prime} \in \mathbb{H}_{c} \mid \mathfrak{r}^{\prime}=\hat{\mathfrak{c}} \mathfrak{r} \hat{\mathfrak{c}}^{*}\right\}
$$

(C) The formula for multiplication of coquaternions $\mathfrak{q}=\left[q_{0}, \mathbf{q}\right]$ and $\mathfrak{c}=\left[c_{0}, \mathbf{c}\right]$ is given by

$$
\mathfrak{q c}=\left[q_{0} c_{0}-q_{1} c_{1}+q_{2} c_{2}+q_{3} c_{3}, q_{0} \mathbf{c}+c_{0} \mathbf{q}+\mathbf{q} \times \mathbf{c}\right]
$$

For a pure coquaternion $\mathfrak{r}=[0, \mathbf{r}]$, unit quaternion $\hat{\mathfrak{c}}=\left[c_{0}, \mathbf{c}\right]$ and its conjugate $\hat{\mathfrak{c}}^{*}=\left[c_{0},-\mathbf{c}\right]$ with $c_{0}^{2}+c_{1}^{2}-c_{2}^{2}-c_{3}^{2}$, a direct calculation yields

$$
\hat{\mathfrak{c}} \hat{\mathbf{c}}^{*}=\left[0,-2 c_{0}(\mathbf{r} \times \mathbf{c})-\mathbf{c} \times(\mathbf{r} \times \mathbf{c})+c_{0}^{2} \mathbf{r}+\left(r_{1} c_{1}-r_{2} c_{2}-r_{3} c_{3}\right) \mathbf{c}\right]
$$

If $\hat{\mathfrak{c}}=\left[\cosh \frac{\theta}{2}, \sinh \frac{\theta}{2} \hat{\mathbf{n}}\right]$ then $|\hat{\mathfrak{c}}|^{2}=1$ and the Euler-Rodrigues formula for the coquaternions is given by the formula

$$
\mathbf{r}^{\prime}=\mathbf{r}-\sinh \theta(\mathbf{r} \times \hat{\mathbf{n}})+(1-\cosh \theta) \hat{\mathbf{n}}(\mathbf{r} \times \hat{\mathbf{n}})
$$

Proof.

$$
\begin{aligned}
{\left[0, \mathbf{r}^{\prime}\right]=\hat{\mathfrak{c}} \hat{\mathfrak{c}}^{*} } & =\left[\cosh \frac{\theta}{2}, \sinh \frac{\theta}{2} \hat{\mathbf{n}}\right][0, \mathbf{r}]\left[\cosh \frac{\theta}{2},-\sinh \frac{\theta}{2} \hat{\mathbf{n}}\right] \\
& =\left[0, \mathbf{r}-\sinh \frac{\theta}{2} \cosh \frac{\theta}{2}(\mathbf{r} \times \hat{\mathbf{n}})-2 \sinh ^{2} \frac{\theta}{2} \hat{\mathbf{n}}(\mathbf{r} \times \hat{\mathbf{n}})\right] \\
& =[0, \mathbf{r}-\sinh \theta(\mathbf{r} \times \hat{\mathbf{n}})+(1-\cosh \theta) \hat{\mathbf{n}}(\mathbf{r} \times \hat{\mathbf{n}})]
\end{aligned}
$$

(D) Set $\xi(s)=\dot{M}(s) M^{-1}(s)$, so that $\dot{M}(s)=\xi(s) M(s)$ is the reconstruction equation. Take $d / d s$ of the defining relation

$$
\begin{aligned}
\frac{d}{d s}\left(M(s) J M(s)^{T}\right) & =\xi J+J \xi^{T} \\
& =\xi J+\left(\xi J^{T}\right)^{T} \\
& =\xi J-(\xi J)^{T}=0
\end{aligned}
$$

Thus, $\xi J=(\xi J)^{T} \in$ sym is symmetric.

- Conjugation by $J$ shows that $J \xi=(J \xi)^{T} \in$ sym is also symmetric.
- Replacing $M \leftrightarrow M^{T}$ gives the corresponding result for left invariant $\Xi(s):=M^{-1}(s) \dot{M}(s)$.
(E) Quotient map,

$$
\begin{aligned}
\pi: T^{*} \mathbb{R}^{2} /\{0\} & \rightarrow \mathbb{R}^{3} /\{0\} \\
\pi:(\mathbf{q}, \mathbf{p}) & \rightarrow \mathbf{Y}=\left(|\mathbf{q}|^{2},|\mathbf{p}|^{2}, \mathbf{q} \cdot \mathbf{p}\right)
\end{aligned}
$$

Orbit manifold (image of the quotient map, which corresponds for $S P(2)$ to the Poincare sphere for $S U(2)$.

$$
|\mathbf{q} \times \mathbf{p}|^{2}=|\mathbf{q}|^{2}|\mathbf{p}|^{2}-(\mathbf{q} \cdot \mathbf{p})^{2}=\text { const }
$$

and the Poincaré sphere for the co-quaternions as representations of the symplectic group $S p(2)$ is given by

$$
\operatorname{det} \frac{1}{2}\left(S m_{0}-\mathbf{Y} \cdot \mathbf{m}\right)=\left[\begin{array}{cc}
S-Y_{3} & Y_{1} \\
-Y_{2} & S+Y_{3}
\end{array}\right]=S^{2}-Y_{3}^{2}+Y_{1} Y_{2}=0
$$

