M4A34 Assessed Homework 1 Spring Term 2011 Darryl Holm

Due at beginning of class Wed 26 Jan 2011

0. Special Euclidean group, $SE(3)$

Problem statement:

[0a] List the subgroups of $SE(3)$

[0b] Write a 4×4 matrix representation of $SE(3)$

[0c] Show that $SE(3)$ is a semidirect-product Lie group

[0d] Compute the tangent space at the identity of $T_I\, SE(3)$ in its 4×4 matrix representation.

[0e] Show that elements of $T_I\, SE(3)$ form a Lie algebra, $se(3)$.

[0f] Compute the ***adjoint actions***, $\mathrm{Ad} : SE(3) \times SE(3) \to SE(3)$, $\mathrm{Ad} : SE(3) \times se(3) \to se(3)$ and $\mathrm{ad} : se(3) \times se(3) \to se(3)$

[0g] Use the hat map to write the ad-action as vector multiplication.

Solution

[0a] The subgroups of $SE(3)$, the ***special Euclidean group*** in three dimensions, may be recognised by their actions on vectors $\mathbf{r} \in \mathbb{R}^3$, written abstractly as $SE(3) \times \mathbb{R}^3 \to \mathbb{R}^3$:

- Spatial translations $g_1(\mathbf{r}_0)$ acting on $\mathbf{r} \in \mathbb{R}^3$ as $g_1(\mathbf{r}_0)\mathbf{r} = \mathbf{r} + \mathbf{r}_0$.
- Proper rotations $g_2(O)$ with $g_2(O)\mathbf{r} = O\mathbf{r}$ where $O^T = O^{-1}$ and $\det O = +1$. This subgroup is called $SO(3)$, the ***special orthogonal group*** in three dimensions.
- Rotations and *reflections* $g_2(O)$ with $O^T = O^{-1}$ and $\det O = \pm 1$. This subgroup is called $O(3)$, the ***orthogonal group*** in three dimensions.
- The full $SE(3)$ group comprises spatial translations $g_1(\mathbf{r}_0)$ wih $\mathbf{r}_0 \in \mathbb{R}^3$ composed with proper rotations $g_2(O) \in SO(3)$ acting on a vector $\mathbf{r} \in \mathbb{R}^3$ as

$$E(O, \mathbf{r}_0)\mathbf{r} = g_1(\mathbf{r}_0)g_2(O)\mathbf{r} = O\mathbf{r} + \mathbf{r}_0\,,$$

where $O^T = O^{-1}$ and $\det O = +1$.

Note: Spatial translations and rotations do not commute in general. That is, $g_1 g_2 \neq g_2 g_1$, unless the direction of translation and axis of rotation are collinear. This brings up the issue of the group action of $SE(3)$ on itself, written abstractly as $SE(3) \times SE(3) \to SE(3)$.

[0b] $SE(3)$ has a 4×4 matrix representation of $SE(3) \times \mathbb{R}^3 \to \mathbb{R}^3$ that may be found by noticing that its right-hand side $E(O, \mathbf{r}_0)\mathbf{r} = O\mathbf{r} + \mathbf{r}_0$ arises in multiplying the matrix times the *extended* vector $(\mathbf{r}, 1)^T$ as

$$\begin{pmatrix} O & \mathbf{r}_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{r} \\ 1 \end{pmatrix} = \begin{pmatrix} O\mathbf{r} + \mathbf{r}_0 \\ 1 \end{pmatrix}.$$

Therefore we may identify a group element of $SE(3)$ with a 4×4 matrix,

$$E(O, \mathbf{r}_0) = \begin{pmatrix} O & \mathbf{r}_0 \\ 0 & 1 \end{pmatrix}.$$

Note: The group $SE(3)$ has six parameters. These are the angles of rotation about each of the three spatial axes by the orthogonal matrix $O \in SO(3)$ with $O^T = O^{-1}$ and the three components of the vector of translations $\mathbf{r}_0 \in \mathbf{R}^3$.

[0c] To show that $SE(3)$ is a semidirect-product Lie group, one notes that the action of the subgroup $SO(3) \subset SE(3)$ on the subgroup $\mathbb{R}^3 \subset SE(3)$ maps the $\mathbb{R}^3$ into itself. This means the translations $\mathbb{R}^3 \subset SE(3)$ form a ***normal***, or ***invariant subgroup*** of the group $SE(3)$.

A Lie group G that may be decomposed uniquely into a normal subgroup N and a subgroup H such that every group element may be written as

$$g = nh \quad \text{or} \quad g = hn \quad \text{(in either order)}, \tag{1}$$

for unique choices of $n \in N$ and $h \in H$ is called a ***semidirect product*** of H and N, denoted by Ⓢ, as in

$$G = H \circledS N\,.$$

Every element of $(O\,,\,\mathbf{r}_0)$ of $SE(3)$ may be expressed uniquely by composing a translation acting from the left on a rotation. That is, each element may be decomposed into

$$(O\,,\,\mathbf{r}_0) = (I\,,\,\mathbf{r}_0)(O\,,\,\mathbf{0})\,,$$

for a *unique* $\mathbf{r}_0 \in \mathbb{R}^3$ and $O \in SO(3)$. Likewise, one may uniquely express the same group element by composition in the opposite order,

$$(O\,,\,\mathbf{r}_0) = (O\,,\,\mathbf{0})(I\,,\,O^{-1}\mathbf{r}_0)\,.$$

This equivalence endows the Lie group $SE(3)$ with a semidirect-product structure,

$$SE(3) = SO(3) \circledS \mathbb{R}^3\,. \tag{2}$$

[0d] A 4×4 matrix representation of the tangent space at the identity of $T_I\,SE(3)$ may be found by first computing the derivative of a general group element $(O(s), \mathbf{r}_0(s))$ along the group path with parameter s and bringing the result back to the identity at $s = 0$,

$$\begin{aligned}\left[\begin{pmatrix} O(s) & \mathbf{r}_0(s) \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} O\,'(s) & \mathbf{r}_0'(s) \\ 0 & 0 \end{pmatrix}\right]_{s=0}\\ = \begin{pmatrix} O^{-1}(0)O\,'(0) & O^{-1}(0)\mathbf{r}_0'(0) \\ 0 & 0 \end{pmatrix} =: \begin{pmatrix} \widehat{\Xi} & \mathbf{r}_0 \\ 0 & 0 \end{pmatrix},\end{aligned}$$

where in the last step we have dropped the unnecessary superscript prime ($\,'$). The quantity $\widehat{\Xi} = O^{-1}(s)O\,'(s)|_{s=0}$ is a 3×3 skew-symmetric matrix, since O is a 3×3 orthogonal matrix.

[0e] Show that elements of $T_I\,SE(3)$ form a Lie algebra, $se(3)$.

The commutator of infinitesimal transformation matrices given by the formula,

$$\left[\begin{pmatrix} \widehat{\Xi}_1 & \mathbf{r}_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \widehat{\Xi}_2 & \mathbf{r}_2 \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} \widehat{\Xi}_1\widehat{\Xi}_2 - \widehat{\Xi}_2\widehat{\Xi}_1 & \widehat{\Xi}_1\mathbf{r}_2 - \widehat{\Xi}_2\mathbf{r}_1 \\ 0 & 0 \end{pmatrix},$$

provides a matrix representation of $T_I\,SE(3)$, which is isomorphic to $se(3)$, the Lie algebra of the Lie group $SE(3)$. In vector notation, this becomes

$$\left[\begin{pmatrix} \boldsymbol{\Xi}_1\times & \mathbf{r}_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \boldsymbol{\Xi}_2\times & \mathbf{r}_2 \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} (\boldsymbol{\Xi}_1 \times \boldsymbol{\Xi}_2)\times & \boldsymbol{\Xi}_1 \times \mathbf{r}_2 - \boldsymbol{\Xi}_2 \times \mathbf{r}_1 \\ 0 & 0 \end{pmatrix}, \tag{3}$$

which answers the question posed for part [0g] in this notation.

The $se(3)$ matrix commutator yields

$$
\begin{aligned}
\Big[(\widehat{\Xi}_1\,,\,\mathbf{r}_1)\,,\,(\widehat{\Xi}_2\,,\,\mathbf{r}_2)\Big] &= \Big(\widehat{\Xi}_1\widehat{\Xi}_2-\widehat{\Xi}_2\widehat{\Xi}_1,\,\widehat{\Xi}_1\mathbf{r}_2-\widehat{\Xi}_2\mathbf{r}_1\Big)\\
&= \Big(\Big[\widehat{\Xi}_1,\,\widehat{\Xi}_2\Big],\,\widehat{\Xi}_1\mathbf{r}_2-\widehat{\Xi}_2\mathbf{r}_1\Big)\,,
\end{aligned}
$$

which is the classic expression for the Lie algebra of a semidirect-product Lie group.

[0f] The adjoint actions, $\mathrm{Ad} : SE(3)\times SE(3)\to SE(3)$, $\mathrm{Ad} : SE(3)\times se(3)\to se(3)$ and $\mathrm{ad} : se(3)\times se(3)\to se(3)$ may all be computed by taking two derivatives of the the AD-operation.

AD: $SE(3)\times SE(3)\mapsto SE(3)$ is conveniently expressed in matrix row notation as

$$
\begin{aligned}
\mathrm{AD}_{(R\,,\,v)}(\tilde{R}\,,\,\tilde{v}) &= (R\,,\,v)(\tilde{R}\,,\,\tilde{v})(R\,,\,v)^{-1}\\
&= (R\,,\,v)(\tilde{R}\,,\,\tilde{v})(R^{-1}\,,\,-R^{-1}v)\\
&= (R\,,\,v)(\tilde{R}R^{-1}\,,\,\tilde{v}-\,\tilde{R}R^{-1}v)\\
&= (R\tilde{R}R^{-1}\,,\,v+R\tilde{v}-R\tilde{R}R^{-1}v)\,.
\end{aligned}
\tag{4}
$$

Ad: $SE(3)\times se(3)\to se(3)$ is found by taking time derivatives of quantities adorned with tilde $(\tilde{\cdot})$ in formula (4) for $\mathrm{AD}_{(R\,,\,v)}(\tilde{R}(t)\,,\,\tilde{v}(t))$ and evaluating at the identity $t=0$. This yields

$$
\mathrm{Ad}_{(R\,,\,v)}(\dot{\tilde{R}}(0)\,,\,\dot{\tilde{v}}(0)) = (\mathrm{Ad}_R\dot{\tilde{R}}(0)\,,\,-\mathrm{Ad}_R\dot{\tilde{R}}(0)v+R\dot{\tilde{v}}(0))\,.
$$

Setting $\dot{\tilde{R}}(0)=\tilde{\xi}$ and $\dot{\tilde{v}}(0)=\tilde{\alpha}$ defines the ***Ad-action*** of $SE(3)$ on its Lie algebra with elements $(\tilde{\xi},\tilde{\alpha})\in se(3)$ as Ad: $SE(3)\times se(3)\to se(3)$

$$
\begin{aligned}
\mathrm{Ad}_{(R\,,\,v)}(\tilde{\xi}\,,\,\tilde{\alpha}) &= (\mathrm{Ad}_R\tilde{\xi}\,,\,-\mathrm{Ad}_R\tilde{\xi}v+R\tilde{\alpha})\\
&= (R\tilde{\xi}R^{-1}\,,\,-R\tilde{\xi}R^{-1}v+R\tilde{\alpha})\,.
\end{aligned}
\tag{5}
$$

The hat map transforms this expression into the vector form,

$$
\mathrm{Ad}_{(R\,,\,\mathbf{v})}(\tilde{\boldsymbol{\xi}}\,,\,\tilde{\boldsymbol{\alpha}}) = (R\tilde{\boldsymbol{\xi}}\,,\,-R\tilde{\boldsymbol{\xi}}\times\mathbf{v}+R\tilde{\boldsymbol{\alpha}})\,.
\tag{6}
$$

ad: $se(3)\times se(3)\to se(3)$, the corresponding ad-operation, is computed by taking time derivatives of *un*adorned quantities of $\mathrm{Ad}_{(R(t)\,,\,v(t))}(\tilde{\xi}\,,\,\tilde{\alpha})$ in equation (5) evaluated at the identity to find

$$
\begin{aligned}
&\mathrm{ad}_{(\dot{R}(0)\,,\,\dot{v}(0))}(\tilde{\xi}\,,\,\tilde{\alpha})\\
&\quad= \bigg(\dot{R}\tilde{\xi}R^{-1}-R\tilde{\xi}R^{-1}\dot{R}R^{-1}\,,\\
&\qquad -\dot{R}\tilde{\xi}R^{-1}v+R\tilde{\xi}R^{-1}\dot{R}R^{-1}v-R\tilde{\xi}R^{-1}\dot{v}+\dot{R}\tilde{\alpha}\bigg)\bigg|_{\mathrm{Id}}\,.
\end{aligned}
$$

As before, one sets $\dot{R}(0)=\xi$, $\dot{v}(0)=\alpha$, $R(0)=\mathrm{Id}$ and $v(0)=0$. In this notation, the ad-operation $\mathrm{ad}_{(\xi\,,\,\alpha)}$ for the Lie-algebra action of $se(3)$ on itself may be rewritten as

$$
\begin{aligned}
\mathrm{ad}_{(\xi\,,\,\alpha)}(\tilde{\xi}\,,\,\tilde{\alpha}) &= \Big(\xi\tilde{\xi}-\tilde{\xi}\xi\,,\,-(\xi\tilde{\xi}-\tilde{\xi}\xi)v-\tilde{\xi}(\xi v+\alpha)+\xi\tilde{\alpha}\Big)\Big|_{\mathrm{Id}} \\
&= \Big([\xi\,,\,\tilde{\xi}]\,,\,-\xi\tilde{\xi}v+\xi\tilde{\alpha}-\tilde{\xi}\alpha\Big)\Big|_{\mathrm{Id}} \\
&= \big(\mathrm{ad}_\xi\,\tilde{\xi}\,,\,\xi\tilde{\alpha}-\tilde{\xi}\alpha\big)\,,
\end{aligned}
$$

where the last step uses $v(0)=0$. The result is just the matrix commutator,

$$
\mathrm{ad}_{(\xi\,,\,\alpha)}(\tilde{\xi}\,,\,\tilde{\alpha}) = \left[\begin{pmatrix}\xi & \alpha\\ 0 & 0\end{pmatrix}\,,\,\begin{pmatrix}\tilde{\xi} & \tilde{\alpha}\\ 0 & 0\end{pmatrix}\right] = \begin{pmatrix}[\xi\,,\,\tilde{\xi}] & \xi\tilde{\alpha}-\tilde{\xi}\alpha\\ 0 & 0\end{pmatrix}.
$$

[0g] The hat map may now be used to write the ad-action as vector multiplication, cf. equation (3),

$$
\left[\begin{pmatrix}\boldsymbol{\xi}\times & \mathbf{r}\\ 0 & 0\end{pmatrix}\,,\,\begin{pmatrix}\tilde{\boldsymbol{\xi}}\times & \tilde{\mathbf{r}}\\ 0 & 0\end{pmatrix}\right] = \begin{pmatrix}(\boldsymbol{\xi}\times\tilde{\boldsymbol{\xi}})\times & \boldsymbol{\xi}\times\tilde{\mathbf{r}}-\tilde{\boldsymbol{\xi}}\times\mathbf{r}\\ 0 & 0\end{pmatrix}. \tag{7}
$$

1. Noether's theorem for Galilean symmetries

A Lie group of transformations of variables $\{t,q\}\in\mathbb{R}\times\mathbb{R}^3$ depending on a single parameter s defined by

$$
\{t,q\}\mapsto\{\,\bar{t}(t,q,s),\,\bar{q}(t,q,s)\}\,, \tag{8}
$$

that leaves invariant the Lagrangian $L(q,\dot{q}):T\mathbb{R}^3\to\mathbb{R}$ in the action $S=\int L(q,\dot{q})\,dt$ for Hamilton's principle is called a ***Lie symmetry of the action***.

According to ***Noether's theorem***, each Lie symmetry of the action for a Lagrangian system defined on a manifold M with Lagrangian $L(q,\dot{q}):TM\to\mathbb{R}$ corresponds to a constant of the motion – [No1918].

Problem statement:

[1a] Write formulas in this notation for the infinitesimal transformations of the tangent lift of the action (8) for an arbitrary Lie group acting on the time, space and velocity variables.

[1b] Derive an explicit expression for the conserved Noether quantity by proving Noether's theorem in this notation.

[1c] From the finite transformations of $G(3)$, compute the infinitesimal transformations of the Galilean group under composition of first rotations, then boosts, then translations in space and time, in the case when they act on a velocity-space-time point $(\dot{\mathbf{q}},\mathbf{q},t)$.

[1d] Since the Galilean transformations form a Lie group, one may expect them to be a source of Lie symmetries of the Lagrangian in Hamilton's principle. Compute the corresponding Noether conservation laws corresponding to the Lie symmetry of the Lagrangian $L(\mathbf{q},\dot{\mathbf{q}}):T\mathbb{R}^3\to\mathbb{R}$ under each one-parameter subgroup of the Galilean Lie group $G(3)$. For symmetry under Galilean boosts, you should consider the Lagrangian for a system of interacting particles.

Solution

[1a] Suppose the identity transformation of the Lie symmetry group is arranged to occur for $s = 0$. The derivatives with respect to the group parameters s at the identity,

$$\begin{aligned} \tau(t,q) &= \frac{d}{ds}\Big|_{s=0} \bar{t}(t,q,s)\,, \\ \xi^a(t,q) &= \frac{d}{ds}\Big|_{s=0} \bar{q}^a(t,q,s)\,, \end{aligned}$$

are called the ***infinitesimal transformations*** of the action of a Lie group on the time and space variables.

Thus, at linear order in a Taylor expansion in the group parameter s one has

$$\bar{t} = t + s\tau(t,q)\,, \quad \bar{q}^a = q^a + s\xi^a(t,q)\,, \tag{9}$$

where τ and ξ^a are functions of coordinates and time, but do not depend on velocities. Then, to first order in s the tangent lift for the velocities of the transformed trajectories are computed as,

$$\frac{d\bar{q}^a}{d\bar{t}} = \frac{\dot{q}^a + s\dot{\xi}^a}{1 + s\dot{\tau}} = \dot{q}^a + s(\dot{\xi}^a - \dot{q}^a\dot{\tau})\,, \tag{10}$$

where order $O(s^2)$ terms are neglected and one defines the total time derivatives

$$\dot{\tau} \equiv \frac{\partial\tau}{\partial t}(t,q) + \dot{q}^b\frac{\partial\tau}{\partial q^b}(t,q) \quad \text{and} \quad \dot{\xi}^a \equiv \frac{\partial\xi^a}{\partial t}(t,q) + \dot{q}^b(t)\frac{\partial\xi^a}{\partial q^b}(t,q)\,.$$

[1b] We are now in a position to prove Noether's Theorem.

Proof. The variation of the action corresponding to the Lie symmetry with infinitesimal transformations (9) is

$$\begin{aligned} \delta S &= \int_{t_1}^{t_2} \Big(\frac{\partial L}{\partial t}\delta t + \frac{\partial L}{\partial q^a}\delta q^a + \frac{\partial L}{\partial \dot{q}^a}\delta\dot{q}^a + L\frac{d\delta t}{dt}\Big)\,dt \\ &= \int_{t_1}^{t_2} \Big(\frac{\partial L}{\partial t}\tau + \frac{\partial L}{\partial q^a}\xi^a + \frac{\partial L}{\partial \dot{q}^a}(\dot{\xi}^a - \dot{q}^a\dot{\tau}) + L\dot{\tau}\Big)\,dt \\ &= \int_{t_1}^{t_2} \Big(\frac{\partial L}{\partial q^a} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^a}\Big)(\xi^a - \dot{q}^a\tau) + \frac{d}{dt}\Big(L\tau + \frac{\partial L}{\partial \dot{q}^a}(\xi^a - \dot{q}^a\tau)\Big)\,dt \\ &= \int_{t_1}^{t_2} [\,L\,]_{q^a}(\xi^a - \dot{q}^a\tau) + \frac{d}{dt}\Big[\frac{\partial L}{\partial \dot{q}^a}\,\xi^a - \Big(\frac{\partial L}{\partial \dot{q}^a}\,\dot{q}^a - L\Big)\tau\Big]\,dt\,. \end{aligned}$$

Thus, stationarity $\delta S = 0$ and the Euler-Lagrange equations $[\,L\,]_{q^a} = 0$ imply

$$0 = \Big[\frac{\partial L}{\partial \dot{q}^a}\,\xi^a - \Big(\frac{\partial L}{\partial \dot{q}^a}\,\dot{q}^a - L\Big)\tau\Big]_{t_1}^{t_2}\,,$$

so that the quantity

$$\begin{aligned} C(t,q,\dot{q}) &= \frac{\partial L}{\partial \dot{q}^a}\,\xi^a - \Big(\frac{\partial L}{\partial \dot{q}^a}\,\dot{q}^a - L\Big)\tau \qquad (11) \\ &\equiv \langle\, p\,,\,\delta q\,\rangle - E\,\delta t\,, \qquad (12) \end{aligned}$$

has the same value at every time along the solution path. That is, $C(t,q,\dot{q})$ is a constant of the motion. When ξ^a (resp. τ) are constants the conserved quantities p_a (resp E) are the components of linear momentum (resp, energy). $\square$

[1c] The composition of translations, $g_1(\mathbf{q}_0(s),t_0(s))$, Galilean boosts $g_3(\mathbf{v}_0(s))$ and rotations $g_2(O(s))$ acting on a velocity-space-time point $(\dot{\mathbf{q}},\mathbf{q},t)$ is given by

$$g_1g_3g_2(\dot{\mathbf{q}},\mathbf{q},t) = \Big(O(s)\dot{\mathbf{q}} + \mathbf{v}_0(s), O(s)\mathbf{q} + t\mathbf{v}_0(s) + \mathbf{q}_0(s), t + t_0(s)\Big)\,.$$

One computes the infinitesimal transformations as

$$\begin{aligned}
\tau &= \frac{dt}{ds}\Big|_{s=0} = t_0\,,\\
\boldsymbol{\xi} &= \frac{d\mathbf{q}}{ds}\Big|_{s=0} = \mathbf{q}_0 + \mathbf{v}_0 t + \boldsymbol{\Xi}\times\mathbf{q}(t)\,,\\
\dot{\boldsymbol{\xi}} - \dot{\mathbf{q}}(t)\dot{\tau} &= \frac{d\dot{\mathbf{q}}}{ds}\Big|_{s=0} = \mathbf{v}_0 + \boldsymbol{\Xi}\times\dot{\mathbf{q}}(t)\,.
\end{aligned}$$

The infinitesimal velocity transformation may also be computed from equation (10).

Consequently, the infinitesimal transformation by the Galilean group of a function $F(t,\mathbf{q},\dot{\mathbf{q}})$ is given by operation of the following vector field, obtained as the first term in a Taylor series,

$$\begin{aligned}
&\frac{d}{ds}\Big|_{s=0} F(t(s),\mathbf{q}(s),\dot{\mathbf{q}}(s)) \qquad (13)\\
&\quad = t_0\frac{\partial F}{\partial t} + \big(\mathbf{q}_0 + \mathbf{v}_0 t + \boldsymbol{\Xi}\times\mathbf{q}\big)\cdot\frac{\partial F}{\partial \mathbf{q}} + \big(\mathbf{v}_0 + \boldsymbol{\Xi}\times\dot{\mathbf{q}}\big)\cdot\frac{\partial F}{\partial \dot{\mathbf{q}}}\,.
\end{aligned}$$

[1d] ***Galiliean Lie symmetries and their Noether conservation laws.***

Space and time translations:
Equation (12) shows that symmetries under space and time translations imply conservation of linear momentum and energy, respectively. Likewise, symmetry under rotations implies conservation of angular momentum.

Rotations:
Suppose the Lagrangian in Hamilton's principle is invariant under S^1 rotations about a spatial direction $\boldsymbol{\Xi}$. The infinitesimal transformation of such a rotation is $\delta\mathbf{q} = \boldsymbol{\Xi}\times\mathbf{q}$. In this case, the conserved Noether quantity (12) is

$$J^{\Xi}(\mathbf{q},\dot{\mathbf{q}}) = \frac{\partial L}{\partial\dot{\mathbf{q}}}\cdot\delta\mathbf{q} = \mathbf{q}\times\frac{\partial L}{\partial\dot{\mathbf{q}}}\cdot\boldsymbol{\Xi}\,, \qquad (14)$$

which is the angular momentum about the $\boldsymbol{\Xi}$-axis.

Galilean boosts:
Finally, symmetry under Galilean boosts implies vanishing of total momentum for a system of N particles, labelled by an index $j = 1,2,\dots,N$. The last statement may be proved explicitly

from Noether's theorem and the infinitesimal Galilean boost transformations, as follows.

$$
\begin{aligned}
\delta S &= \int_{t_1}^{t_2} \Big(\frac{\partial L}{\partial t}\delta t + \sum_j \Big(\frac{\partial L}{\partial \mathbf{q}^j}\cdot \delta\mathbf{q}^j + \frac{\partial L}{\partial \dot{\mathbf{q}}^j}\cdot \delta\dot{\mathbf{q}}^j \Big) + L\frac{d\delta t}{dt} \Big)\, dt \\
&= \int_{t_1}^{t_2} \sum_j \Big(\frac{\partial L}{\partial \mathbf{q}^j}\cdot \mathbf{v}_0 t + \frac{\partial L}{\partial \dot{\mathbf{q}}^j}\cdot \mathbf{v}_0 \Big)\, dt \\
&= \int_{t_1}^{t_2} \sum_j \Big(\frac{\partial L}{\partial \mathbf{q}^j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}^j} \Big)\cdot \mathbf{v}_0 t + \frac{d}{dt}\sum_j \Big(\frac{\partial L}{\partial \dot{\mathbf{q}}^j}\cdot \mathbf{v}_0 t \Big)\, dt \\
&= \int_{t_1}^{t_2} \sum_j [\, L\,]_{\mathbf{q}^j}\cdot \mathbf{v}_0 t\, dt + \Big[\Big(\sum_j \frac{\partial L}{\partial \dot{\mathbf{q}}^j} \Big)\cdot \mathbf{v}_0 t \Big]_{t_1}^{t_2}.
\end{aligned}
$$

So the Euler-Lagrange equations $[\, L\,]_{\mathbf{q}^j} = 0$ and stationarity $\delta S = 0$ for any time $t \in [t_1, t_2]$ together imply

$$
\mathbf{P}t\cdot \mathbf{v}_0 = \text{constant}, \quad \text{with} \quad \mathbf{P} := \sum_j \frac{\partial L}{\partial \dot{\mathbf{q}}^j}\,.
$$

Let's explore the meaning of this result for a system of N particles with constant total mass $M = \sum_j m_j$. The centre of mass of the system is defined as

$$
\mathbf{Q}_{CM} := M^{-1}\sum_j m_j \mathbf{q}^j.
$$

For a simple mechanical system with Lagrangian

$$
L(\mathbf{q},\, \dot{\mathbf{q}}) = \frac{1}{2}\Big(\sum_j m_j |\dot{\mathbf{q}}^j|^2) - V(\{\mathbf{q}^j\} \Big)
$$

one finds that

$$
\mathbf{P} = \sum_j \frac{\partial L}{\partial \dot{\mathbf{q}}^j} = \sum_j m_j \dot{\mathbf{q}}^j = \frac{d}{dt}\sum_j m_j \mathbf{q}^j(t) =: \frac{d}{dt}\left(M\mathbf{Q}_{CM} \right).
$$

Hence, for such a system Noether's theorem yields

$$
\mathbf{P}t - M\mathbf{Q}_{CM} = 0,
$$

provided the Lagrangian is *also* invariant under spatial translations, so that $\mathbf{P}$ is *also* a constant of the motion. Consequently, Noether's theorem for symmetry under combined Galilean boosts and spatial translations produces a constant of motion that depends explicitly on time. The meaning of it is that the centre of mass position $\mathbf{Q}_{CM}$ moves at constant velocity $\mathbf{P}/M$.

2. Monopole Kepler problem

Consider the Kepler problem with a magnetic monopole, whose dynamical equation is,

$$\ddot{\mathbf{r}} + \frac{\lambda}{r^3}\mathbf{L} + \Big(\frac{\mu}{r^3} - \frac{\lambda^2}{r^4}\Big)\mathbf{r} = 0\,. \tag{15}$$

for real positive constants λ and μ. When $\lambda = 0$ this equation governs the Kepler problem for planetary motion.

Problem statement.

[2a] Does equation (15) conserve an energy?

[2b] Take vector cross products of equation (15) with $\mathbf{r}$ and $\mathbf{L} = \mathbf{r}\times\dot{\mathbf{r}}$ to find its two additional *conserved vectors.*

[2c] Is angular momentum also conserved?

[2d] Are the orbits still conic sections?

[2e] Write the Lagrangian for which equation arises from Hamilton's principle. Hint: Assume there exists a generalised vector function $\mathbf{A}(\mathbf{r}) : \mathbb{R}^3 \to \mathbb{R}^3$ whose curl satisfies curl$\mathbf{A} = \mathbf{r}/r^3$.

[2f] Write the Hamiltonian and Poisson bracket for equation (15).

Solution

[2a] The scalar product of equation (15) with $\mathbf{r}$ shows conservation of the energy

$$E(\mathbf{r},\dot{\mathbf{r}}) = \frac{1}{2}|\dot{\mathbf{r}}|^2 + \frac{\lambda^2}{r^3} - \frac{\mu}{r}$$

[2b] The vector cross products of equation (15) with $\mathbf{r}$ and $\mathbf{L} = \mathbf{r}\times\dot{\mathbf{r}}$ produce two additional conserved vectors,

$$\mathbf{P} = \mathbf{L} - \lambda\hat{\mathbf{r}} \quad \text{and} \quad \mathbf{J} = \dot{\mathbf{r}}\times\mathbf{L} + \frac{\lambda}{r}\mathbf{L} - \mu\hat{\mathbf{r}}$$

These vectors satisfy

$$P^2 = L^2 + \lambda^2\,, \quad \mathbf{J}\cdot\mathbf{P} = \lambda\mu$$

and they are related to the conserved energy H by

$$J^2 = 2HL^2 + \mu^2 \quad \text{and} \quad |\mathbf{J}\times\mathbf{P}|^2 = (L^2+\lambda^2)(2HL^2+\mu^2) - \lambda^2\mu^2\,.$$

[2c] Because $P^2 = L^2 + \lambda^2$, the *magnitude* of angular momentum L is conserved, but its direction is *not* conserved,

$$\frac{d\mathbf{L}}{dt} = \mathbf{r}\times\ddot{\mathbf{r}} = \frac{\lambda}{r^3}\,\mathbf{r}\times\mathbf{L} = \lambda\frac{d\hat{\mathbf{r}}}{dt} \quad \text{so} \quad \frac{dL^2}{dt} = 2\mathbf{L}\cdot\frac{d\mathbf{L}}{dt} = 0\,.$$

This differs from the Kepler case, but the constancy of L^2 is enough for the orbits to remain conic sections. Constancy of magnitude L also means the orbit sweeps out equal areas in equal times. This is ***Kepler's Second Law***.

[2d] Taking the scalar product

$$\mathbf{r}\cdot\mathbf{J} = rJ\cos\theta = \mathbf{r}\cdot(\dot{\mathbf{r}}\times\mathbf{L} - \mu\,\mathbf{r}/r) = L^2 - \mu r\,,$$

eliminates the λ-dependence and allows the orbit $r(\theta)$ to be written in plane polar coordinates, as

$$r(\theta) = \frac{L^2}{\mu + J\cos\theta} = \frac{l_\perp}{1 + e\cos\theta}\,, \tag{16}$$

with *constant* eccentricity $e = J/\mu$ and *constant* semi latus rectum $l_\perp = L^2/\mu$. The expression $r(\theta)$ for the Kepler orbit is the formula for a conic section. This is ***Kepler's First Law***. So the orbits are still conic sections. Kepler's first two laws and the geometric properties of an elliptic orbit with semi-axes a and b for $-2H > 0$ yield the relation $T^2/a^3 = constant$, which is ***Kepler's Third Law***.

[2e] The Lagrangian for which equation (15) arises from Hamilton's principle is

$$L(\mathbf{r},\dot{\mathbf{r}}) = \frac{1}{2}|\dot{\mathbf{r}}|^2 + \lambda A(\mathbf{r})\cdot\dot{\mathbf{r}} - V(r) \quad\text{with}\quad V(r) = -\,\frac{\mu}{r} + \frac{\lambda^2}{2r^2}$$

as may be shown by a direct calculation.

[2f] The Legendre transform with

$$\mathbf{p} = \frac{\partial L}{\partial\dot{\mathbf{r}}} = \dot{\mathbf{r}} + \lambda\mathbf{A}(\mathbf{r}) \quad\text{with}\quad \dot{\mathbf{r}}(\mathbf{r},\mathbf{p}) = \mathbf{p} - \lambda\mathbf{A}(\mathbf{r})$$

yields the Hamiltonian

$$H(\mathbf{r},\mathbf{p}) = \mathbf{p}\cdot\dot{\mathbf{r}} - L(\mathbf{r},\dot{\mathbf{r}}) = \frac{1}{2}\big|\mathbf{p} - \lambda\mathbf{A}(\mathbf{r})\big|^2 + V(r) = \frac{1}{2}|\dot{\mathbf{r}}|^2 + V(r) = E(\mathbf{r},\dot{\mathbf{r}})\,,$$

in agreement with the conserved energy $E(\mathbf{r},\dot{\mathbf{r}})$. The canonical Poisson bracket in $\{r_i, p_j\} = \delta_{ij}$ recovers equation (15) in canonical Hamiltonian form, upon assuming $\mathrm{curl}\mathbf{A} = \mathbf{r}/|\mathbf{r}|^3$.

Interestingly, the Poisson brackets with $\dot{\mathbf{r}}(\mathbf{r},\mathbf{p})$ are not canonical, but have a *magnetic term*,

$$\{r_i, \dot{r}_j\} = \delta_{ij} \quad\text{and}\quad \{r_i, r_j\} = 0, \quad\text{but}\quad \{\dot{r}_i, \dot{r}_j\} = \lambda\epsilon_{ijk}\frac{r_k}{r^3} \neq 0\,.$$

Nonetheless, the Poisson bracket relations for the conserved vectors $\mathbf{P}$ and $\mathbf{J}$ mimic the corresponding Poisson brackets in the Kepler case that arises when $\lambda = 0$. Namely,

$$\{P_i, P_j\} = \epsilon_{ijk}P_k\,,\quad \{P_i, J_j\} = \epsilon_{ijk}J_k\,,\quad \{J_i, J_j\} = -\,2H\epsilon_{ijk}P_k\,.$$

Thus,

$$\{P_i, P^2\} = 2\epsilon_{ijk}P_jP_k = 0\,,\quad \{J_i, P^2\} = -2\epsilon_{ijk}J_jJ_k = 0$$

and

$$\{P_i, \mathbf{J}\cdot\mathbf{P}\} = \epsilon_{ijk}(P_jJ_k + J_jP_k) = 0\,,\quad \{J_i, \mathbf{J}\cdot\mathbf{P}\} = \epsilon_{ijk}(-2HP_jP_k + J_jJ_k) = 0\,.$$

So the two conserved scalar quantities, P^2 and $\mathbf{J}\cdot\mathbf{P}$ are Casimirs for these Poisson brackets.

For a review of the literature for this system, see Leach, P. G. and Flessa, G. P. [2003] Generalisations of the Laplace-Runge-Lenz vector. *J. Nonlin. Math.l Phys.* **10**, 340-423.

3. Higher order (HO) mechanics

The measured precession of the perihelion of Mercury requires a correction to Newton's equation for the Kepler problem. One possible correction is given by HO mechanics,

$$\ddot{\mathbf{q}} + \frac{\mu}{|\mathbf{q}|^3}\mathbf{q} = \epsilon \ddddot{\mathbf{q}}(t)\,, \tag{17}$$

for a small real positive constant $\epsilon \ll 1$. The question here is, "Would this work?"

This modification suggests a series of general questions about how such HO mechanics would influence the spatial path $\mathbf{q}(t) \in \mathbb{R}^3$ of a particle of mass $m \in \mathbb{R}$ as a function of time $t \in \mathbb{R}$. Namely, consider the motion equation

$$\epsilon \ddddot{\mathbf{q}}(t) = \frac{d}{dt}\frac{\partial V}{\partial \dot{\mathbf{q}}} - \frac{\partial V}{\partial \mathbf{q}} = \mathbf{F}(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}})\,, \tag{18}$$

for a scalar "potential" $V(\mathbf{q}, \dot{\mathbf{q}}) \in \mathbb{R}$. For example, setting $V(\mathbf{q}, \dot{\mathbf{q}}) = \frac{m}{2}|\dot{\mathbf{q}}|^2 - W(\mathbf{q})$ produces the HO simple mechanical system.

$$\epsilon \ddddot{\mathbf{q}}(t) = m\ddot{\mathbf{q}} + \frac{\partial W}{\partial \mathbf{q}}\,, \tag{19}$$

[3a] How would Newton's three Laws of motion change for HO dynamics of simple mechanical systems?

- [3a.1] What is HO uniform motion?
- [3a.2] What is HO "force"?
- [3a.3] What are HO action and reaction?

[3b] What is Hamilton's principle for HO mechanics?

[3c] What is Noether's theorem for HO mechanics?

[3d] What are Galilean transformations for HO mechanics?

[3e] What is the Legendre transformation for HO mechanics?

[3f] What are Hamilton's canonical equations for HO mechanics?

[3g] In the limit $\epsilon \to 0$, do solutions HO mechanics converge to solutions of Newtonian mechanics?

Solution

[3a] How would Newton's three Laws of motion change for HO dynamics of simple mechanical systems?

[3a.1] HO "uniform motion" has cubic time dependence

$$\ddddot{\mathbf{q}}(t) = 0 \quad \text{so that} \quad \mathbf{q}(t) = \mathbf{a}t^3 + \mathbf{b}t^2 + \mathbf{c}t + \mathbf{d}\,,$$

for constant vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$.

[3a.2] It would seem that HO "force" should be given by

$$\epsilon \ddddot{\mathbf{q}}(t) = \mathbf{F}(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}) = \frac{d}{dt}\frac{\partial V}{\partial \dot{\mathbf{q}}} - \frac{\partial V}{\partial \mathbf{q}}\,, \tag{20}$$

for a scalar "potential" $V(\mathbf{q}, \dot{\mathbf{q}}) \in \mathbb{R}$ that depends on position and velocity. Then, an object in HO "uniform motion" would remain in HO "uniform motion" ......

[3a.3] HO action and reaction must be given by conservation of total linear momentum, but as we shall see, there will be two different kinds of linear momentum.

[3b] Hamilton's principle for HO mechanics is found by first calculating in general that

$$
\begin{aligned}
\delta \int_{t_1}^{t_2} L(\mathbf{q},\dot{\mathbf{q}},\ddot{\mathbf{q}})\,dt &= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial \mathbf{q}}\cdot\delta\mathbf{q} + \frac{\partial L}{\partial \dot{\mathbf{q}}}\cdot\delta\dot{\mathbf{q}} + \frac{\partial L}{\partial \ddot{\mathbf{q}}}\cdot\delta\ddot{\mathbf{q}} \right) dt \\
&= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial \mathbf{q}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{\mathbf{q}}} \right)\cdot\delta\mathbf{q}\,dt \\
&\quad + \left[\left(\frac{\partial L}{\partial \dot{\mathbf{q}}} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{\mathbf{q}}} \right)\cdot\delta\mathbf{q} + \frac{\partial L}{\partial \ddot{\mathbf{q}}}\cdot\delta\dot{\mathbf{q}} \right]_{t_1}^{t_2} .
\end{aligned}
\tag{21}
$$

Then HO mechanics emerges from $\delta S = 0$ with

$$
S = \int_{t_1}^{t_2} \left(\epsilon\frac{1}{2}|\ddot{\mathbf{q}}|^2 + V(\mathbf{q},\dot{\mathbf{q}}) \right) dt
$$

for variations $\delta\mathbf{q}$ and $\delta\dot{\mathbf{q}}$ that both vanish at the endpoints in time.

[3c] What is Noether's theorem for HO mechanics?

Noether's theorem for HO mechanics produces two different types of linear momentum conservation: one for spatial shifts $\delta\mathbf{q} = const$ and one for Galilean boosts $\delta\dot{\mathbf{q}} = const$. Likewise, for angular momentum. We ignore time shifts for a moment and look to get conservation of energy from the Hamiltonian formulation later.

[3d] What are Galilean transformations for HO mechanics?

One would imagine that Galilean transformations for HO mechanics would be the same as for Newtonian mechanics, but extended to allow for constant boosts in acceleration ($\ddot{\mathbf{q}}$) and "jerk" ($\dddot{\mathbf{q}}$), as well as the usual Galilean boost in velocity ($\dot{\mathbf{q}}$). Here is a matrix representation

$$
\begin{bmatrix} \mathbf{q}' \\ (t^3/3)' \\ (t^2/2)' \\ t' \\ 1 \end{bmatrix}
=
\begin{bmatrix} O & \mathbf{j}_0 & \mathbf{a}_0 & \mathbf{v}_0 & \mathbf{q}_0 \\ 0 & 1 & 2t_0 & t_0^2 & t_0^3/3 \\ 0 & 0 & 1 & t_0 & t_0^2/2 \\ 0 & 0 & 0 & 1 & t_0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} \mathbf{q} \\ t^3/3 \\ t^2/2 \\ t \\ 1 \end{bmatrix}
$$

[3e] What is the Legendre transformation for HO mechanics?

The natural candidates for canonically conjugate variables are the pairs that appear in Noether's theorem. Namely canonically conjugate coordinates $\mathbf{q}$ and $\dot{\mathbf{q}}$ and their corresponding momenta

$$
\mathbf{p}_1 := \frac{\partial L}{\partial \dot{\mathbf{q}}} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{\mathbf{q}}} \quad \text{and} \quad \mathbf{p}_2 := \frac{\partial L}{\partial \ddot{\mathbf{q}}}
$$

for which Hamilton's principle implies

$$
\dot{\mathbf{p}}_1 = \frac{\partial L}{\partial \mathbf{q}} \quad \text{and} \quad \dot{\mathbf{p}}_2 = \frac{\partial L}{\partial \dot{\mathbf{q}}} - \mathbf{p}_1 \,. \tag{22}
$$

The Legendre transformation for HO mechanics should naturally be defined by

$$
H(\mathbf{q},\mathbf{p}_1,\dot{\mathbf{q}},\mathbf{p}_2) := \mathbf{p}_1\cdot\dot{\mathbf{q}} + \mathbf{p}_2\cdot\ddot{\mathbf{q}} - L(\mathbf{q},\dot{\mathbf{q}},\ddot{\mathbf{q}})\,. \tag{23}
$$

The corresponding ***energy*** that will be conserved when the Lagrangian has no explicit time dependence is given by

$$E = \left(\frac{\partial L}{\partial \dot{\mathbf{q}}} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{\mathbf{q}}} \right) \cdot \dot{\mathbf{q}} + \frac{\partial L}{\partial \ddot{\mathbf{q}}} \cdot \ddot{\mathbf{q}} - L(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}) \,. \tag{24}$$

[3f] What are Hamilton's canonical equations for HO mechanics?

By taking the differential of the Legendre transformation (23), we find

$$\begin{aligned} dH &= \dot{\mathbf{q}} \cdot d\mathbf{p}_1 + \ddot{\mathbf{q}} \cdot d\mathbf{p}_2 - \frac{\partial L}{\partial \mathbf{q}} \cdot d\mathbf{q} + \left(\mathbf{p}_1 - \frac{\partial L}{\partial \dot{\mathbf{q}}} \right) \cdot d\dot{\mathbf{q}} + \left(\mathbf{p}_2 - \frac{\partial L}{\partial \ddot{\mathbf{q}}} \right) \cdot d\ddot{\mathbf{q}} \\ &= \frac{\partial H}{\partial \mathbf{p}_1} \cdot d\mathbf{p}_1 + \frac{\partial H}{\partial \mathbf{p}_2} \cdot d\mathbf{p}_2 + \frac{\partial H}{\partial \mathbf{q}} \cdot d\mathbf{q} + \frac{\partial H}{\partial \dot{\mathbf{q}}} \cdot d\dot{\mathbf{q}} \,. \end{aligned}$$

Identifying coefficients and using the result of Hamilton's principle in equations (22) yields the canonical Hamiltonian equations

$$\begin{aligned} \dot{\mathbf{q}} &= \frac{\partial H}{\partial \mathbf{p}_1} \,, \qquad & \dot{\mathbf{p}}_1 &= -\frac{\partial H}{\partial \mathbf{q}} \,, \\ \ddot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}_2} &= \frac{d}{dt}\frac{\partial H}{\partial \mathbf{p}_1} \,, \qquad & \dot{\mathbf{p}}_2 &= -\frac{\partial H}{\partial \dot{\mathbf{q}}} \,. \end{aligned}$$

The corresponding Poisson bracket is

$$\frac{dF}{dt} = \{F, H\} = \frac{\partial F}{\partial \mathbf{q}}\frac{\partial H}{\partial \mathbf{p}_1} - \frac{\partial F}{\partial \mathbf{p}_1}\frac{\partial H}{\partial \mathbf{q}} + \frac{\partial F}{\partial \dot{\mathbf{q}}}\frac{\partial H}{\partial \mathbf{p}_2} - \frac{\partial F}{\partial \mathbf{p}_2}\frac{\partial H}{\partial \dot{\mathbf{q}}} \,,$$

which satisfies

$$X_H \,\lrcorner \left(d\mathbf{q} \wedge d\mathbf{p}_1 + d\dot{\mathbf{q}} \wedge d\mathbf{p}_2 \right) = dH \quad \text{with} \quad X_H := \{\,\cdot\,, H\} \,.$$

[3g] In the limit $\epsilon \to 0$, do solutions HO mechanics converge to solutions of Newtonian mechanics?

For a simple mechanical system with potential $W(\mathbf{q})$, we have

$$\left(1 - \frac{\epsilon}{m}\frac{d^2}{dt^2} \right) m\ddot{\mathbf{q}} = -\frac{\partial W}{\partial \mathbf{q}}$$

Hence, the force in HO mechanics is formally given by a *history-dependent* expression

$$m\ddot{\mathbf{q}}(t) = -\left(1 - \frac{\epsilon}{m}\frac{d^2}{dt^2} \right)^{-1} * \frac{\partial W}{\partial \mathbf{q}}$$

where one might imagine computing the inverse by using a Fourier transform, denoted with $(\widehat{\cdot})$. Namely,

$$m\omega^2 \widehat{\mathbf{q}}(\omega) = \frac{1}{1 + \frac{\epsilon}{m}\omega^2} \left(\widehat{\frac{\partial W}{\partial \mathbf{q}}}(\omega) \right) = \left(1 - \frac{\epsilon}{m}\omega^2 \right) \left(\widehat{\frac{\partial W}{\partial \mathbf{q}}}(\omega) \right) + O(\epsilon^2).$$

So it seems that as $\epsilon \to 0$ the solutions of the HO theory might have a chance of converging to Newtonian mechanics, provided the solutions continue to satisfy $\epsilon\,\omega^2/m \ll 1$. To investigate whether this situation holds, let's consider the HO Harmonic Oscillator.

HOHO example. For the higher order harmonic oscillator (HOHO) in one dimension we have

$$L(z,\dot z,\ddot z)=\frac12\Big(\epsilon\ddot z^2+m\dot z^2-kz^2\Big)$$

$$\left(1-\frac{\epsilon}{m}\frac{d^2}{dt^2}\right)m\ddot z=-\,kz$$

with spectrum

$$\Big(1+\frac{\epsilon}{m}\omega^2\Big)\,\omega^2=\frac{k}{m}\quad\Longrightarrow\quad\omega^2=k/m+O(\epsilon),\ \text{and}\ -\frac{m}{\epsilon}\left(1+\frac{k\epsilon}{m^2}+O(\epsilon)\right)$$

So the spectrum contains a stable mode of oscillation with ($\pm$) frequency of order $O(1)$ on the real line and also a growing/decaying pair of modes on the imaginary axis whose growth/decay rate is of order $O(1/\sqrt{\epsilon})$. That is,

$$\omega=\pm\sqrt{\frac{k}{m}}+O(\epsilon),\ \text{and}\ \omega=\pm\,i\,\sqrt{\frac{m}{\epsilon}}\big(1+O(\epsilon)\big).$$

One approach for regularising the rapid unstable growth and decay of the unstable modes arises in the process of approximating the inverse operator, for which one finds

$$\omega^2=\frac{k}{m}\left(1-\frac{\epsilon}{m}\omega^2\right)+O(\epsilon^2)\quad\Longrightarrow\quad\omega^2=\frac{k}{m}\underbrace{\left(1-\frac{k\epsilon}{m^2}\right)}_{O(\epsilon)\text{ shift}}+O(\epsilon^2).$$

Thus, a solution of the type we seek does exist, but the process of going to HO dynamics is a *singular perturbation* that has the possibility of introducing rapidly growing unstable modes.

- Thus, the HO terms represent a singular perturbation that may be regularised, for example, by approximating the inverse operator. The result then has the effect of only slightly shifting the eigenfrequencies and a regular limit arises as its coefficient tends to zero.
- Therefore, the modification to include the 4^{th}-order derivative might actually do what was intended; namely, to cause a slight shift in timing, that might, say, affect the precession of a perihelion. But this is possible only after the instability introduced by the unstable modes injected by the singular perturbation has been regularised.

Energetics. The Hamiltonian for HOHO dynamics is

$$H(z,p_1,\dot z,p_2)=p_1\dot z+p_2\ddot z-L(z,\dot z,\ddot z)=p_1\dot z+\frac{1}{2\epsilon}\,p_2^2-\frac12 m\dot z^2+\frac12 kz^2$$

and its canonical Hamiltonian equations are

$$\dot z=\frac{\partial H}{\partial p_1}\,,\qquad \dot p_1=-\,\frac{\partial H}{\partial z}=-\,kz\,,$$
$$\ddot z=\frac{\partial H}{\partial p_2}=p_2/\epsilon\,,\qquad \dot p_2=-\,\frac{\partial H}{\partial \dot z}=-\,p_1+m\dot z\,.$$

These canonical Hamiltonian equations do recover the HOHO dynamics as,

$$\epsilon\ddot{\ddot z}=\ddot p_2=-\,\dot p_1+m\ddot z=kz+m\ddot z$$

by taking another derivative of the p_2-equation. However, the Hamiltonian above is indefinite as a quadratic form. One might hope that perhaps a canonical transformation would help. We do have

$$p_1 = m\dot{z} - \dot{p}_2 = \frac{d}{dt}(mz - p_2)$$

so maybe defining mixed variables $m\dot{z} - p_1$ would be helpful. We do have the symplectic transform

$$\omega = dz \wedge dp_1 + d\dot{z} \wedge dp_2 = \frac{-1}{2m}\big[d(mz+p_2) \wedge d(m\dot{z} - p_1) + d(m\dot{z} + p_1) \wedge d(mz - p_2)\big]$$

so mixed variables are available, but they cannot change the signature of the quadratic form. Moreover, if we go back to the energy in the original variables, we have

$$\begin{aligned} E(z,\dot{z},\ddot{z}) &= (m\dot{z} - \epsilon\dddot{z})\dot{z} + \epsilon\ddot{z}^2 - \frac{1}{2}(\epsilon\ddot{z}^2 + m\dot{z}^2 - kz^2) \\ &= \frac{1}{2}(\epsilon\ddot{z}^2 + m\dot{z}^2 + kz^2) + \epsilon\ddot{z}^2 - \epsilon\frac{d}{dt}(\ddot{z}\dot{z}) \end{aligned}$$

and we see that the last term has the highest derivatives and is completely uncontrolled in sign. As a quadratic form, the Hamiltonian is

$$H(z,p_1,\dot{z},p_2) = \frac{1}{2}\begin{bmatrix} z \\ p_1 \\ \dot{z} \\ p_2 \end{bmatrix}^T \begin{bmatrix} k & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -m & 0 \\ 0 & 0 & 0 & \epsilon^{-1} \end{bmatrix} \begin{bmatrix} z \\ p_1 \\ \dot{z} \\ p_2 \end{bmatrix} = \frac{1}{2m}p_1^2 + \frac{1}{2\epsilon}p_2^2 - \frac{1}{2m}(m\dot{z} - p_1)^2 + \frac{1}{2}kz^2$$

whose diagonalisation obtained by completing squares shows that it is not positive definite. This implies that the linearised equations around the trivial equilibrium may be unstable due to negative energy modes. As we saw earlier, these negative energy modes are present in the spectrum and they must be handled with care, because they have large growth rates that diverge as $\epsilon \to 0$.