1. (a) (i) **I:** P(exactly 1 event occurs in $[t, t+\delta t)) = \lambda\delta t + o(\delta t)$,
$[o(\delta t)/\delta t \to 0$ as $\delta t \to 0]$.

II: P(2 or more events occur in $[t, t+\delta t)) = o(\delta t)$.

III: Occurrence of events after time t is independent of occurrence of events before t.

seen ⇓

4

(ii) Let $X(t)$ be the number of realizations by time t and let $p(t) = \mathsf{P}(X(t) = 0)$

sim. seen ⇓

$$\begin{aligned}
p(t+\delta t) &= \mathsf{P}(0 \text{ realizations in } [0,t) \text{ and } 0 \text{ realizations in } [t, t+\delta t)) \\
&= p(t)(1 - \lambda\delta t + o(\delta t)) \qquad \text{(from axioms)} \\
\frac{p(t+\delta t) - p(t)}{\delta t} &= -p(t)\lambda + \frac{o(\delta t)}{\delta t} \\
\Rightarrow \frac{dp(t)}{dt} &= -p(t)\lambda \\
\Rightarrow -\log(p(t)) &= \lambda t + c
\end{aligned}$$

Now $p(0) = 1$, so $c = 0$ giving $p(t) = e^{-\lambda t}$ as required.

5

(iii) For a non-homogeneous process we have

method seen ⇓

$$p(t) = e^{-\mu(t)} \quad \text{where } \mu(t) = \int_0^t \lambda(u)\, du$$

2

So,

$$\mu(t) = \int_0^t 1 + \sin(u)\, du = [u - \cos(u)]_0^t = t - \cos(t) + 1,$$

giving

$$p(t) = \exp(\cos(t) - t - 1).$$

3

(b) Let $X(t_1, t_2)$ be the number of questions answered in in $[t_1, t_2)$, then

method seen ⇓

$$\mathsf{E}(X(t, t+\delta t)) = \frac{2}{1+t}\,\delta t + o(\delta t).$$

Let $D(t)$ = number questions answered by t in deterministic model. Then

$$\begin{aligned}
D(t+\delta t) &= D(t) + \frac{2}{1+t}\,\delta t + o(\delta t) \\
\Rightarrow \frac{D(t+\delta t) - D(t)}{\delta t} &= \frac{2}{1+t} + \frac{o(\delta t)}{\delta t} \\
\Rightarrow \frac{dD}{dt} &= \frac{2}{1+t} \\
\Rightarrow D(t) &= 2\log(1+t) + c
\end{aligned}$$

and since $D(0) = 0$, we have $c = 0$

$$D(t) = 2\log(1+t)$$

6

2. (a) (i) Let $\Pi(s)$ be the pgf for X, and let $\mu = \mathsf{E}(X)$ and $\mu_n = \mathsf{E}(Z_n)$, so $\boxed{\text{part seen} \Downarrow}$

$$\mu = \Pi^{'}(1) \qquad \mu_n = \Pi_n^{'}(1)$$

where $\Pi_n^{'}(s)$ is the pgf of $Z_n = Y_1 + \ldots + Y_{Z_{n-1}}$ with Y_i being the number of offspring of individual i in generation $n-1$. From standard pgf results, we have

$$\begin{aligned}
\Pi_n(s) &= \Pi_{n-1}\left[\Pi(s)\right] \\
\Rightarrow \Pi_n^{'}(s) &= \Pi_{n-1}^{'}\left[\Pi(s)\right]\Pi^{'}(s) \\
\Pi_n^{'}(1) &= \Pi_{n-1}^{'}\left[\Pi(1)\right]\Pi^{'}(1) \\
&= \Pi_{n-1}^{'}(1)\Pi^{'}(1) \\
\text{so} \quad \mu &= \mu_{n-1}\mu = \mu_{n-2}\mu^2 = \ldots = \mu^n.
\end{aligned}$$

as $\mathsf{E}(X) = 1$ we have $\mathsf{E}(Z_n) = 1^n = 1$. $\boxed{5}$

(ii) Let $\sigma^2 = \mathsf{var}(X)$ and let $\sigma_n^2 = \mathsf{var}(Z_n)$.

$$\begin{aligned}
\Pi_n^{'}(s) &= \Pi_{n-1}^{'}\left[\Pi(s)\right]\Pi^{'}(s) \\
\Pi_n^{''}(s) &= \Pi_{n-1}^{''}\left[\Pi(s)\right]\Pi^{'}(s)^2 + \Pi_{n-1}^{'}\left[\Pi(s)\right]\Pi^{''}(s) \qquad (1)
\end{aligned}$$

Now $\Pi(1) = 1, \Pi^{'}(1) = \mu, \Pi^{''}(1) = \sigma^2 - \mu + \mu^2$.
Also, since $\sigma_n^2 = \Pi_n^{''}(1) + \mu_n - \mu_n^2$, we have

$$\begin{aligned}
\Pi_n^{''}(1) &= \sigma_n^2 - \mu^n + \mu^{2n} \\
\text{and} \quad \Pi_{n-1}^{''}(1) &= \sigma_{n-1}^2 - \mu^{n-1} + \mu^{2n-2}.
\end{aligned}$$

From (1),

$$\begin{aligned}
\Pi_n^{''}(1) &= \Pi_{n-1}^{''}(1)\Pi^{'}(1)^2 + \Pi_{n-1}^{'}(1)\Pi^{''}(1) \\
\sigma_n^2 - \mu^n + \mu^{2n} &= (\sigma_{n-1}^2 - \mu^{n-1} + \mu^{2n-2})\mu^2 + \mu^{n-1}(\sigma^2 - \mu + \mu^2) \\
\Rightarrow \sigma_n^2 &= \mu^2\sigma_{n-1}^2 + \mu^{n-1}\sigma^2
\end{aligned}$$

Leading to

$$\sigma_n^2 = \mu^{n-1}\sigma^2(1 + \mu + \mu^2 + \ldots + \mu^{n-1})$$

So, as $\mu = 1$ we have

$$\mathsf{var}(Z_n) = \sigma_n^2 = n\sigma^2.$$

$\boxed{7}$

(b) (i) We have $\boxed{\text{method seen} \Downarrow}$

$$\Pi(s) = \sum_{i=0}^{\infty} \mathsf{P}(X = i)s^i = \alpha + (1-\alpha)s^3.$$

Giving,

$$\Pi^{'}(s) = 3(1-\alpha)s^2$$

So $\mu = \Pi^{'}(1) = 3(1-\alpha)$.
Let P(ultimate extinction) $= \theta^*$, then

1. $\mu \leq 1 \Rightarrow \theta^* = 1 \Rightarrow$ ultimate extinction certain.
2. $\mu > 1 \Rightarrow \theta^* < 1 \Rightarrow$ ultimate extinction not certain.

$\mu > 1$ when $3(1-\alpha) > 1 \Rightarrow \alpha < \frac{2}{3}$.
So, when $\alpha < \frac{2}{3}$ ultimate extinction is not certain. $\boxed{4}$

(ii) $\theta^* =$ smallest positive solution of $\theta = \Pi(\theta)$, and $\theta = \frac{1}{2}$:

$$\begin{aligned} \theta &= \alpha + (1-\alpha)\theta^3 \\ \frac{1}{2}\theta^3 - \theta + \frac{1}{2} &= 0 \\ \theta^3 - 2\theta + 1 &= 0 \end{aligned}$$

We know that $\theta = 1$ is a solution:

$$\theta^3 - 2\theta + 1 = (\theta - 1)(\theta^2 + \theta - 1)$$

roots of $\theta^2 + \theta - 1$ are $-\frac{1}{2} \pm \frac{\sqrt{5}}{2}$, therefore

$$\text{Probability of extinction} = \frac{\sqrt{5}-1}{2}$$

$\boxed{4}$

3. (a) (i) Show that $\boxed{\text{sim. seen} \Downarrow}$

$$
\begin{aligned}
\mathsf{P}(\text{ever reach } 0 \mid \text{starts at } i) &= \mathsf{P}(\text{ever reach } i-1 \mid \text{starts at } i) \\
&\quad \times \mathsf{P}(\text{ever reach } i-2 \mid \text{starts at } i-1) \\
&\quad \vdots \\
&\quad \times \mathsf{P}(\text{ever reach } 0 \mid \text{starts at } 1) \\
&= \underbrace{A_1 \times A_1 \times \ldots \times A_1}_{i \text{ times}} \qquad \text{(spatial homogeneity)} \\
&= (A_1)^i
\end{aligned}
$$

$\boxed{3}$

(ii) condition on first step:

$$
\begin{aligned}
A_1 &= \mathsf{P}(\text{visit origin} \mid \text{start from 1}) \\
&= q\mathsf{P}(\text{visit origin} \mid \text{start from 0}) + p\mathsf{P}(\text{visit origin} \mid \text{start from 3}) \\
&= q + pA_3 \\
\Rightarrow pA_1^3 - A_1 + q &= 0
\end{aligned}
$$

$\boxed{3}$

(iii) Solving $pA_1^3 - A_1 + q = 0$ gives $\boxed{\text{unseen} \Downarrow}$

$$
A_1 = 1 \qquad \text{or} \qquad A_1 = \frac{-1 \pm \sqrt{1 + \frac{4q}{p}}}{2}
$$

Now look for solutions in $[0, 1]$:

$$
\frac{-1 - \sqrt{1 + \frac{4q}{p}}}{2} < 0,
$$

also,

$$
\begin{aligned}
\frac{-1+\sqrt{1+\frac{4q}{p}}}{2} &\leq 1 \Rightarrow 1 + \frac{4q}{p} \leq 9 \\
\Rightarrow \frac{1-p}{p} &\leq 2 \Rightarrow p \geq \frac{1}{3}
\end{aligned}
$$

take positive solutions:

$$
A_1 = \begin{cases} \frac{-1+\sqrt{1+\frac{4q}{p}}}{2} & \text{if } p > 1/3 \\ 1 & \text{if } p \leq 1/3 \end{cases}
$$

(Noting that when $p = 1/3$, $A_1 = 1$). $\boxed{5}$

(b) (i) Let X_t be the value of the share on day t. $\boxed{\text{sim. seen} \Downarrow}$

Let Y_t be the change in value of the share on day t.

Then,

$$X_t = X_0 + Y_1 + \ldots + Y_t.$$

$$X_0 = 100 \quad Y_i = \begin{cases} 0.4 & \text{with probability } 0.5 \\ -0.2 & \text{with probability } 0.3 \\ 0 & \text{with probability } 0.2 \end{cases}$$

$$\begin{aligned}
\mathsf{E}(Y_i) &= 0.4 \times 0.5 - 0.2 \times 0.3 = 0.2 - 0.06 = 0.14. \\
\mathsf{var}(Y_i) &= \mathsf{E}(Y_i^2) - \mathsf{E}^2(Y_i) \\
\mathsf{E}(Y_i^2) &= 0.4^2 \times 0.5 + (0.2)^2 \times 0.3 = 0.16 \times 0.5 + 0.04 \times 0.3 \\
&= 0.08 + 0.012 = 0.092 \\
\mathsf{var}(Y_i) &= 0.092 - 0.14^2 = 0.092 - 0.0196 = 0.0724.
\end{aligned}$$

Giving

$$\begin{aligned}
\mathsf{E}(X_t) &= 100 + 0.14t = \mu_t \\
\mathsf{var}(X_t) &= 0.0724t = \sigma_t^2
\end{aligned}$$

$\boxed{5}$

(ii) For large t

$$\begin{aligned}
X_t - X_0 = \sum_{i=1}^{t} Y_i &\sim N(0.014t, 0.0724t) \\
\Rightarrow \frac{X_t - X_0 - 0.14t}{\sqrt{0.0724t}} &\sim N(0, 1)
\end{aligned}$$

$$\begin{aligned}
\mathsf{P}(X_{365} > 110) &= 1 - \Phi\left(\frac{110 - (100 + 0.14 \times 365)}{\sqrt{0.0724 \times 365}}\right) \\
&= 1 - \Phi\left(\frac{110 - \mu_{365}}{\sqrt{\sigma_{365}^2}}\right)
\end{aligned}$$

$\boxed{4}$

4. (a) (i) A Markov chain is irreducible if it has only one communicating class, i.e. there is a path of non-zero probability from state i to state j and back again for all i, j in the sample space. $\boxed{\text{seen} \Downarrow}$ $\boxed{2}$

(ii) A Markov chain is aperiodic if all states have period 1, i.e. if

$$\text{gcd}\{n : p_{ij}^{(n)} > 0\} = 1.$$

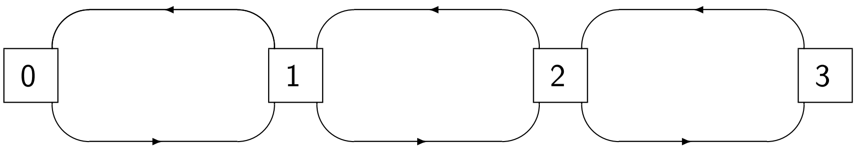
where $p_{ij}^{(n)}$ is the probability of going from i to j in n steps. $\boxed{2}$

(b) (i) State space $= \{0, 1, 2, 3\}$ (Number of balls in the first urn). $\boxed{\text{sim. seen} \Downarrow}$

$$P = \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$\boxed{4}$

(ii) transition diagram:



$\boxed{2}$

(iii) Irreducible, finite state space $\Rightarrow$ there is a unique stationary distribution. $\boxed{1}$

(iv) Need aperiodicity for this distribution to also be limiting. Here the Markov chain is periodic with period 2, so the stationary distribution is not limiting. $\boxed{2}$

(v) Find stationary distribution, $\boldsymbol{\pi}$ from, $\boldsymbol{\pi} = \boldsymbol{\pi} P, \quad \sum_{i=0}^{3} \pi_i = 1$

$$\left.\begin{array}{rcl} \pi_0 & = & \frac{1}{3}\pi_1 \\ \pi_0 + \frac{2}{3}\pi_2 & = & \pi_1 \\ \frac{2}{3}\pi_1 + \pi_3 & = & \pi_2 \\ \frac{1}{3}\pi_2 & = & \pi_3 \end{array}\right\} \Rightarrow \begin{array}{rcl} \pi_1 & = & 3\pi_0 \\ \pi_2 & = & 3\pi_0 \\ \pi_3 & = & \pi_0 \end{array}$$

and

$$\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1 \Rightarrow \pi_0 + 3\pi_0 + 3\pi_0 + \pi_0 = 1 \Rightarrow \pi_0 = \frac{1}{8}$$

Giving,

$$\boldsymbol{\pi} = \left(\frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8}\right)$$

$\boxed{5}$

Mean recurrence time to state 0, μ_0 is given by:

$$\mu_0 = \frac{1}{\pi_0} = 8$$

$\boxed{2}$

5. (a) (i) Define

$$Q = \frac{d}{dt}P(t)\bigg|_{t=0}$$

the transition rate matrix with elements q_{ij}, and let $P(t)$ have element $p_{ij}(t)$, i, j in the sample space. The forward differential equations are given by:

$$\begin{aligned} \frac{d}{dt}P(t) &= P(t)Q \\ \Rightarrow \frac{d}{dt}p_{ij}(t) &= \sum_k p_{ik}(t)q_{jk} \quad \forall\, i,j. \end{aligned}$$

seen ⇓ 2

(ii)

$$\sum_i \pi_i = 1, \quad \boldsymbol{\pi} Q = \mathbf{0} \quad (\text{or } \boldsymbol{\pi} = P(t)\boldsymbol{\pi}, \forall t).$$

2

(b) (i) Let state 0 $\equiv$ good mood, and state 1 $\equiv$ bad mood.

sim. seen ⇓

$$\begin{aligned} p_{01}(\delta t) = \mathsf{P}(0 \to 1 \quad \text{in } [t, t+\delta t)) &= \alpha\,\delta t + o(\delta t) \\ p_{10}(\delta t) = \mathsf{P}(1 \to 0 \quad \text{in } [t, t+\delta t)) &= \beta\,\delta t + o(\delta t) \end{aligned}$$

We have, by definition,

$$p_{ij}(\delta t) = \begin{cases} 1 + \delta t\; q_{ii} + o(\delta t) & i = j \\ \delta t\; q_{ij} + o(\delta t) & i \neq j \end{cases} \text{ small } \delta t$$

So, as the rows of Q sum to zero, we have

$$Q = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}.$$

3

(ii) assume true for $n = k$, let $n = k + 1$

unseen ⇓

$$\begin{aligned} Q^{k+1} &= Q^k Q \\ &= ((-\alpha - \beta)^{k-1} Q)Q \quad \text{from assumption} \\ &= (-\alpha - \beta)^{k-1} Q^2 \end{aligned}$$

$$Q^2 = \begin{pmatrix} \alpha^2 + \alpha\beta & -\alpha^2 - \alpha\beta \\ -\alpha\beta - \beta^2 & \alpha\beta + \beta^2 \end{pmatrix} = (-\alpha-\beta) \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix} = (-\alpha-\beta)Q$$

$$Q^{k+1} = (-\alpha - \beta)^{k-1}(-\alpha - \beta)Q = (-\alpha - \beta)^k Q$$

true for $n = k+1$ if true for $n = k$, as true for $n = 1$ ($Q^1 = (-\alpha-\beta)^0 Q$), result follows by induction. 4

$$
\begin{aligned}
\exp(tQ) &= I + \sum_{n=1}^{\infty} \frac{t^n}{n!}(-\alpha-\beta)^{n-1}Q \\
&= I + \frac{Q}{-\alpha-\beta} \sum_{n=0}^{\infty} \frac{(t(-\alpha-\beta))^{n-1}}{n!} \\
&= I + \frac{Q}{-\alpha-\beta} \{\exp(t(-\alpha-\beta) - 1\} \\
&= I + \frac{Q}{\alpha+\beta} + \frac{Q}{-\alpha-\beta} \exp(t(-\alpha-\beta))
\end{aligned}
$$

$\boxed{3}$

(iii) Backward differential equations:

$$
\begin{aligned}
\frac{d}{dt}P(t) &= \frac{d}{dt}\left\{ I + \frac{Q}{\alpha+\beta} + \frac{Q}{-\alpha-\beta} \exp(t(-\alpha-\beta)) \right\} \\
&= \exp(t(-\alpha-\beta))Q \\
QP(t) &= Q\left\{ I + \frac{Q}{\alpha+\beta} + \frac{Q}{-\alpha-\beta} \exp(t(-\alpha-\beta)) \right\} \\
&= Q + \frac{Q^2}{\alpha+\beta} + \frac{Q^2}{-\alpha-\beta} \exp(t(-\alpha-\beta)) \\
&= Q + \frac{(-\alpha-\beta)Q}{\alpha+\beta} + \frac{(-\alpha-\beta)Q}{-\alpha-\beta} \exp(t(-\alpha-\beta)) \\
&= Q\exp(t(-\alpha-\beta))
\end{aligned}
$$

i.e. $P(t)$ satisfies the backward differential equations:

$$
\frac{d}{dt}P(t) = QP(t).
$$

$\boxed{3}$

(iv) stationary distribution satisfies

$$
(\pi_0 \;\; \pi_1)\begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix} = (0 \;\; 0) \quad \pi_0 + \pi_1 = 1.
$$

$$
\begin{aligned}
-\alpha\pi_0 + \beta\pi_1 &= 0 \quad \text{and} \quad \pi_0 + \pi_1 = 1 \\
-\alpha\pi_0 + \beta(1-\pi_0) &= 0 \Rightarrow \pi_0 = \frac{\beta}{\alpha+\beta} \\
\pi_1 &= 1 - \frac{\beta}{\alpha+\beta} = \frac{\alpha}{\alpha+\beta}
\end{aligned}
$$

giving

$$
\boldsymbol{\pi} = \left(\frac{\beta}{\alpha+\beta}, \;\; \frac{\alpha}{\alpha+\beta} \right)
$$

$\boxed{3}$