UNIVERSITY OF LONDON

IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

QUESTIONS

BSc and MSci EXAMINATIONS (MATHEMATICS)

MAY–JUNE 2002

This paper is also taken for the relevant examination for the Associateship.

M3S8/M4S8 TIME SERIES

DATE: Tuesday, 28th May 2001 TIME: 10 am – 12 noon

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used. Statistical tables will not be available.

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Note: Throughout this paper $\{\epsilon_t\}$ is a sequence of uncorrelated random variables having zero mean and variance σ_ϵ^2, unless stated otherwise.

1. *a)* What is meant by saying that a stochastic process is second-order stationary?

b) Consider the following ARMA(1,1) model,

$$X_t = \alpha X_{t-1} + \epsilon_t - 2\alpha\epsilon_{t-1}.$$

i) Express the model using the backward shift operator B.

ii) Determine constraints on α which ensure that the model is both stationary and invertible.

iii) Assuming stationarity express the model in an infinite MA representation.

iv) Determine $\text{var}\{X_t\}$.

2. *a)* Let $\{\varepsilon_t\}$ and $\{\eta_t\}$ be zero-mean white noise processes with variances 1 and θ^2 respectively, $|\theta| < 1,\ \theta \neq 0$.

Show that the MA(1) processes,

$$X_t = \varepsilon_t + \theta\varepsilon_{t-1} \quad \text{and} \quad Y_t = \eta_t + \frac{1}{\theta}\eta_{t-1},$$

have the same autocovariance function.

Are either $\{X_t\}$ or $\{Y_t\}$ invertible?

b) Consider the process defined by

$$X_t = \epsilon_t + (-1)^{t-1}\epsilon_{t-1}.$$

i) Determine $\text{E}\{X_t\}$ and $\text{cov}\{X_t, X_{t+\tau}\}$ for $\tau = 0, \pm1, \pm2, \ldots$.

ii) Giving full justification, determine whether the process is second order stationary.

3. *a)* What are the three properties of a linear time-invariant (LTI) filter?

b) *i)* Determine the spectral density function of the white noise process $\{\epsilon_t\}$.

ii) Find the frequency response functions associated with the following LTI filters

$$\begin{aligned} L_1\{\{\epsilon_t\}\} &= \epsilon_t - 0.2\epsilon_{t-1}, \\ L_2\{\{X_t\}\} &= X_t - 2X_{t-1} + X_{t-2}. \end{aligned}$$

iii) By determining their gain functions, describe the nature of $L_1\{\cdot\}$ and $L_2\{\cdot\}$.

iv) Determine the spectral density function for the ARMA(2,1) process,

$$X_t = 2X_{t-1} - X_{t-2} + \epsilon_t - 0.2\epsilon_{t-1}.$$

4. Assume that $X_1, X_2, \ldots, X_N$ is a sample from a zero mean AR(2) process, with defining equation,

$$X_t = \phi_{1,2}X_{t-1} + \phi_{2,2}X_{t-2} + \epsilon_t.$$

a) Show that

$$\widehat{s}_\tau = \frac{1}{N}\sum_{t=1}^{N-|\tau|} X_t X_{t+|\tau|} \qquad \tau = 0, \pm 1, \pm 2, \ldots, \pm(N-1),$$

is a biased estimator of $s_\tau = \text{cov}\{X_t, X_{t+\tau}\}$.

b) By multiplying the defining equation by X_{t-k} and taking expectations for $k = 1, 2$, derive the Yule-Walker estimators of $\phi_{1,2}$ and $\phi_{2,2}$.

c) Show that the corresponding Yule-Walker estimator of σ_ϵ^2 is given by

$$\widehat{\sigma}_\epsilon^2 = \frac{\widehat{s}_0^3 - 2\widehat{s}_0\widehat{s}_1^2 + 2\widehat{s}_1^2\widehat{s}_2 - \widehat{s}_0\widehat{s}_2^2}{\widehat{s}_0^2 - \widehat{s}_1^{\,2}}.$$

d) Describe the relationship between Yule-Walker estimators and least-squares estimators of the AR parameters.

5. Consider the following periodogram estimator of the spectral density function of $\{X_t\}$ at the Fourier frequencies $f_k = \frac{k}{N}$, $k = 0, 1, \ldots, \frac{N}{2}$,

$$\widehat{S}^{(p)}(f_k) = \frac{1}{N}\left|\sum_{t=1}^{N}(X_t - c)e^{-i2\pi f_k t}\right|^2,$$

where c is some constant.

a) Show that

$$\widehat{S}^{(p)}(0) = N(\overline{X} - c)^2,$$

where $\overline{X} = (1/N)\sum_{t=1}^{N} X_t$.

b) Show that the periodogram does not depend on c for all non-zero Fourier frequencies.

What are the implications of this result?

[You may use the fact that for any complex number $z \neq 1$, then, $\sum_{t=1}^{N} z^t = \frac{z - z^{N-1}}{1-z}$.]

c) What type of processes are likely to produce a periodogram which is a biased estimator of the true spectrum due to sidelobe leakage?

d) Describe a method to reduce leakage bias in the periodogram, and explain why it is effective.