UNIVERSITY OF LONDON

IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY AND MEDICINE

SOLUTIONS

BSc and MSci EXAMINATIONS (MATHEMATICS)
MAY–JUNE 2002

This paper is also taken for the relevant examination for the Associateship.

M3S8/M4S8 TIME SERIES

DATE: Tuesday, 28th May 2001 TIME: 10 am – 12 noon

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used. Statistical tables will not be available.

1. *a)* $\{X_t\}$ is second-order stationary if $E\{X_t\}$ is a finite constant for all t, var$\{X_t\}$ is a finite constant for all t, and cov$\{X_t, X_{t+\tau}\} = s_\tau$, a finite quantity depending only on τ and not on t. seen ⇓

4

b) sim. seen ⇓

$$X_t = \alpha X_{t-1} + \epsilon_t - 2\alpha\epsilon_{t-1}.$$

i)

$$\begin{aligned}(1-\alpha B)X_t &= (1-2\alpha B)\epsilon_t\\ \Phi(B)X_t &= \Theta(B)\epsilon_t.\end{aligned}$$

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ii) For stationarity, roots of $\Phi(z)$ must lie outside the unit circle:

$$1-\alpha z = 0 \quad \Rightarrow \quad z = \frac{1}{\alpha}.$$

$$\left|\frac{1}{\alpha}\right| > 1 \quad \Rightarrow \quad |\alpha| < 1.$$

For invertibility, roots of $\Theta(z)$ must lie outside the unit circle: 2

$$1-2\alpha z = 0 \quad \Rightarrow \quad z = \frac{1}{2\alpha}.$$

$$\left|\frac{1}{2\alpha}\right| > 1 \quad \Rightarrow \quad |\alpha| < \frac{1}{2}.$$

To ensure both stationarity and invertibility we must have 2

$$-\frac{1}{2} < \alpha < \frac{1}{2}.$$

1

iii) unseen ⇓

$$\begin{aligned}X_t &= \Theta(B)\Phi^{-1}(B)\epsilon_t\\ &= (1-2\alpha B)(1+\alpha B+\alpha^2B^2+\alpha^3B^3+\ldots)\epsilon_t\\ &= \left[1+B(\alpha-2\alpha)+B^2(\alpha^2-2\alpha^2)+B^3(\alpha^3-2\alpha^3)+\ldots\right]\epsilon_t\\ &= \left[1-\alpha B-\alpha^2B^2-\alpha^3B^3-\ldots\right]\epsilon_t\\ &= \epsilon_t - \sum_{j=1}^{\infty}\alpha^j\epsilon_{t-j}.\end{aligned}$$

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iv)

$$\begin{aligned}\mathrm{var}\{X_t\} &= \mathrm{var}\left\{\epsilon_t - \sum_{j=1}^{\infty}\alpha^j\epsilon_{t-j}\right\}\\ &= \sigma_\epsilon^2 + \sum_{j=1}^{\infty}\alpha^{2j}\sigma_\epsilon^2\\ &= \sigma_\epsilon^2\sum_{j=0}^{\infty}\alpha^{2j} = \frac{\sigma_\epsilon^2}{1-\alpha^2}.\end{aligned}$$

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2. *a)* We have that $\sigma_\epsilon^2 = 1$ and $\sigma_\eta^2 = \theta^2$. $\boxed{\text{sim. seen} \Downarrow}$

$$\begin{aligned} s_{\tau,X} &= \text{cov}\{X_t, X_{t+\tau}\} = \text{E}\{X_t X_{t+\tau}\} \\ &= \text{E}\{(\epsilon_t + \theta\epsilon_{t-1})(\epsilon_t + \theta\epsilon_{t-1})\} \\ &= \text{E}\{\epsilon_t\epsilon_{t+\tau}\} + \theta\text{E}\{\epsilon_t\epsilon_{t+\tau-1}\} + \theta\text{E}\{\epsilon_{t-1}\epsilon_{t+\tau}\} + \theta^2\text{E}\{\epsilon_{t-1}\epsilon_{t+\tau-1}\}. \end{aligned}$$

Above expectations are non-zero in the following cases:

$$\begin{aligned} t = t+\tau \Rightarrow \tau = 0; \qquad & t-1 = t+\tau \Rightarrow \tau = -1; \\ t = t+\tau-1 \Rightarrow \tau = 1; \qquad & t-1 = t+\tau-1 \Rightarrow \tau = 0. \end{aligned}$$

Giving,

$$s_{\tau,X} = \begin{cases} \sigma_\epsilon^2 + \sigma_\epsilon^2\theta^2 = 1 + \theta^2 & \tau = 0 \\ \theta\sigma_\epsilon^2 = \theta & \tau = \pm 1 \\ 0 & \text{otherwise,} \end{cases}$$

and,

$$s_{\tau,Y} = \begin{cases} \sigma_\eta^2 + \sigma_\eta^2\frac{1}{\theta^2} = \theta^2 + 1 & \tau = 0 \\ \frac{1}{\theta}\sigma_\eta^2 = \theta & \tau = \pm 1 \\ 0 & \text{otherwise,} \end{cases}$$

so, $s_{\tau,X} = s_{\tau,Y}$. $\boxed{8}$

For $\{X_t\}$ to be invertible the roots of $1 + \theta z$ must lie outside the unit circle, i.e., $|\theta| < 1$.

For $\{Y_t\}$ to be invertible the roots of $1 + \frac{1}{\theta}z$ must lie outside the unit circle, i.e., $|\theta| > 1$.

So, $\{X_t\}$ is invertible and $\{Y_t\}$ is not. $\boxed{2}$

b) $\boxed{\text{unseen} \Downarrow}$

$$X_t = \epsilon_t + (-1)^{t-1}\epsilon_{t-1}.$$

i)

$$\text{E}\{X_t\} = \text{E}\{\epsilon_t\} + (-1)^{t-1}\text{E}\{\epsilon_{t-1}\} = 0.$$

$\boxed{1}$

$$\begin{aligned} \text{cov}\{X_t, X_{t+\tau}\} &= \text{E}\{X_t X_{t+\tau}\} \\ &= \text{E}\{(\epsilon_t + (-1)^{t-1}\epsilon_{t-1})(\epsilon_{t+\tau} + (-1)^{t+\tau-1}\epsilon_{t+\tau-1})\} \\ &= \text{E}\{\epsilon_t\epsilon_{t+\tau}\} + (-1)^{t-1}\text{E}\{\epsilon_{t-1}\epsilon_{t+\tau}\} + \\ &\quad (-1)^{t+\tau-1}\text{E}\{\epsilon_t\epsilon_{t+\tau-1}\} + (-1)^{t-1}(-1)^{t+\tau-1}\text{E}\{\epsilon_{t-1}\epsilon_{t+\tau-1}.\} \end{aligned}$$

Above expectations are non-zero when $\tau = 0, \pm 1$ (as in part (a)). Giving, $\boxed{3}$

$$\begin{aligned} \tau = 0 \quad \text{cov}\{X_t, X_t\} &= \sigma_\epsilon^2 + (-1)^{t-1+t-1}\sigma_\epsilon^2 \\ &= \sigma_\epsilon^2(1 + (-1)^{2(t-1)}) = 2\sigma_\epsilon^2. \\ \tau = 1 \quad \text{cov}\{X_t, X_{t+1}\} &= (-1)^t\sigma_\epsilon^2 \\ \tau = -1 \quad \text{cov}\{X_t, X_{t-1}\} &= (-1)^{t-1}\sigma_\epsilon^2 \end{aligned}$$

Giving,

$$\text{cov}\{X_t, X_{t+\tau}\} = \begin{cases} 2\sigma_\epsilon^2 & \tau = 0 \\ (-1)^t\sigma_\epsilon^2 & \tau = 1 \\ (-1)^{t-1}\sigma_\epsilon^2 & \tau = -1 \\ 0 & \text{otherwise.} \end{cases}$$

$\boxed{5}$

ii) $\{X_t\}$ is not second order stationary as $\text{cov}\{X_t, X_{t+\tau}\}$ depends on t. $\boxed{1}$

3. *a)* A digital filter L that transforms an input sequence $\{x_t\}$ into an output sequence $\{y_t\}$ is called a linear time-invariant (LTI) digital filter if it has the following three properties: $\boxed{\text{seen} \Downarrow}$

[1] Scale-preservation:

$$L\{\{\alpha x_t\}\} = \alpha L\{\{x_t\}\}.$$

[2] Superposition:

$$L\{\{x_{t,1} + x_{t,2}\}\} = L\{\{x_{t,1}\}\} + L\{\{x_{t,2}\}\}.$$

[3] Time invariance:
If

$$L\{\{x_t\}\} = \{y_t\}, \quad \text{then} \quad L\{\{x_{t+\tau}\}\} = \{y_{t+\tau}\}.$$

Where τ is integer-valued, and the notation $\{x_{t+\tau}\}$ refers to the sequence whose t-th element is $x_{t+\tau}$. $\boxed{3}$

b) *i)*

$$s_{\tau,\epsilon} = \begin{cases} \sigma_\epsilon^2 & \tau = 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$S_\epsilon(f) = \sum_{\tau=-\infty}^{\infty} s_{\tau,\epsilon} e^{-i2\pi f\tau} = \sigma_\epsilon^2 \qquad |f| \leq \frac{1}{2}.$$

i.e., a constant spectrum. $\boxed{3}$

ii) $\boxed{\text{sim. seen} \Downarrow}$

$$\begin{aligned} L_1\{\{e^{i2\pi ft}\}\} &= e^{i2\pi ft} - 0.2e^{i2\pi f(t-1)} \\ &= e^{i2\pi ft}(1 - 0.2e^{-i2\pi f}) \\ &= e^{i2\pi ft}G_1(f); \\ L_2\{\{e^{i2\pi ft}\}\} &= e^{i2\pi ft} - 2e^{i2\pi f(t-1)} + e^{i2\pi f(t-2)} \\ &= e^{i2\pi ft}(1 - 2e^{-i2\pi f} + e^{-i4\pi f}) \\ &= e^{i2\pi ft}(1 - e^{-i2\pi f})^2 \\ &= e^{i2\pi ft}G_2(f). \end{aligned}$$

so,

$$G_1(f) = 1 - 0.2e^{-i2\pi f} \quad \text{and} \quad G_2(f) = (1 - e^{-i2\pi f})^2.$$

$\boxed{4}$

iii)

$$\begin{aligned} |G_1(f)|^2 &= |1 - 0.2\cos(2\pi f) + i0.2\sin(2\pi f)|^2 \\ &= (1 - 0.2\cos(2\pi f))^2 + 0.04\sin^2(2\pi f) \\ &= 1 - 0.4\cos(2\pi f) + 0.04 = 1.04 - 0.4\cos(2\pi f). \end{aligned}$$

$$\begin{aligned} |G_2(f)|^2 &= |1 - e^{-i2\pi f}|^4 \\ &= ((1 - \cos(2\pi f))^2 + \sin^2(2\pi f))^2 \\ &= (1 - 2\cos(2\pi f) + 1)^2 \\ &= 4(1 - \cos(2\pi f))^2 = 4(2\sin^2(\pi f))^2 = 16\sin^4(\pi f). \end{aligned}$$

Both $|G_1(f)|^2$ and $|G_2(f)|^2$ increase as the frequency increases from 0 to 0.5, so both are high pass filters. $\boxed{6}$

iv) From the properties of LTI filters, we have,

$$\begin{aligned} |G_2(f)|^2 S_X(f) &= |G_1(f)|^2 S_\epsilon(f) \\ S_X(f) &= \frac{(1.04 - 0.4\cos(2\pi f))\sigma_\epsilon^2}{16\sin^4(\pi f)}. \end{aligned}$$

Note here that $\{X_t\}$ is not stationary. $\boxed{4}$

4. *a)* seen ⇓

$$\begin{aligned}
\mathrm{E}\{\widehat{s}_\tau\} &= \frac{1}{N}\sum_{t=1}^{N-|\tau|} \mathrm{E}\{X_t X_{t+|\tau|}\} \\
&= \frac{1}{N}\sum_{t=1}^{N-|\tau|} s_\tau \\
&= \frac{(N-|\tau|)}{N} s_\tau \neq s_\tau .
\end{aligned}$$

therefore, $\widehat{s}_\tau$ is a biased estimator of s_τ. 4

b) sim. seen ⇓

$$\begin{aligned}
\mathrm{E}\{X_t X_{t-k}\} &= \phi_{1,2}\mathrm{E}\{X_{t-1}X_{t-k}\} + \phi_{2,2}\mathrm{E}\{X_{t-2}X_{t-k}\} + \mathrm{E}\{\epsilon_t X_{t-k}\} \\
s_k &= \phi_{1,2}s_{k-1} + \phi_{2,2}s_{k-2} + \mathrm{E}\{\epsilon_t X_{t-k}\}.
\end{aligned}$$

For $k = 1, 2$, 3

$$\begin{aligned}
s_1 &= \phi_{1,2}s_0 + \phi_{2,2}s_1 \\
s_2 &= \phi_{1,2}s_1 + \phi_{2,2}s_0
\end{aligned}$$

Substitute estimators into the equations to obtain,

$$\begin{pmatrix} \widehat{s}_1 \\ \widehat{s}_2 \end{pmatrix} = \begin{pmatrix} \widehat{s}_0 & \widehat{s}_1 \\ \widehat{s}_1 & \widehat{s}_0 \end{pmatrix} \begin{pmatrix} \widehat{\phi}_{1,2} \\ \widehat{\phi}_{2,2} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \widehat{\phi}_{1,2} \\ \widehat{\phi}_{2,2} \end{pmatrix} = \frac{1}{\widehat{s}_0^2 - \widehat{s}_1^2} \begin{pmatrix} \widehat{s}_0 & -\widehat{s}_1 \\ -\widehat{s}_1 & \widehat{s}_0 \end{pmatrix} \begin{pmatrix} \widehat{s}_1 \\ \widehat{s}_2 \end{pmatrix}.$$

The Yule-Walker estimators are,

$$\widehat{\phi}_{1,2} = \frac{\widehat{s}_0\widehat{s}_1 - \widehat{s}_1\widehat{s}_2}{\widehat{s}_0^2 - \widehat{s}_1^2}; \qquad \widehat{\phi}_{2,2} = \frac{\widehat{s}_0\widehat{s}_2 - \widehat{s}_1^2}{\widehat{s}_0^2 - \widehat{s}_1^2}.$$

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c) Letting $k = 0$,

$$s_0 = \phi_{1,2}s_1 + \phi_{2,2}s_2 + \sigma_\epsilon^2.$$

The Yule-Walker estimator of σ_ϵ^2 is given by,

$$\begin{aligned}
\widehat{\sigma}_\epsilon^2 &= \widehat{s}_0 - \widehat{\phi}_{1,2}\widehat{s}_1 - \widehat{\phi}_{2,2}\widehat{s}_2 \\
&= \widehat{s}_0 - \frac{\widehat{s}_0\widehat{s}_1^2 - \widehat{s}_1^2\widehat{s}_2}{\widehat{s}_0^2 - \widehat{s}_1^2} - \frac{\widehat{s}_0\widehat{s}_2^2 - \widehat{s}_2\widehat{s}_1^2}{\widehat{s}_0^2 - \widehat{s}_1^2} \\
&= \frac{\widehat{s}_0^3 - 2\widehat{s}_0\widehat{s}_1^2 + 2\widehat{s}_1^2\widehat{s}_2 - \widehat{s}_0\widehat{s}_2^2}{\widehat{s}_0^2 - \widehat{s}_1^2},
\end{aligned}$$

as required. 4

d) The Yule-Walker estimators can be formulated as a least squares problem by explicitly adding zeros to the observations at the beginning and end of the data.

3

5. *a)* unseen $\Downarrow$

$$\begin{aligned}
\widehat{S}^{(p)}(0) &= \frac{1}{N}\left|\sum_{t=1}^{N}(X_t - c)\right|^2 \\
&= \frac{1}{N}\left(\left[\sum_{t=1}^{N} X_t\right] - Nc\right)^2 \\
&= \frac{1}{N}(N\overline{X} - Nc)^2 = N(\overline{X} - c)^2.
\end{aligned}$$

4

b)

$$\begin{aligned}
\widehat{S}^{(p)}(f_k) &= \frac{1}{N}\left|\sum_{t=1}^{N}(X_t - c)e^{-i2\pi f_k t}\right|^2 \\
&= \left|\sum_{t=1}^{N} X_t e^{-i2\pi f_k t} - c\sum_{t=1}^{N} e^{-i2\pi f_k t}\right|^2.
\end{aligned}$$

From the hint:

$$\begin{aligned}
\sum_{t=1}^{N} e^{-i2\pi f_k t} &= \frac{e^{-i2\pi f_k} - e^{-i2\pi f_k (N+1)}}{1 - e^{-i2\pi f_k}} \\
&= \frac{e^{-i2\pi k/N} - e^{-i2\pi k}e^{-i2\pi k/N}}{1 - e^{-i2\pi f_k}} \\
&= \frac{e^{-i2\pi k/N}(1 - e^{-i2\pi k})}{1 - e^{-i2\pi f_k}} \\
&= \frac{e^{-i2\pi k/N}(1 - 1)}{1 - e^{-i2\pi f_k}} = 0.
\end{aligned}$$

Therefore,

$$\widehat{S}^{(p)}(f_k) = \left|\sum_{t=1}^{N} X_t e^{-i2\pi f_k t}\right|^2,$$

which is independent of c. 8

If the time series is demeaned (regardless of the estimator of the mean), this only affects the periodogram at frequency 0. If $c = \overline{X}, \widehat{S}^{(p)}(0) = 0$. 2

c) Processes with a large dynamic range are more likely to produce periodograms which are biased – this is due to sidelobe leakage via the sidelobes of Féjer's kernel. seen $\Downarrow$

2

d) One way of reducing the bias is by tapering – this has the effect of changing the shape of the Féjer's kernel in the expression for the expectation of the periodogram. The spectral estimator becomes (for a zero mean process)

$$\widehat{S}^{(d)}(f_k) = \left|\sum_{t=1}^{N} h_t X_t e^{-i2\pi f_k t}\right|^2.$$

where the real-valued sequence $\{h_1, h_2, \ldots, h_N\}$ is the data taper. This taper attenuates the values of the time series at the beginning and end, thus making the transition to zero implied in the truncated sum of the definition of the peridogram less abrupt. 4