UNIVERSITY OF LONDON

IMPERIAL COLLEGE LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)
MAY–JUNE 2003

This paper is also taken for the relevant examination for the Associateship.

M3S8/M4S8 (SOLUTIONS) TIME SERIES

DATE: Tuesday, 27th May 2003 TIME: 10 am – 12 noon

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used. Statistical tables will not be available.

1. *a)* $\{X_t\}$ is second-order stationary if $E\{X_t\}$ is a finite constant for all t, $var\{X_t\}$ is a finite constant for all t, and $cov\{X_t, X_{t+\tau}\}$, is a finite quantity depending only on τ and not on t. seen ⇓

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b) We can write the AR(p) process as

$$\begin{aligned} X_t - \phi_{1,p}X_{t-1} - \phi_{2,p}X_{t-2} - \ldots - \phi_{p,p}X_{t-p} &= \epsilon_t \\ (1 - \phi_{1,p}B - \phi_{2,p}B^2 \ldots \phi_{p,p}B^p)X_t &= \epsilon_t. \end{aligned}$$

The condition for stationarity is that the roots of the characteristic polynomial

$$1 - \phi_{1,p}z - \phi_{2,p}z^2 \ldots \phi_{p,p}z^p = \Phi(z)$$

lie OUTSIDE the unit circle.

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c) *i)* sim. seen ⇓

$$\begin{aligned} X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} &= \epsilon_t \\ (1 - \phi_1 B - \phi_2 B^2)X_t &= \epsilon_t \\ (1 - \rho_1 B)(1 - \rho_2 B)X_t &= \epsilon_t \\ X_t &= \frac{1}{(1 - \rho_1 B)(1 - \rho_2 B)}\epsilon_t \end{aligned}$$

$$\frac{1}{(1 - \rho_1 B)(1 - \rho_2 B)} = \frac{A}{1 - \rho_1 B} + \frac{C}{1 - \rho_2 B}$$

from which we obtain,

$$\begin{aligned} A + C = 1 &\Rightarrow A\rho_2 = (1 - C)\rho_2 \\ -(A\rho_2 + C\rho_1) = 0 &\Rightarrow (1 - C)\rho_2 = -C\rho_1 \\ &\Rightarrow C = \rho_2/(\rho_2 - \rho_1). \end{aligned}$$

Hence, $A = 1 - C = -\rho_1/(\rho_2 - \rho_1)$. Thus

$$\frac{1}{(1 - \rho_1 B)(1 - \rho_2 B)} = \left(\frac{1}{\rho_2 - \rho_1}\right)\left[\frac{\rho_2}{1 - \rho_2 B} - \frac{\rho_1}{1 - \rho_1 B}\right],$$

and hence,

$$X_t = \sum_{k=0}^{\infty} \frac{\rho_2^{k+1} - \rho_1^{k+1}}{\rho_2 - \rho_1}\epsilon_{t-k}.$$

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ii) Using the MA representation, and properties of the zero-mean white noise process, we have unseen $\Downarrow$

$$E\{X_t\} = 0, \quad var\{X_t\} = \frac{1}{(\rho_2 - \rho_1)^2} \sum_{k=0}^{\infty} \left(\rho_2^{k+1} - \rho_1^{k+1}\right)^2 \sigma_\epsilon^2,$$

and

$$\begin{aligned} s_\tau &= E\{X_t X_{t+\tau}\} = \frac{1}{(\rho_2 - \rho_1)^2} E\{\sum_{k=0}^{\infty} \left(\rho_2^{k+1} - \rho_1^{k+1}\right) \epsilon_{t-k} \sum_{j=0}^{\infty} \left(\rho_2^{j+1} - \rho_1^{j+1}\right) \epsilon_{t+\tau-j} \\ &= \frac{1}{(\rho_2 - \rho_1)^2} \sum_{k=0}^{\infty} \left(\rho_2^{k+1} - \rho_1^{k+1}\right) \left(\rho_2^{\tau+k+1} - \rho_1^{\tau+k+1}\right) \sigma_\epsilon^2, \end{aligned}$$

as $E\{\epsilon_{t-k}\epsilon_{t+\tau-j}\}$ is only non-zero when $j = \tau + k$. For stationarity, the above variance and covariance terms must be finite, this occurs when the modulus of the multipliers in the geometric progressions are less than one, this is equivalent to

$$|\rho_1| < 1 \quad \text{and} \quad |\rho_2| < 1.$$

The condition in part *b)* is that the roots of $\Phi(z)$ lie outside the unit circle, *i.e.*

$$\Phi(z) = (1 - \phi_1 z - \phi_2 z^2) = (1 - \rho_1 z)(1 - \rho_2 z) = 0$$

$$\Rightarrow z = \frac{1}{\rho_1}, \quad z = \frac{1}{\rho_2}$$

$$|z| > 1 \Rightarrow |\rho_1| < 1 \quad \text{and} \quad |\rho_2| < 1,$$

as required. $\boxed{6}$

2. *a)* seen ⇓

$$
\begin{aligned}
Z_t &= \Delta^2 X_t = (1-B)^2 X_t = (1-2B+B^2)X_t = X_t - 2X_{t-1} + X_{t-2} \\
&= (\alpha + \beta t + Y_t) - 2(\alpha + \beta(t-1) + Y_{t-1}) + (\alpha + \beta(t-2) + Y_{t-2}) \\
&= Y_t - 2Y_{t-1} + Y_{t-2} = \Delta^2 Y_t.
\end{aligned}
$$

which does not depend on α or β, as required. 2

If we can show that taking first differences of a stationary process produces a stationary process, then we will have shown that $\{Z_t\}$ is also stationary (as $Z_t = \Delta^2 Y_t = \Delta(\Delta Y_t)$ and Y_t is stationary). $\{Y_t\}$ is second-order stationary, therefore unseen ⇓

$$
E\{Y_t\} = \mu, \quad var\{Y_t\} = \sigma_Y^2, \quad cov\{Y_t, Y_{t+\tau}\} = s_{Y,\tau}.
$$

are finite constants, not depending on t. Let $X_t = \Delta Y_t$,

$$
\begin{aligned}
E\{X_t\} &= E\{Y_t - Y_{t-1}\} = 0; \\
var\{X_t\} &= E\{(Y_t - Y_{t-1})^2\} = \sigma_Y^2 - 2s_{Y,1} + \sigma_Y^2 = 2(\sigma_Y^2 - s_{Y,1}); \\
cov\{X_t, X_{t+\tau}\} &= E\{(Y_t - Y_{t-1})(Y_{t+\tau} - Y_{t+\tau-1}\} \\
&= s_{Y,\tau} - s_{Y,\tau-1} - s_{Y,\tau+1} + s_{Y,\tau} = 2s_{Y,\tau} - s_{Y,\tau-1} - s_{Y,\tau+1}.
\end{aligned}
$$

so, the mean and variance of the first difference of a stationary process are also finite constants and the covariances are finite quantities which do not depend on t

$\Rightarrow X_t$ is second-order stationary $\Rightarrow Z_t\ (= \Delta X_t)$ is second-order stationary. 5

b) *i)* A digital filter L that transforms an input sequence $\{x_t\}$ into an output sequence $\{y_t\}$ is called a linear time-invariant (LTI) digital filter if it has the following three properties: seen ⇓

[1] Scale-preservation:

$$
L\{\{\alpha x_t\}\} = \alpha L\{\{x_t\}\}.
$$

[2] Superposition:

$$
L\{\{x_{t,1} + x_{t,2}\}\} = L\{\{x_{t,1}\}\} + L\{\{x_{t,2}\}\}.
$$

[3] Time invariance:
If

$$
L\{\{x_t\}\} = \{y_t\}, \quad \text{then} \quad L\{\{x_{t+\tau}\}\} = \{y_{t+\tau}\}.
$$

Where τ is integer-valued, and the notation $\{x_{t+\tau}\}$ refers to the sequence whose t-th element is $x_{t+\tau}$. 3

ii) Let $\boxed{\text{part seen} \Downarrow}$

$$L\{Y_t\} = Y_t - 2Y_{t-1} + Y_{t-2},$$
$$L\{e^{i2\pi ft}\} = e^{i2\pi ft}(1 - 2e^{-i2\pi f} + e^{-i4\pi f}) = e^{i2\pi ft}G(f).$$

So, the transfer function

$$|G(f)|^2 = |1 - 2e^{-i2\pi f} + e^{-14\pi f}|^2.$$

We have $\boxed{2}$

$$\begin{aligned}
S_Z(f) &= |G(f)|^2 S_Y(f) = |1 - 2e^{-i2\pi f} + e^{-14\pi f}|^2 S_Y(f) \\
&= |(1 - e^{-i2\pi f})^2|^2 S_Y(f) = |(e^{-i\pi f}(e^{i\pi f} - e^{-i\pi f}))^2|^2 S_Y(f) \\
&= |(2\sin(\pi f))^2|^2 = 16\sin^4(\pi f) S_Y(f).
\end{aligned}$$

$\boxed{4}$

iii)

$$Y_t = \epsilon_t - \theta_{1,2}\epsilon_{t-1} - \theta_{2,2}\epsilon_{t-2},$$

We have

$$S_Y(f) = |G_\theta(f)|^2 S_\epsilon(f),$$

where $G_\theta(f)$ is the transfer function of

$$\begin{aligned}
L\{\epsilon_t\} &= \epsilon_t - \theta_{1,2}\epsilon_{t-1} - \theta_{2,2}\epsilon_{t-2} \\
L\{e^{i2\pi ft}\} &= e^{i2\pi ft}(1 - \theta_{1,2}e^{-i2\pi f} - \theta_{2,2}e^{-i4\pi f}) = e^{i2\pi ft}G_\theta(f).
\end{aligned}$$

So,

$$S_Y(f) = |1 - \theta_{1,2}e^{-i2\pi f} - \theta_{2,2}e^{-i4\pi f}|^2\sigma_\epsilon^2,$$

and

$$S_Z(f) = 16\sin^4(\pi f)|1 - \theta_{1,2}e^{-i2\pi f} - \theta_{2,2}e^{-i4\pi f}|^2\sigma_\epsilon^2.$$

$\boxed{4}$

3. *a)* A stochastic process is invertible if it can be written in autoregressive form. *i.e.*, consider, $\boxed{\text{seen} \Downarrow}$

$$X_t = \sum_{k=0}^{\infty} g_k \epsilon_{t-k} = \sum_{k=0}^{\infty} g_k B^k \epsilon_t = G(B)\epsilon_t,$$

so $\epsilon_t = G^{-1}(B)X_t$, the expansion of $G^{-1}(B)$ in powers of B gives the required AR form, provided $G^{-1}(B)$ admits a power series expansion,
$G^{-1}(z) = \sum_{k=0}^{\infty} h_k z^k$. $\boxed{3}$

b) Assume $\{X_t\}$ is stationary AR(p), then

$$X_t = \sum_{j=1}^{p} \phi_{j,p} X_{t-j} + \epsilon_t \Rightarrow X_t \left(1 - \sum_{j=1}^{p} \phi_{j,p} B^j\right) = \epsilon_t.$$

Therefore $G^{-1}(z) = \sum_{j=0}^{p} \phi_{j,p} z^j$, where $\phi_{0,p} = 1$, and so $G^{-1}(z)$ admits a power series expansion, therefore $\{X_t\}$ is invertible. $\boxed{3}$

c) We can write the MA(q) process as,

$$\begin{aligned} X_t &= \epsilon_t - \theta_{1,q}\epsilon_{t-1} - \theta_{2,q}\epsilon_{t-2} - \ldots - \theta_{q,q}\epsilon_{t-q} \\ &= (1 - \theta_{1,q}B - \theta_{2,q}B^2 - \ldots - \theta_{q,q}B^q)\epsilon_t. \end{aligned}$$

The condition for stationarity is that the roots of the characteristic polynomial

$$1 - \theta_{1,q}z - \theta_{2,q}z^2 \ldots \theta_{q,q}z^q = \Theta(z)$$

lie OUTSIDE the unit circle. $\boxed{3}$

d) *i)*

$$\begin{aligned} s_\tau &= cov\{X_t, X_{t-\tau}\} = E\{(\epsilon_t - \theta_{1,1}\epsilon_{t-1})(\epsilon_{t-\tau} - \theta_{1,1}\epsilon_{t-\tau-1})\} \\ &= E\{\epsilon_t\epsilon_{t-\tau}\} - E\{\epsilon_t\theta_{1,1}\epsilon_{t-\tau-1}\} - E\{\theta_{1,1}\epsilon_{t-1}\epsilon_{t-\tau}\} + E\{\theta_{1,1}\epsilon_{t-1}\theta_{1,1}\epsilon_{t-\tau-1}\}. \end{aligned}$$

Therefore, s_τ will be non-zero when $\tau = 0$ or $\tau = \pm 1$, giving,

$$s_\tau = \begin{cases} \sigma_\epsilon^2(1 + \theta_{1,1}^2) & \tau = 0; \\ -\sigma_\epsilon^2\theta_{1,1} & \tau = \pm 1; \\ 0 & \text{otherwise.} \end{cases}$$

$\boxed{5}$

ii) $\boxed{\text{unseen} \Downarrow}$

$$s_0 = \sigma_\epsilon^2(1+\theta_{1,1}^2) = 5.0 \qquad s_1 = -\sigma_\epsilon^2\theta_{1,1} = 2.0,$$

So,

$$\frac{s_0}{s_1} = \frac{1+\theta_{1,1}^2}{-\theta_{1,1}} = \frac{5}{2},$$

$$\begin{aligned} \Rightarrow 2\theta_{1,1}^2 + 5\theta_{1,1} + 2 &= 0 \\ (2\theta_{1,1}+1)(\theta_{1,1}+2) &= 0 \\ \Rightarrow \theta_{1,1} = -1/2 \quad \text{or} \quad \theta_{1,1} = -2. \end{aligned}$$

i.e., there is more than one process with $s_0 = 5.0$ and $s_1 = 2.0$.
For invertibility the roots of $(1-\theta_{1,1}z)$ must lie outside the unit circle, *i.e.* $|\theta_{1,1}| < 1$, this is satisfied when $\theta_{1,1} = -1/2$, but not when $\theta_{1,1} = -2$, so only one of these processes is invertible. $\boxed{6}$

4. *a)* meth. seen ⇓

$$Z_t - 0.99Z_{t-3} = \epsilon_t.$$

Let $L\{Z_t\} = Z_t - 0.99Z_{t-3}$, then

$$L\{e^{i2\pi ft}\} = e^{i2\pi ft}(1 - 0.99e^{-i6\pi f}) = e^{i2\pi ft}G_Z(f).$$

So,

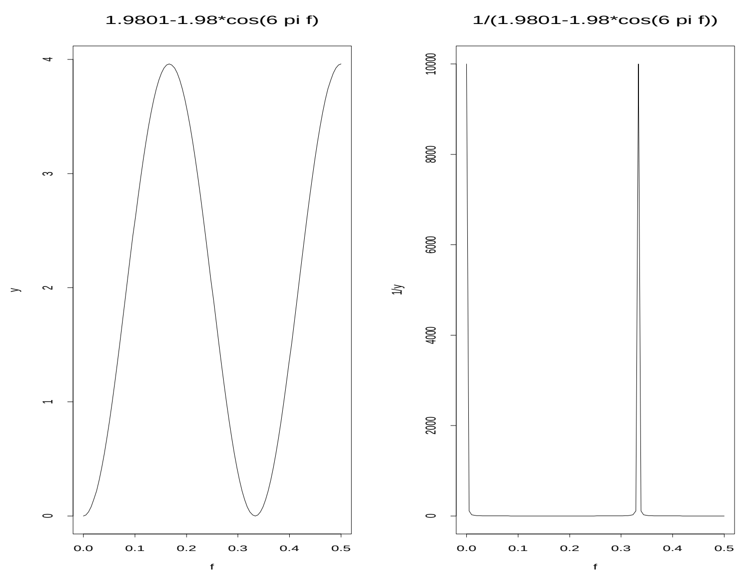
$$\begin{aligned}
|G_Z(f)|^2 &= |1 - 0.99e^{-i6\pi f}|^2 \\
&= (1 - 0.99\cos(6\pi f))^2 + 0.99^2\sin^2(6\pi f) \\
&= 1 - 1.98\cos(6\pi f) + 0.99^2 = 1.9801 - 1.98\cos(6\pi f). \\
|G_Z(f)|^2 S_Z(f) &= S_\epsilon(f) \\
\Rightarrow S_Z(f) &= \frac{\sigma_\epsilon^2}{1.9801 - 1.98\cos(6\pi f)}.
\end{aligned}$$

From the form of $S_Z(f)$, the maximum value will occur when the denominator is at its smallest, this will occur when $\cos(6\pi f)$ takes the value 1, *i.e.* when $f = 0$ or $f = 1/3$. Alternatively, 5

unseen ⇓

$$\frac{d}{df}S_Z(f) \propto \frac{\sin(6\pi f)}{(1.9801 - 1.98\cos(6\pi f))^2}$$

t.p.s occur when $\sin(6\pi f) = 0$, *i.e.* when $f = 0$ (max), $f = 1/6$ (min) and $f = 1/3$ (max). Indicating periodic behaviour with $f = 1/3$.



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b) For $\tau = 0$,

$$\begin{aligned} s_\tau &= \int_{-1/2}^{1/2} S(f)\, df = \int_{-1/2}^{1/2} 4\sigma^2 \left(\frac{1}{2} - |f|\right) df \\ &= 8\sigma^2 \int_0^{1/2} \left(\frac{1}{2} - f\right) df = 8\sigma^2 \left[\frac{f}{2} - \frac{f^2}{2}\right]_0^{1/2} \\ &= 8\sigma^2 \left(\frac{1}{4} - \frac{1}{8}\right) = \sigma^2. \end{aligned}$$

For $\tau \neq 0$, $\boxed{3}$

$$\begin{aligned} s_\tau &= \int_{-1/2}^{1/2} e^{i2\pi f\tau} S(f)\, df \\ &= \int_{-1/2}^{1/2} e^{i2\pi f\tau} 4\sigma^2 \left(\frac{1}{2} - |f|\right) df \\ &= \frac{4\sigma^2}{2} \int_{-1/2}^{1/2} e^{i2\pi f\tau} - 4\sigma^2 \int_{-1/2}^{1/2} |f| e^{i2\pi f\tau}\, df \\ &= 0 - 4\sigma^2 \left\{ \int_{-1/2}^{0} -f e^{i2\pi f\tau}\, df + \int_0^{1/2} f e^{i2\pi f\tau}\, df \right\} \\ &= -4\sigma^2 \int_0^{1/2} f(e^{-i2\pi f\tau} + e^{i2\pi f\tau})\, df \\ &= -4\sigma^2 \int_0^{1/2} 2f \cos(2\pi f\tau)\, df \\ &= -4\sigma^2 \left\{ \left[\frac{f \sin(2\pi f\tau)}{\pi\tau}\right]_0^{1/2} - \int_0^{1/2} \frac{\sin(2\pi f\tau)}{\pi\tau}\, df \right\} \\ &= \frac{2\sigma^2}{(\pi\tau)^2} \left[-\cos(2\pi f\tau)\right]_0^{1/2} = \frac{2\sigma^2}{(\pi\tau)^2} \left\{1 - \cos(\pi\tau)\right\}. \end{aligned}$$

Therefore,

$$s_\tau = \begin{cases} \sigma^2 & \tau = 0; \\ \dfrac{4\sigma^2}{(\pi\tau)^2} & |\tau| = 1, 3, 5, \ldots; \\ 0 & \text{otherwise,} \end{cases}$$

as required. $\boxed{8}$

5. *a)* *i)* We know from the spectral representation theorem that, $\boxed{\text{seen} \Downarrow}$

$$X_t = \int_{-1/2}^{1/2} e^{i2\pi f' t}\, dZ(f'),$$

so that,

$$\begin{aligned} J(f) &= \sum_{t=1}^{N} \left(\int_{-1/2}^{1/2} \frac{1}{\sqrt{N}} e^{i2\pi f' t}\, dZ(f') \right) e^{-i2\pi f t} \\ &= \int_{-1/2}^{1/2} \sum_{t=1}^{N} \frac{1}{\sqrt{N}} e^{-i2\pi (f - f')t}\, dZ(f') \end{aligned}$$

We find that,

$$\begin{aligned} &\mathrm{E}\{\widehat{S}^{(p)}(f)\} = \mathrm{E}\{|J(f)|^2\} = \mathrm{E}\{J^*(f)J(f)\} \\ &= \mathrm{E}\left\{ \int_{-1/2}^{1/2} \sum_{t=1}^{N} \frac{1}{\sqrt{N}} e^{i2\pi (f - f')t}\, dZ^*(f') \int_{-1/2}^{1/2} \sum_{s=1}^{N} \frac{1}{\sqrt{N}} e^{-i2\pi (f - f'')s}\, dZ(f'') \right\} \\ &= \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \sum_{t=1}^{N} \frac{1}{\sqrt{N}} e^{i2\pi (f - f')t} \sum_{s=1}^{N} \frac{1}{\sqrt{N}} e^{-i2\pi (f - f'')s} \mathrm{E}\{dZ^*(f')\, dZ(f'')\} \\ &= \int_{-1/2}^{1/2} \mathcal{F}(f - f') S(f')\, df', \end{aligned}$$

where $\mathcal{F}$ is Féjer's kernel defined by

$$\mathcal{F}(f) = \left| \sum_{t=1}^{N} \frac{1}{\sqrt{N}} e^{-i2\pi f t} \right|^2 .$$

$\boxed{6}$

ii) The periodogram is likely to be biased if the process has a large dynamic range, *i.e.*, when

$$10 \log_{10} \frac{\max_f S(f)}{\min_f S(f)}$$

is large.

The expected value of the periodogram is the true spectrum convolved with Féjer's kernel. Féjer's kernel has a broad central lobe and $N-2$ sidelobes, if a process has a large dynamic range then power can "leak" from parts of the spectrum where $S(f)$ is large via the sidelobes to other frequencies where $S(f)$ is small $\boxed{3}$

b) *i)* We can construct a direct spectral estimator, $\widehat{S}^{(d)}(f)$ as

$$\widehat{S}^{(d)}(f) = \left| \sum_{t=1}^{N} h_t X_t e^{-i2\pi f t} \right|^2 .$$

In this case, we will have

$$\mathrm{E}\{\widehat{S}^{(d)}(f)\} = \int_{-1/2}^{1/2} \mathcal{H}(f - f') S(f')\, df',$$

where the kernel, $\mathcal{H}(f)$ is now,

$$\mathcal{H}(f) = \left| \sum_{t=1}^{N} h_t e^{-i2\pi f t} \right|^2 .$$

Careful choice of taper can lead to a kernel with lower sidelobes and thus reduce the bias due to sidelobe leakage. 4

ii) unseen ⇓

$$\begin{aligned}
E\{\widehat{S}^{(d)}(0)\} &= \int_{-1/2}^{1/2} \mathcal{H}(-f')S(f')\,df', = \int_{-1/2}^{1/2} \mathcal{H}(f)\sigma_X^2\,df \\
&= \sigma_X^2 \int_{-1/2}^{1/2} \left| \sum_{t=1}^{N} h_t e^{-i2\pi f t} \right|^2 df \\
&= \sigma_X^2 \int_{-1/2}^{1/2} \left(\sum_{t=1}^{N} h_t e^{-i2\pi f t} \right) \left(\sum_{s=1}^{N} h_s e^{i2\pi f s} \right) df \\
&= \sigma_X^2 \sum_{t=1}^{N} \sum_{s=1}^{N} h_t h_s \int_{-1/2}^{1/2} e^{-i2\pi f(t-s)}\,df
\end{aligned}$$

Now,

$$\int_{-1/2}^{1/2} e^{-i2\pi f(t-s)} = \begin{cases} 1 & \text{if } t = s; \\ 0 & \text{otherwise.} \end{cases}$$

So,

$$E\{\widehat{S}^{(d)}(0)\} = \sigma_X^2 \sum_{t=1}^{N} h_t^2.$$

As $\{X_t\}$ is white noise, then $S_X(f) = \sigma_X^2$ for all f. To ensure that the $\widehat{S}^{(d)}(0)$ is unbiased, we need the constraint,

$$\sum_{t=1}^{N} h_t^2 = 1.$$

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