IMPERIAL COLLEGE LONDON

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)
MAY–JUNE 2005

M3S8 (Solutions)

Time Series

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1. (a) $\boxed{\text{seen} \Downarrow}$

$$\begin{aligned}\mathbb{E}(X_t^2) &= \mathbb{E}(\sigma_t^2\varepsilon_t^2)\\ &= \mathbb{E}((a_0 + a_1X_{t-1}^2)\varepsilon_t^2)\\ &= (a_0 + a_1\mathbb{E}(X_{t-1}^2))\mathbb{E}(\varepsilon_t^2)\\ &= a_0 + a_1\mathbb{E}(X_t^2),\end{aligned}$$

which yields

$$\mathbb{E}(X_t^2) = \frac{a_0}{1-a_1}.$$

$\boxed{4}$

(b) $\boxed{\text{method seen} \Downarrow}$

$$\begin{aligned}X_t^4 &= (a_0 + a_1X_{t-1}^2)^2\varepsilon_t^4\\ &= (a_0^2 + 2a_0a_1X_{t-1}^2 + a_1^2X_{t-1}^4)\varepsilon_t^4.\end{aligned}$$

Hence,

$$\mathbb{E}(X_t^4) = 3\left(a_0^2 + 2a_0a_1\frac{a_0}{1-a_1} + a_1^2\mathbb{E}(X_t^4)\right),$$

which gives

$$\mathbb{E}(X_t^4) = \frac{3a_0^2(1+a_1)}{(1-a_1)(1-3a_1^2)}.$$

$\boxed{5}$

(c) $\boxed{\text{unseen} \Downarrow}$

$$\kappa_{X_t} = \frac{\mathbb{E}(X_t^4)}{(\mathbb{E}(X_t^2))^2} = \frac{3a_0^2(1+a_1)}{(1-a_1)(1-3a_1^2)}\frac{(1-a_1)^2}{a_0^2} = \frac{3(1-a_1^2)}{1-3a_1^2}.$$

Note that $\kappa_{X_t} > 3$ as $1-a_1^2 > 1-3a_1^2$ for $a_1 > 0$. On the other hand, if $Z \sim N(0,\sigma^2)$, then

$$\kappa_Z = \frac{\mathbb{E}(Z^4)}{(\mathbb{E}(Z^2))^2} = \frac{3\sigma^4}{(\sigma^2)^2} = 3.$$

Therefore, X_t is not Normally distributed. $\boxed{6}$

(d) $\boxed{\text{method seen} \Downarrow}$

$$\rho_1 = \frac{\mathbb{E}(X_t^2X_{t-1}^2) - (\mathbb{E}(X_t^2))^2}{\mathbb{E}(X_t^4) - (\mathbb{E}(X_t^2))^2} =: \frac{A - B^2}{C - B^2}.$$

We have already computed B and C. It remains to compute A.

$$\begin{aligned}\mathbb{E}(X_t^2X_{t-1}^2) &= \mathbb{E}(\varepsilon_t^2)(a_0\mathbb{E}(X_{t-1}^2) + a_1\mathbb{E}(X_{t-1}^4))\\ &= a_0\frac{a_0}{1-a_1} + a_1\frac{3a_0^2(1+a_1)}{(1-a_1)(1-3a_1^2)}\\ &= \frac{a_0^2(1+3a_1)}{(1-a_1)(1-3a_1^2)}.\end{aligned}$$

Substituting into the above, we obtain $\rho_1 = a_1$. $\boxed{5}$

2. (a) We have

method seen ⇓

$$\begin{bmatrix} \widehat{s}_1 \\ \widehat{s}_2 \end{bmatrix} = \begin{bmatrix} \widehat{s}_0 & \widehat{s}_1 \\ \widehat{s}_1 & \widehat{s}_0 \end{bmatrix} \begin{bmatrix} \widehat{a}^{YW} \\ \widehat{b}^{YW} \end{bmatrix},$$

which gives

$$\begin{bmatrix} \widehat{a}^{YW} \\ \widehat{b}^{YW} \end{bmatrix} = \frac{1}{\widehat{s}_0^2 - \widehat{s}_1^2} \begin{bmatrix} \widehat{s}_0 & -\widehat{s}_1 \\ -\widehat{s}_1 & \widehat{s}_0 \end{bmatrix} \begin{bmatrix} \widehat{s}_1 \\ \widehat{s}_2 \end{bmatrix} = \begin{bmatrix} \frac{\widehat{s}_0\widehat{s}_1 - \widehat{s}_1\widehat{s}_2}{\widehat{s}_0^2 - \widehat{s}_1^2} \\ \frac{\widehat{s}_0\widehat{s}_2 - \widehat{s}_1^2}{\widehat{s}_0^2 - \widehat{s}_1^2} \end{bmatrix}.$$

Substituting

$$\widehat{s}_\tau = \frac{1}{n} \sum_{t=1}^{n-|\tau|} x_t x_{t+|\tau|},$$

we obtain

$$\begin{aligned}
\widehat{a}^{YW} &= \frac{\sum_{t=1}^{n-1} x_t x_{t+1} \left(\sum_{t=1}^{n} x_t^2 - \sum_{t=1}^{n-2} x_t x_{t+2} \right)}{\left(\sum_{t=1}^{n} x_t^2\right)^2 - \left(\sum_{t=1}^{n-1} x_t x_{t+1} \right)^2} \\
\widehat{b}^{YW} &= \frac{\sum_{t=1}^{n} x_t^2 \sum_{t=1}^{n-2} x_t x_{t+2} - \left(\sum_{t=1}^{n-1} x_t x_{t+1} \right)^2}{\left(\sum_{t=1}^{n} x_t^2\right)^2 - \left(\sum_{t=1}^{n-1} x_t x_{t+1} \right)^2}
\end{aligned}$$

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(b) We have

method seen ⇓

$$\begin{bmatrix} \widehat{a}^{LS} \\ \widehat{b}^{LS} \end{bmatrix} = (F^T F)^{-1} F^T \mathbf{x}_{(2)},$$

where

$$\begin{aligned}
F^T &= \begin{bmatrix} x_2 & \cdots & x_{n-1} \\ x_1 & \cdots & x_{n-2} \end{bmatrix} \\
\mathbf{x}_{(2)}^T &= (x_3, \ldots, x_n).
\end{aligned}$$

Thus,

$$\begin{aligned}
\begin{bmatrix} \widehat{a}^{LS} \\ \widehat{b}^{LS} \end{bmatrix} &= \begin{bmatrix} \sum_{t=2}^{n-1} x_t^2 & \sum_{t=1}^{n-2} x_t x_{t+1} \\ \sum_{t=1}^{n-2} x_t x_{t+1} & \sum_{t=1}^{n-2} x_t^2 \end{bmatrix}^{-1} \begin{bmatrix} \sum_{t=2}^{n-1} x_t x_{t+1} \\ \sum_{t=1}^{n-2} x_t x_{t+2} \end{bmatrix} \\
&= \frac{1}{\sum_{t=2}^{n-1} x_t^2 \sum_{t=1}^{n-2} x_t^2 - \left(\sum_{t=1}^{n-2} x_t x_{t+1} \right)^2} \\
&\times \begin{bmatrix} \sum_{t=1}^{n-2} x_t^2 & -\sum_{t=1}^{n-2} x_t x_{t+1} \\ -\sum_{t=1}^{n-2} x_t x_{t+1} & \sum_{t=2}^{n-1} x_t^2 \end{bmatrix} \begin{bmatrix} \sum_{t=2}^{n-1} x_t x_{t+1} \\ \sum_{t=1}^{n-2} x_t x_{t+2} \end{bmatrix}.
\end{aligned}$$

This leads to

$$\begin{aligned}
\widehat{a}^{LS} &= \frac{\sum_{t=1}^{n-2} x_t^2 \sum_{t=2}^{n-1} x_t x_{t+1} - \sum_{t=1}^{n-2} x_t x_{t+1} \sum_{t=1}^{n-2} x_t x_{t+2}}{\sum_{t=2}^{n-1} x_t^2 \sum_{t=1}^{n-2} x_t^2 - \left(\sum_{t=1}^{n-2} x_t x_{t+1} \right)^2} \\
\widehat{b}^{LS} &= \frac{\sum_{t=2}^{n-1} x_t^2 \sum_{t=1}^{n-2} x_t x_{t+2} - \sum_{t=1}^{n-2} x_t x_{t+1} \sum_{t=2}^{n-1} x_t x_{t+1}}{\sum_{t=2}^{n-1} x_t^2 \sum_{t=1}^{n-2} x_t^2 - \left(\sum_{t=1}^{n-2} x_t x_{t+1} \right)^2}.
\end{aligned}$$

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unseen $\Downarrow$

(c) (i) For $n = 3$, $\widehat{a}^{LS}$ does not make sense as we have $\widehat{a}^{LS} = 0/0$, so Yule-Walker should be used. $\boxed{3}$

(ii) As n tends to infinity, from the above formulae it is apparent that the corresponding Yule-Walker and least squares estimates become "closer and closer" to each other. Therefore, it "does not matter" which one is used in the case $n = 10^{10}$ as the estimated values will probably be very similar for both methods. $\boxed{3}$

3. (a) For $\tau > 0$, unseen $\Downarrow$

$$\mathbb{E}(X_t X_{t-\tau}) = \mathbb{E}(a_t)\mathbb{E}(X_{t-1}X_{t-\tau}) + \sqrt{8/9}\mathbb{E}(\varepsilon_t)\mathbb{E}(X_{t-\tau}),$$

which, from the properties of the Uniform distribution, gives

$$s_\tau = \frac{1}{2}s_{\tau-1}.$$

Also,

$$s_0 = \mathbb{E}(X_t^2) = \mathbb{E}(a_t^2 X_{t-1}^2 + 2\sqrt{8/9}a_t X_{t-1}\varepsilon_t + 8/9\,\varepsilon_t^2) = 1/3\,\mathbb{E}(X_t^2) + 8/9,$$

which yields $s_0 = 4/3$. Thus,

$$s_\tau = \frac{4}{3}\left(\frac{1}{2}\right)^{|\tau|}.$$

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(b) To compute the autocovariance function $\widetilde{s}_\tau$ of Y_t, we use exactly the same technique as in (a) above and find that seen $\Downarrow$

$$\widetilde{s}_\tau = \frac{4}{3}\left(\frac{1}{2}\right)^{|\tau|} = s_\tau.$$

Thus, the autocovariance functions are the same. 4

(c) The autocovariance function of X_t is the same as that of Y_t (see above). Therefore, the spectral densities will also be the same (as the spectral density is the Fourier transform of the autocovariance sequence). Denoting the spectral density of a process Z_t by $S_Z(f)$, we have seen $\Downarrow$

$$|1 - 1/2\exp(-i2\pi f)|^2 S_X(f) = S_\varepsilon(f),$$

which simplifies to

$$(5/4 - \cos(2\pi f))S_X(f) = 1,$$

and therefore

$$S_X(f) = (5/4 - \cos(2\pi f))^{-1}.$$

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(d) The distributions are not identical. To see this, it is enough to show, for example, that $\mathbb{E}(X_t^4) \neq \mathbb{E}(Y_t^4)$. We compute unseen $\Downarrow$

$$\begin{aligned}\mathbb{E}(X_t^4) &= \mathbb{E}\left(a_t^4 X_{t-1}^4 + 4a_t^3 X_{t-1}^3\sqrt{8/9}\varepsilon_t + 6a_t^2 X_{t-1}^2 8/9\varepsilon_t^2 + 4a_t X_{t-1}(8/9)^{3/2}\varepsilon_t^3 + (8/9)^2\varepsilon_t^4\right)\\ &= 1/5\mathbb{E}(X_t^4) + 6\cdot 1/3\cdot 4/3\cdot 8/9 + 3(8/9)^2,\end{aligned}$$

which gives $\mathbb{E}(X_t^4) = 160/27$. On the other hand,

$$\begin{aligned}\mathbb{E}(Y_t^4) &= \mathbb{E}\left(1/16Y_{t-1}^4 + 4/8Y_{t-1}^3\varepsilon_t + 6/4Y_{t-1}^2\varepsilon_t^2 + 4/2Y_{t-1}\varepsilon_t^3 + \varepsilon_t^4\right)\\ &= 1/16\mathbb{E}(Y_t^4) + 6/4\cdot 4/3 + 3,\end{aligned}$$

which gives $\mathbb{E}(Y_t^4) = 16/3 \neq 160/27$. 6

4. (a) sim. seen ⇓

$$\begin{aligned}
\mathsf{MSE}(\widehat{\mu},\mu) &= n^{-2}\sum_{s,t=1}^{n} s_{t-s} \\
&= n^{-2}\left(ns_0 + 2(n-1)s_1 + \ldots + 2s_{n-1}\right) \\
&\le 2n^{-1}\left(s_0 + \ldots + s_{n-1}\right) \\
&= 2s_0 n^{-1} + 2n^{-1}\sum_{\tau=1}^{n-1}\frac{1}{\tau}\left(1 + \ldots + \frac{1}{\tau}\right) \\
&= 2s_0 n^{-1} + 2n^{-1}\sum_{\tau=1}^{n-1}\frac{1}{\tau}O(\log(\tau)) \\
&\le 2s_0 n^{-1} + 2O(\log(n))n^{-1}\sum_{\tau=1}^{n-1}\frac{1}{\tau} \\
&= 2s_0 n^{-1} + 2O(\log^2(n))n^{-1} \\
&\to 0 \quad \mathsf{as} \quad n \to \infty.
\end{aligned}$$

5

(b) unseen ⇓

$$\begin{aligned}
0 &= n\left(\frac{1}{n}\sum_{s=1}^{n}(x_s - \widehat{\mu})\right)^2 \\
&= \frac{1}{n}\sum_{s=1}^{n}\sum_{t=1}^{n}(x_s - \widehat{\mu})(x_t - \widehat{\mu}) \\
&= \frac{1}{n}\sum_{|\tau|<n}\sum_{t=1}^{n-|\tau|}(x_t - \widehat{\mu})(x_{t+|\tau|} - \widehat{\mu}).
\end{aligned}$$

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(c) Recall the spectral representation theorem seen ⇓

$$X_t = \int_{-1/2}^{1/2} e^{i2\pi f' t}\, dZ(f').$$

We have $\mathbb{E}(\widehat{S}(f)) = \mathbb{E}|J(f)|^2$, where

$$\begin{aligned}
J(f) &= \frac{1}{\sqrt{n}}\sum_{t=1}^{n} X_t e^{-i2\pi f t} \\
&= \sum_{t=1}^{n}\left(\int_{-1/2}^{1/2}\frac{1}{\sqrt{n}}e^{i2\pi f' t}\, dZ(f')\right)e^{-i2\pi f t} \\
&= \int_{-1/2}^{1/2}\sum_{t=1}^{n}\frac{1}{\sqrt{n}}e^{-i2\pi(f-f')t}\, dZ(f')
\end{aligned}$$

We find that,

$$
\begin{aligned}
\mathbb{E}\{\widehat{S}(f)\} &= \mathbb{E}\{|J(f)|^2\} = \mathbb{E}\{J^*(f)J(f)\} \\
&= \mathbb{E}\left\{\int_{-1/2}^{1/2}\sum_{t=1}^{n}\frac{1}{\sqrt{n}}e^{i2\pi(f-f')t}\,dZ^*(f')\int_{-1/2}^{1/2}\sum_{s=1}^{n}\frac{1}{\sqrt{n}}e^{-i2\pi(f-f'')s}\,dZ(f'')\right\} \\
&= \int_{-1/2}^{1/2}\int_{-1/2}^{1/2}\sum_{t=1}^{n}\frac{1}{\sqrt{n}}e^{i2\pi(f-f')t}\sum_{s=1}^{n}\frac{1}{\sqrt{n}}e^{-i2\pi(f-f'')s}\mathbb{E}\{dZ^*(f')\,dZ(f'')\} \\
&= \int_{-1/2}^{1/2}\mathcal{F}(f-f')S(f')\,df',
\end{aligned}
$$

where

$$\mathcal{F}(f) = \left|\sum_{t=1}^{n}\frac{1}{\sqrt{n}}e^{-i2\pi ft}\right|^2.$$

$\boxed{5}$

(d) Define the process $\boxed{\text{sim. seen} \Downarrow}$

$$X_t = \sum_{k=1}^{\infty}\frac{1}{k}\varepsilon_{t-k+1}.$$

We have

$$s_0 = \mathsf{Var}(X_t) = \sum_{k=1}^{\infty}k^{-2} = \pi^2/6,$$

and for $\tau \geq 1$,

$$
\begin{aligned}
s_\tau &= \mathbb{E}\left(\sum_{k=1}^{\infty}\frac{1}{k}\varepsilon_{t-k+1}\sum_{l=1}^{\infty}\frac{1}{l}\varepsilon_{t+\tau-l+1}\right) \\
&= \sum_{k=1}^{\infty}\frac{1}{k(k+\tau)} \\
&= \sum_{k=1}^{\infty}\frac{1}{\tau}\left(\frac{1}{k}-\frac{1}{k+\tau}\right) \\
&= \frac{1}{\tau}\left(1+\frac{1}{2}+\ldots+\frac{1}{\tau}\right).
\end{aligned}
$$

So s_τ is a valid autocovariance function. $\boxed{5}$

5. (a) We want to minimize, seen $\Downarrow$

$$\begin{aligned}
\mathsf{E}\{(X_{t+l} - X_t(l))^2\} &= \mathsf{E}\left\{\left(\sum_{k=0}^{\infty}\psi_k\varepsilon_{t+l-k} - \sum_{k=0}^{\infty}\delta_k\varepsilon_{t-k}\right)^2\right\} \\
&= \mathsf{E}\left\{\left(\sum_{k=0}^{l-1}\psi_k\varepsilon_{t+l-k} + \sum_{k=0}^{\infty}[\psi_{k+l} - \delta_k]\varepsilon_{t-k}\right)^2\right\} \\
&= \sigma_\varepsilon^2\left\{\left(\sum_{k=0}^{l-1}\psi_k^2\right) + \sum_{k=0}^{\infty}(\psi_{k+l} - \delta_k)^2\right\}.
\end{aligned}$$

The first term is independent of the choice of the $\{\delta_k\}$ and the second term is clearly minimized by choosing $\delta_k = \psi_{k+l}, k = 0, 1, 2, \ldots$. 6

(b) We have $X_t = \Psi(B)\varepsilon_t \Rightarrow \varepsilon_t = \Psi^{-1}(B)X_t$, and so method seen $\Downarrow$

$$\begin{aligned}
X_t(l) &= \sum_{k=0}^{\infty}\psi_{k+l}\varepsilon_{t-k} = \Psi^{(l)}(B)\varepsilon_t \quad [= \delta(B)\varepsilon_t] \\
&= \Psi^{(l)}(B)\Psi^{-1}(B)X_t = G^{(l)}(B)X_t
\end{aligned}$$

Now

$$X_t - aX_{t-1} = \varepsilon_t \;\Rightarrow\; (1 - aB)\,X_t = \varepsilon_t$$

So that,

$$\Rightarrow \Psi(B) = (1 - aB)^{-1} = 1 + aB + a^2B^2 + \ldots$$

So $\delta_k = \psi_{k+l} = a^{k+l}$ and when $l = 1$, $\delta_k = a^{k+1}$ giving

$$\Psi^{(1)}(B) = \left(a + a^2B + a^3B^2 + \ldots\right)$$

We have,

$$G^{(1)}(B) = \Psi^{(1)}(B)\Psi^{-1}(B) = \left(a + a^2B + a^3B^2 + \ldots\right)(1 - aB) = a.$$

Giving

$$X_t(1) = G^{(1)}(B)X_t = aX_t$$

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(c) Recalling that $s_\tau = a^{|\tau|}/(1 - a^2)$, it is easily seen that unseen $\Downarrow$

$$\Gamma_{(n)}\mathbf{a} = \gamma_{(n)}.$$

The result follows upon applying $\Gamma_{(n)}^{-1}$ to both sides of the above equation. 8