3. Consider the following integral:

$$\theta = \int_0^1 \frac{e^u - 1}{e - 1}\,\mathrm{d}u \;=\; \int_0^1 \phi(u) f(u)\,\mathrm{d}u.$$

a) Write down an expression for the Monte Carlo estimator, $\widehat{\theta}_1$, of θ, using uniform random variates, $U_1, \ldots, U_n \sim U(0,1)$. That is, $f(u)$ is the uniform density on $[0,1]$. Determine an expression for the variance of the estimator $\widehat{\theta}_1$.

b) Give an expression for an estimator, $\widehat{\theta}_2$, of θ, composed using antithetic variates U_i and $1 - U_i$, $i = 1, \ldots, n$.
State why antithetic variates are suitable for estimating θ and explain why $\widehat{\theta}_2$ will have lower variance than $\widehat{\theta}_1$.

c) Determine the variance of the estimator, $\widehat{\theta}_3$, of θ, composed using U as the control variate, such that,

$$\widehat{\theta}_3 = \frac{1}{n}\sum_{i=1}^{n}\left\{\frac{e^{U_i} - 1}{e - 1} - \beta[U_i - \mathrm{E}(U_i)]\right\}.$$

What value of β minimizes this variance?