M3S4/M4S4: Applied probability: 2007-8
Assessed Coursework 1
Distributed: Wednesday, Feb 6   Due: Wednesday, Feb 20

1. (a) If exactly one event of a Poisson process took place in an interval $[0, t]$ derive the distribution of the time at which that event took place.

   (b) If X and Y are independent Poisson random variables with means μ_X and μ_Y respectively, find the distribution of $Z = X + Y$.
   What is the conditional distribution of X, given that $X + Y = z$?

2. Given
$$Z = X_1 + \ldots + X_N.$$
Find the mean and variance of Z if $X_i \sim Poisson(\mu)$ and $N \sim G_1(p)$ (all independent).

3. In a Poisson process with rate λ, define $\mathrm{P}_n(t) = \mathrm{P}\{N(t) = n\}$, where $N(t)$ is the number of events which have occurred by time t, and suppose that $N(0) = 0$.

   (a) Using the axioms of the Poisson process and by expressing
   $\mathrm{P}_0(t + \delta d) = \mathrm{P}\{N(t + \delta t) = 0\}$ in terms of the number of events up to time t and the number between times t and $t + \delta t$ show that
$$\mathrm{P}_0'(t) = -\lambda \mathrm{P}_0(t).$$
   Hence show that $\mathrm{P}_0(t) = Ke^{-\lambda t}$, and find the value of K.

   (b) Show that, for $n \geq 1$
$$\mathrm{P}_n'(t) = -\lambda \mathrm{P}_n(t) + \lambda P_{n-1}(t).$$
   Hence deduce that $P_1(t) = \lambda t e^{-\lambda t}$.

   (c) Use induction to show that
$$P_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}.$$

4. A branching process is called *binary fission* if the offspring probability distribution has non-zero probabilities only for 0 or 2 offspring. Such a process starts at generation 0, with a single individual, and the probability of each individual producing 0 offspring is p.

   (a) Find the mean and variance of the size of the population at generation n.

   (b) Carefully distinguishing between the cases where ultimate extinction is certain or not, find the probability of ultimate extinction in terms of p.