M3S4/M4S4: Applied probability: 2007-8
Assessed Coursework 1: SOLUTIONS

1. (a) If exactly one event of a Poisson process took place in an interval $[0, t]$ find and name the distribution of the time at which the event occurred.

Let $N(t_1, t_2) =$ number of events in $[t_1, t_2]$
$X =$ time first event occurred

We have $N(t_1, t_2) \sim Poisson(\lambda(t_2 - t_1))$.

$$\begin{aligned}
P(X < s \,|\, N(0,t) = 1) &= \frac{P(X < s \cap N(0,t) = 1)}{P(N(0,t) = 1)} \\
&= \frac{P(N(0,s) = 1 \cap N(s,t) = 0)}{P(N(0,t) = 1)} \\
&= \frac{P(N(0,s) = 1)P(N(0,t-s) = 0)}{P(N(0,t) = 1)} \\
&= \frac{\lambda s e^{-\lambda s} e^{-\lambda(t-s)}}{\lambda t e^{-\lambda t}} \\
&= \frac{s}{t}
\end{aligned}$$

i.e. X is $Uniform[0, t]$ $\boxed{2}$

(b) If X and Y are independent Poisson random variables with means μ_X and μ_Y respectively, find the distribution of $Z = X + Y$.
What is the conditional distribution of X, given that $X + Y = z$?

We have

$$\begin{aligned}
\Pi_Z(s) &= \Pi_X(s)\Pi_Y(s) = e^{-\mu_x(1-s)}e^{-\mu_Y(1-s)} \\
&= e^{-(\mu_X+\mu_Y)(1-s)}
\end{aligned}$$

So $Z \sim Poisson(\mu_X + \mu_Y)$. $\boxed{1}$

$$\begin{aligned}
P(X = x \,|\, X + Y = z) &= \frac{P(X = x \cap X + Y = z)}{P(X + Y = z)} \\
&= \frac{P(X = x \cap Y = z - x)}{P(Z = z)} \\
&= \frac{\frac{e^{-\mu_X}(\mu_X)^x}{x!}\frac{e^{-\mu_Y}(\mu_Y)^{z-x}}{(z-x)!}}{\frac{e^{-(\mu_X+\mu_Y)}(\mu_X+\mu_Y)^z}{z!}} \\
&= \frac{z!}{x!(z-x)!}\left(\frac{\mu_X}{\mu_X+\mu_Y}\right)^x\left(1 - \frac{\mu_X}{\mu_X+\mu_Y}\right)^{z-x}
\end{aligned}$$

which is $Binomial(z, \mu_X/(\mu_X + \mu_Y))$. $\boxed{2}$

2. Given

$$Z = X_1 + \ldots + X_N.$$

Find the mean and variance of Z if $X_i \sim Poisson(\mu)$ (independent) and $N \sim G_1(p)$. We have $\Pi_X(s) = e^{-\mu(1-s)}$ and $\Pi_N(s) = ps/(1-qs)$.

$$\begin{aligned}
\Pi_Z(s) &= \Pi_N(\Pi_X(s)) \\
&= \frac{pe^{-\mu(1-s)}}{1-qe^{-\mu(1-s)}} \\
&= \frac{p}{e^{\mu(1-s)}-q} = p(e^{\mu(1-s)}-q)^{-1} \\
\Pi_Z'(s) &= p\mu e^{\mu(1-s)}(e^{\mu(1-s)}-q)^{-2} \\
\Pi_Z''(s) &= 2p\mu^2 e^{2\mu(1-s)}(e^{\mu(1-s)}-q)^{-3} - p\mu^2 e^{\mu(1-s)}(e^{\mu(1-s)}-q)^{-2}
\end{aligned}$$

$$\begin{aligned}
\mathrm{E}(Z) &= \Pi_Z'(1) = p\mu(1-q)^{-2} = \frac{\mu}{p} \\
\mathrm{var}(Z) &= \Pi_Z''(1) + \Pi_Z'(1) - (\Pi_Z'(1))^2 \\
&= \frac{2\mu^2 p}{p^3} - \frac{\mu^2 p}{p^2} + \frac{\mu}{p} - \frac{\mu^2}{p^2} \\
&= \frac{2\mu^2 - \mu^2 p + \mu p - \mu^2}{p^2} \\
&= \frac{\mu^2 - \mu^2 p + \mu p}{p^2}
\end{aligned}$$

$\boxed{4}$

3. In a Poisson process with rate λ, define $\mathrm{P}_n(t) = \mathrm{P}\{N(t) = n\}$, where $N(t)$ is the number of events which have occurred by time t, and suppose that $N(0) = 0$.

   (a) Using the axioms of the Poisson process and by expressing $\mathrm{P}_0(t+h) = \mathrm{P}\{N(t+h) = 0\}$ in terms of the number of events up to time t and the number between times t and $t+h$ show that

   $$\mathrm{P}_0'(t) = -\lambda \mathrm{P}_0(t).$$

   Hence show that $\mathrm{P}_0(t) = Ke^{-\lambda t}$, and find the value of K.

$$\begin{aligned}
\mathrm{P}_0(t+h) &= \mathrm{P}\{N(t+h) = 0\} \\
&= \mathrm{P}\{N(t) = 0 \cap N(t+h) - N(t) = 0\} \\
&= \mathrm{P}\{N(t) = 0)\mathrm{P}\{N(t+h) - N(t) = 0\}
\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{P}_0(t)(1-\lambda h + o(h)) \\
\Rightarrow \frac{\mathrm{P}_0(t+h)-\mathrm{P}_0(t)}{h} &= -\lambda \mathrm{P}_0(t) + \mathrm{P}_0(t)\frac{o(h)}{h}
\end{aligned}
$$

So,

$$
\begin{aligned}
\lim_{h\to 0}\frac{\mathrm{P}_0(t+h)-\mathrm{P}_0(t)}{h} &= -\lambda \mathrm{P}_0(t) \\
\mathrm{P}_0'(t) &= -\lambda \mathrm{P}_0(t)
\end{aligned}
$$

as required.

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{P}_0(t) &= -\lambda \mathrm{P}_0(t) \\
\int \frac{1}{\mathrm{P}_0(t)}\,\mathrm{dP}_0(t) &= \int -\lambda\,\mathrm{d}t \\
\log(\mathrm{P}_0(t)) &= -\lambda t + K_1 \\
\mathrm{P}_0(t) &= Ke^{-\lambda t}
\end{aligned}
$$

We know that $N(0) = 0$, so $\mathrm{P}_0(0) = 1$, so $K = 1$. $\boxed{2}$

(b) Show that, for $n \geq 1$

$$\mathrm{P}_n'(t) = -\lambda \mathrm{P}_n(t) + \lambda \mathrm{P}_{n-1}(t).$$

Hence deduce that $\mathrm{P}_1(t) = \lambda t e^{-\lambda t}$.

$$
\begin{aligned}
\mathrm{P}_n(t+h) &= \mathrm{P}\{N(t+h)=n\} \\
&= \mathrm{P}\{N(t)=n \cap N(t+h)-N(t)=0\} + \\
&\quad \mathrm{P}\{N(t)=n-1 \cap N(t+h)-N(t)=1\} + \\
&\quad \ldots + \mathrm{P}\{N(t)=0 \cap N(t+h)-N(t)=n\} \\
&= \mathrm{P}_n(t)(1-\lambda h+o(h)) + \mathrm{P}_{n-1}(t)(\lambda h + o(h)) + 0 \\
\Rightarrow \lim_{h\to 0}\frac{\mathrm{P}_n(t+h)-\mathrm{P}_n(t)}{h} &= -\lambda \mathrm{P}_n(t) + \lambda \mathrm{P}_{n-1}(t)
\end{aligned}
$$

So,

$$\mathrm{P}_n'(t) = -\lambda \mathrm{P}_n(t) + \lambda \mathrm{P}_{n-1}(t)$$

as required. $\boxed{2}$

$$
\begin{aligned}
\mathrm{P}_1'(t) &= -\lambda \mathrm{P}_1(t) + \lambda \mathrm{P}_0(t) \\
&= -\lambda \mathrm{P}_1(t) + \lambda e^{-\lambda t} \\
\mathrm{P}_1'(t) + \lambda \mathrm{P}_1(t) &= \lambda e^{-\lambda t}
\end{aligned}
$$

$$
\begin{aligned}
e^{\lambda t}\mathrm{P}_1'(t) + \lambda e^{\lambda t}\mathrm{P}_1(t) &= \lambda \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\lambda t}\mathrm{P}_1(t)\right) &= \lambda \\
e^{\lambda t}\mathrm{P}_1(t) &= \lambda t + c
\end{aligned}
$$

Now $P_1(0) = 0 \Rightarrow c = 0$, so

$$\mathrm{P}_1(t) = \lambda t e^{-\lambda t}$$

as required. $\boxed{1}$

(c) Use induction to show that

$$\mathrm{P}_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}.$$

Assume true for $n = k - 1$, let $n = k$,

$$
\begin{aligned}
\mathrm{P}_k'(t) &= -\lambda \mathrm{P}_k(t) + \lambda \mathrm{P}_{k-1}(t) \\
&= -\lambda \mathrm{P}_k(t) + \lambda \frac{e^{-\lambda t}(\lambda t)^{k-1}}{(k-1)!} \\
e^{\lambda t}\mathrm{P}_k'(t) + \lambda e^{\lambda t}\mathrm{P}_k(t) &= \lambda \frac{(\lambda t)^{k-1}}{(k-1)!} \\
\frac{\mathrm{d}}{\mathrm{d}t} e^{\lambda t}\mathrm{P}_k(t) &= \lambda \frac{(\lambda t)^{k-1}}{(k-1)!} \\
\mathrm{P}_k(t) &= e^{-\lambda t}\lambda \frac{\lambda^{k-1}}{(k-1)!}\frac{t^k}{k} + c = \frac{e^{-\lambda t}(\lambda t)^k}{k!}
\end{aligned}
$$

$c = 0$ as $\mathrm{P}_k(0) = 0$.

True for $n = 0, 1$ and result follows by induction. $\boxed{2}$

4. A branching process is called *binary fission* if the offspring probability distribution has non-zero probabilities only for 0 or 2 offspring. Given that such a process starts at generation 0, with a single individual, and that the probability of each individual producing 0 offspring is p, calculate:

(a) The mean and variance of the size of the population at generation n.

Let $q = 1 - p$,

$$
\begin{aligned}
\mu &= \mathrm{E}(X) = 0 \times p + 2 \times q = 2q \\
\sigma^2 &= \mathrm{E}(X^2) - \mu^2 = 4q - 4q^2 = 4pq
\end{aligned}
$$

then we have $\mu_n = (2q)^n$

Also, using

$$\sigma_n^2 = \begin{cases} \mu^{n-1}\sigma^2 \dfrac{1-\mu^n}{1-\mu} & \mu \neq 1 \\ n\sigma^2 & \mu = 1 \end{cases}$$

giving,

$$\sigma_n^2 = \begin{cases} 2^{n+1} q^n p \dfrac{1-(2q)^n}{1-2q} & q \neq \frac{1}{2} \\ 4pqn & q = \frac{1}{2} \end{cases}$$

$\boxed{2}$

(b) $\mu = 2q$. Ultimate extinction is certain if $\mu \leq 1$ i.e. if $2q \leq 1 \Rightarrow p \geq \frac{1}{2}$.

If $\mu > 1$ i.e. $p < \frac{1}{2}$ the probability of ultimate extinction is given by the smallest positive solution of $\theta = \Pi(\theta)$.

$$\begin{aligned} \Pi(\theta) &= p + q\theta^2 \\ \theta &= p + q\theta^2 \\ q\theta^2 - \theta + p &= 0 \\ (\theta - 1)(q\theta - p) &= 0 \end{aligned}$$

roots of $q\theta^2 - \theta + p$ are $\theta = 1$ and $\theta = p/q$, If $p < 1/2$ then $p/q < 1$ so

$$\mathrm{P}(\text{ultimate extinction}) = \frac{p}{q}$$

$\boxed{2}$