M3S4/M4S4: Applied probability: 2007-8
Problems 3: Pgfs and branching processes

1. If
$$Y = aX + b \qquad a, b \in \{0, 1, 2, 3, \ldots\}$$
show that
$$\Pi_Y(s) = s^b \Pi_X(s^a).$$

2. Prove that if $X \sim Poisson(\mu)$ then,
$$\Pi_X(s) = \exp(-\mu(1-s)).$$

3. A gambler keeps placing bets until he wins once and then he stops. What is the pgf of the total number of bets he places if each bet has probability p of winning?

4. Use the probability generating function to find the mean and variance of a Poisson distribution with mean μ .

5. Use pgfs to find the distribution of the sum of n independent Poisson distributions with parameters $\mu_i, \quad i = 1, \ldots, n$.

6. A Poisson process runs for a time t. Each event has a probability p of being observed and a probability $q = 1 - p$ of being missed. What is the distribution of the number of events which are observed in time t?

7. If $X_i \sim Binomial(n, p)$ and $N \sim Poisson(\mu)$ use probability generating function arguments to derive the mean of $Z = \sum_{i=1}^{N} X_i$.

8. In a branching process, if the number of offspring of an individual has a geometric distribution $G_0(p)$, find the mean and variance of the number of individuals in the nth generation. Calculate their values when $n = 5$ and

   (a) $p = 1/3$ (b) $p = 1/2$ (c) $p = 2/3$.

9. If, in a branching process, the number of offspring of an individual is $Poisson(0.5)$, find the probability that extinction has occurred by the 1st, 2nd, 3rd, 4th, and 5th generations.

10. If, in a branching process, the number of offspring of an individual has a $G_0(0.6)$ distribution,

    (a) Calculate the probability that the process becomes extinct by the 6th generation.

    (b) Calculate the probability that the process becomes extinct at the 6th generation.

11. Suppose each individual in a branching process can have only 0, 1, or 2 offspring, with respective probabilities r, q, and p. Show whether extinction is certain if

    (a) $p > r$ (b) $p = r$ (c) $p < r$.

    In any case in which extinction is not certain, give the probability that it will occur.

12. Suppose the number of offspring of each individual in a branching process has a $G_0(p)$ distribution. Calculate the probability of ultimate extinction.