M3S4/M4S4: Applied probability: 2007-8
Problems 5: Markov Chains

1. (a) Two players, A and B, play a series of games, A starts with £j and B starts with £$(a-j)$. A has a probability p of winning and a probability $q = 1-p$ of losing each game. If A ever loses his last £1, B returns it to him and they play on. However, if B loses his last £1, the game ends.

   Write down the transition matrix for the size of A's fortune after n games.

   (b) In each of a sequence of independent trials, a ball is placed at random in one of six containers. What is the transition matrix for the number of containers that have at least one ball in them after n trials?

2. For a Markov chain with transition matrix P and where $\boldsymbol{a}^{(n)}$ is the distribution of states at time n, show that $\boldsymbol{a}^{(n+1)} = \boldsymbol{a}^{(n)}P$.

3. Use the conditions $\boldsymbol{\pi} = \boldsymbol{\pi}P$, $\pi_j \geq 0$ and $\sum \pi_j = 1$ to find the limiting distribution of

$$P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix},$$

   where $0 \leq \alpha, \beta \leq 1$ and where α and β are not both equal to zero and are not both equal to 1.

4. A study of occupational transitions from generation to generation suggests that the transition probabilities between occupational status levels 1, 2 and 3 are given by

$$P = \begin{pmatrix} 0.5 & 0.4 & 0.1 \\ 0.2 & 0.6 & 0.2 \\ 0.1 & 0.2 & 0.7 \end{pmatrix}.$$

   What proportion of the population would one expect to be in level 1 after many generations have elapsed?

5. List the communicating classes for the gambler's ruin Markov chain and say whether they are closed or not.

6. For Markov chains represented by the following transition matrices:

   (i) Decide whether or not they have unique stationary distributions,

   (ii) Find a stationary distribution for each of them by finding a solution of the conditions $\boldsymbol{\pi} = \boldsymbol{\pi} P$, $\pi_j \geq 0$ and $\sum \pi_j = 1$ and show that it is not unique where appropriate.

$$\text{(a)} \begin{pmatrix} 0.5 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{(b)} \begin{pmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0.5 & 0.5 \end{pmatrix} \quad \text{(c)} \begin{pmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.3 & 0.3 & 0.4 & 0 \\ 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0.5 \end{pmatrix}$$

7. For Markov chains represented by the following transition matrices:

   (i) Classify the communicating classes and determine the periodicity of each class.

   (ii) Decide whether or not it has a unique stationary distribution. For each irreducible chain, decide whether or not it has a limiting distribution.

$$\text{(a)} \begin{pmatrix} 0 & 0.5 & 0.5 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{(b)} \begin{pmatrix} 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & 0 \end{pmatrix}$$

$$\text{(c)} \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{(d)} \begin{pmatrix} 0.8 & 0 & 0 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.1 & 0 & 0.9 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0.9 & 0 & 0 & 0.1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0.3 & 0.5 & 0 & 0 & 0 & 0.2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

8. The transition matrix of a Markov chain is

$$P = \begin{pmatrix} q & p & 0 & 0 & 0 & \dots \\ q & 0 & p & 0 & 0 & \dots \\ q & 0 & 0 & p & 0 & \dots \\ q & 0 & 0 & 0 & p & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

(a) Find the first return probabilities for state 0 (i.e. find $f_{00}^{(n)}$ for $n \geq 1$).

(b) Show that state 0 is recurrent.

9. During the lectures we explored the Markov chain with transition matrix

$$P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}.$$

This Markov chain is finite and irreducible and hence is recurrent. Using probability generating functions, find its mean return time to state 0.