M3S4/M4S4: Applied probability: 2007-8
Solutions 1: Introduction

1.

$$\begin{aligned}
\mathrm{E}(X) = \int_0^\infty [1 - F(x)]\, dx &= \int_0^\infty \int_x^\infty f(y)\, dy\, dx = \int_0^\infty \int_0^y f(y)\, dx\, dy \\
&= \int_0^\infty f(y) \int_0^y dx\, dy = \int_0^\infty f(y) y\, dy = E(X).
\end{aligned}$$

2. $T > t$ if and only if $T_i > t$ for $i = 1, \ldots, n$. Thus the event $[T > t]$ is the same as the joint event $[T_1 > t] \cap [T_2 > t] \cap \ldots \cap [T_n > t]$. Since the T_i are independent, we have that

$$\begin{aligned}
\mathrm{P}[T > t] &= \prod_{i=1}^n \mathrm{P}[T_i > t] = \prod_{i=1}^n e^{-\lambda_i t} \\
\Rightarrow F(t) &= 1 - \mathrm{P}[T > t] = 1 - \exp\left(-t \sum_{i=1}^n \lambda_i\right) \\
\Rightarrow f(t) &= \left(\sum_{i=1}^n \lambda_i\right) \exp\left(-t \sum_{i=1}^n \lambda_i\right).
\end{aligned}$$

That is, T has an exponential distribution with parameter $\sum_{i=1}^n \lambda_i$.

3. T_k has a geometric $G_1(p)$ distribution, where p is the probability of success in each trial.

4. After $(n-1)$ births there are n individuals alive, each of which is independently giving birth at according to a Poisson process at rate β . Thus, for each individual, the time to the next birth is exponential with parameter β . Thus, overall, the time to the next birth is the time to the minimum of n exponential variates with parameter β . We have already seen (Q2 above) that this is an exponential distribution with parameter $n\beta$. The mean of such a distribution is $1/n\beta$.

5. If there are x individuals at time t in a simple birth process, then the probability of one birth in interval $[t, t + \delta t]$ is $\beta x \delta t + o(\delta t)$, and the probability of no births is $1 - \beta x \delta t + o(\delta t)$, so that the expected number of births in this interval is $\beta x \delta t + o(\delta t)$. So,

$$\begin{aligned}
x(t + \delta t) &= x(t) + \beta x \delta t + o(\delta t) \\
\Rightarrow \frac{x(t + \delta t) - x(t)}{\delta t} &= \beta x + \frac{o(\delta t)}{\delta t}.
\end{aligned}$$

Letting $\delta t \to 0$ gives

$$\frac{dx}{dt} = \beta x$$

Solving this gives

$$\ln(x) = \beta t + c,$$

using x(0)=1 leads to

$$x = \exp(\beta t).$$

6. Letting $D(t)$ be the number of drops by time t, we have

$$D(t+\delta t) = D(t) + \frac{5}{1+10t}\delta t + o(\delta t)$$

yielding

$$\frac{dD}{dt} = \frac{5}{1+10t},$$

so that

(a)

$$D(t) = \int_0^1 \frac{5}{1+10t}\, dt = [0.5\ln(1+10t)]_0^1 = 0.5\ln 11.$$

(b)

$$D(t) = \int_4^5 \frac{5}{1+10t}\, dt = [0.5\ln(1+10t)]_4^5 = 0.5(\ln 51 - \ln 41).$$

7. (a)

$$x_n = sx_{n-1} + b.$$

(b)

$$x_n = s(sx_{n-2} + b) + b = b + sb + s^2 b + \ldots s^{n-1}b + s^n.$$