M3S4/M4S4: Applied probability: 2007-8
Solutions 2: Point Processes

1. (a) T_i is the sum of three independent Poisson variables, each with rate λ. Hence the distribution of T_i is $Gamma(3, \lambda)$.

   (b) $W_n = T_1 + \ldots + T_n \sim Gamma(3n, \lambda)$.

2. The distribution between any two consecutive events is exponential with rate λ . Thus the distribution between the ith and jth is the sum of $(j - i)$ exponential variates with rate λ. This is $Gamma(j - i, \lambda)$.

3. Sum of n independent χ_1^2 variables is χ_n^2 , so the answer is χ_2^2. But $\chi_2^2 = Exponential(1/2)$ (the exponential distribution with parameter 1/2, so that the process is a Poisson process.

4. (a) and (e)

5. (a) The number, X, of customers arriving in 30 seconds has a Poisson distribution with parameter $10 \times 0.5 = 5$. So $\mathrm{P}(X > 5) = 1 - \mathrm{P}(X \leq 5) = 0.3840$.

   (b) Each type of customer arrives according to a Poisson process, independently of the other two types. Thus

$$\begin{aligned} \mathrm{P}(X_A = 6 \,|\, \mu = 6)\mathrm{P}(X_B = 3 \,|\, \mu = 3)\mathrm{P}(X_C = 1 \,|\, \mu = 1) &= 0.1606 \times 0.2240 \times 0.3679 \\ &= 0.0132. \end{aligned}$$

   (c) As 20 events are known to have occurred, time is irrelevant. The number X_C of type C customers is distributed as $Binomial(20, 0.1)$ so that

$$\mathrm{P}(X_C = 1) = \binom{20}{1}(0.1)(0.9)^{19} = 0.270.$$

   (d) The probability that any customer requires a type A transaction is 0.6. Hence the probability that the first three customers are type A is $(0.6)^3 = 0.216$.

   (e) Suppose that the required time is t. The probability that at least one type A customer has arrived by time t is $(1 - e^{-6t})$ and for type B it is $(1 - e^{-3t})$. Since the two Poisson processes are independent, we must find t such that

$$(1 - e^{-6t})(1 - e^{-3t}) = 0.9.$$

   Solving this numerically gives $t = 0.794$ minutes.

6. (a)

$$\mu(t) = \int_0^t \lambda(u)\, du = \int_0^t 10(1+2u)\, du = 10(t+t^2).$$

Five minutes is 1/12 hours, so that the number of customers arriving between 10.00 and 10.05 is a Poisson process with rate $\mu(1/12) = 0.9028$. It follows that the probability that two customers have arrived by 10.05 is

$$\mu(1/12)^2 \exp(-\mu(1/12))/2 = 0.1698.$$

(b) Converting to the units we need, we want to know the probability that 6 customers arrive in the interval [0.75, 1.00]. This will have a Poisson distribution with parameter $\mu(0.75, 1) = \mu(1) - \mu(0.75) = 6.875$. From which it follows that the probability that 6 customers will arrive is

$$\exp(-6.875)(6.875)^6/6! = 0.1515.$$

(c) The number of customers arriving between these times will have a Poisson distribution with parameter $\mu(1,2) = \mu(2) - \mu(1) = 40$. The probability that more than 50 arrive can be obtained by summing the relevant probabilities from a table of the $Poisson(40)$ distribution, or by using a calculator.

(d) The cdf to the time of the first arrival is $F(t) = 1 - \exp(-10(t+t^2))$. The median of this is given by the value of t for which $1 - \exp(-10(t+t^2)) = 0.5$, yielding 0.065, so that the median time of arrival of the first customer is about 3.9 minutes after 10.

(e) The cdf of the time to the first arrival after 11 is $F(t) = 1 - \exp(-\mu(1,t))$, with $\mu(1,t) = 10(t+t^2) - 20$. Putting $F(t) = 0.95$ leads to $t = 1.097$, so that there is a probability of 0.95 that at least one customer will have arrived after 11.00 and before 11.06.

7. Let $S(t)$ be the number of events which have occurred by time t. Then,

$$\begin{aligned}
\mathrm{var}(S(t)) &= \mathrm{E}(S^2(t)) - \mathrm{E}^2(S(t)) \\
\mathrm{E}(S^2(t)) &= \sum_x \mathrm{P}(X(t) = x) \sum_x s^2 \mathrm{P}(S(t) = s \mid X(t) = x) \\
&= \sum_x \mathrm{P}(X(t) = x) \mathrm{E}(S(t)^2 \mid X(t) = x)
\end{aligned}$$

Now

$$\mathrm{E}(S^2(t)\,|\,X(t)=x) = \mathrm{E}[(Y_1+\ldots+Y_x)^2] = \mathrm{E}[T_1+T_2],$$

where $T_1 = \sum_i Y_i^2$ and $T_2 = \sum_{i,j,i\neq j} Y_iY_j$.

Now $\mathrm{E}(Y_i^2) = \sigma^2+\mu^2$, and by the independence of Y_i and Y_j, $\mathrm{E}(Y_iY_j) = \mu^2$, and there is a total of $x(x-1)$ such pairs.

Hence,

$$\mathrm{E}(S^2(t))\,|\,X(t)=x) = x(\sigma^2+\mu^2) + x(x-1)\mu^2 = x\sigma^2 + x^2\mu^2.$$

Thus,

$$\mathrm{E}(S^2(t)) = \sum_x \mathrm{P}(X(t)=x)(x\sigma^2+x^2\mu^2) = \sigma^2\mathrm{E}(X(t)) + \mu^2\mathrm{E}(X^2(t)).$$

Since $X(t)$ is Poisson, we know from coursework that $\mathrm{E}(X(t)) = \lambda t$ and that $\mathrm{E}(X^2(t)) = \lambda t + \lambda^2t^2$, so that

$$\begin{aligned}
\mathrm{E}(S^2(t)) &= \sigma^2\lambda t + \mu^2(\lambda t+\lambda^2t^2) \\
\mathrm{var}(S(t)) &= \mathrm{E}(S^2(t)) - \mathrm{E}^2(S(t)) \\
&= \sigma^2\lambda t + \mu^2(\lambda t+\lambda^2t^2) - (\mu\lambda t)^2 \\
&= \lambda t(\sigma^2+\mu^2)
\end{aligned}$$

8. For a non-homogeneous Poisson process $X(t) \sim \mathit{Poisson}(\mu(t))$ with $\mu(t) = \int_0^t \lambda(t)\,dt$. Thus, since both the mean time function and the mean variance function are equal to $\mu(t)$, the index of dispersion is 1.

9. Mean $= \mu\lambda t = 4p\lambda t$.
   Variance $= \lambda t(\sigma^2+\mu^2) = \lambda t(4p(1-p)+16p^2)$

10. Mean $= \mu\lambda t = \frac{1}{p}\lambda t$
    (Check following result)
    Variance $= \lambda t(\sigma^2+\mu^2) = \lambda t\left(\frac{1-p}{p^2}+\frac{1}{p^2}\right) = \lambda t\left(\frac{2-p}{p^2}\right)$

11. (a) When $Y \sim G_0(p)$ we have $\mu = p/q$ and $\sigma^2 = p/q^2$. Hence the index of dispersion is given by $I(t) = \mu + \sigma^2/\mu = (p+1)/q$.
    (b) When $Y \sim \mathit{Poisson}(\mu)$ we have $\mu = \sigma^2$, so that $I(t) = 1+\mu$.