M3S4/M4S4: Applied probability: 2007-8
Solutions 3: pgfs and branching processes

1.

$$\Pi_Y(s) = \mathrm{E}(s^Y) = \mathrm{E}^(s^{aX+b}) = s^b\mathrm{E}(s^{aX}) = s^b\Pi_X(s^a).$$

2.

$$\Pi(s) = \sum_{x=0}^{\infty} p(x)s^x = \sum \frac{e^{-\mu}\mu^x}{x!}s^x = e^{-\mu}\sum \frac{(\mu s)^x}{x!} = e^{-\mu}e^{\mu s} = \exp(-\mu(1-s)).$$

3. This is a $G_1(p)$ distribution, with $p(x) = q^{x-1}p$. So,

$$\Pi(s) = \sum_{x=1}^{\infty} q^{x-1}ps^x = ps\sum(qs)^{x-1} = \frac{ps}{1-qs}.$$

4.

$$\Pi(s) = \exp(-\mu(1-s)), \quad \Pi'(s) = \mu\exp(-\mu(1-s)), \quad \Pi''(s) = \mu^2\exp(-\mu(1-s)),$$

so that,

$$\Pi'(1) = \mu, \quad \Pi''(1) = \mu^2$$

and

$$\sigma^2 = \Pi''(1) + \Pi'(1) - (\Pi'(1))^2 = \mu^2 + \mu - \mu^2 = \mu.$$

5. Let $Y = \sum_{i=1}^n X_i$ where $X_i \sim Poisson(\mu_i)$, Then

$$\begin{aligned}
\Pi_Y(s) &= \Pi_1(s)\ldots\Pi_n(s) \\
&= \exp(-\mu_1(1-s))\ldots\exp(-\mu_n(1-s)) \\
&= \exp(-(\mu_1+\ldots+\mu_n)(1-s))
\end{aligned}$$

which is the pgf of a $Poisson(\mu_1+\ldots+\mu_n)$.

6.

$$Z = \sum_{i=1}^{n} X_i.$$

$$\begin{array}{ll}
N \sim Poisson(\lambda t) & \Pi_N(s) = \exp(-\lambda t(1-s)) \\
X_i \sim Binomial(1,p) & \Pi_{X_i}(s) = q + ps
\end{array}$$

$$\begin{aligned}
\Pi_Z(s) &= \Pi_N\left[\Pi_X(s)\right] \\
&= \exp(-\lambda t(1-(q+ps))) \\
&= \exp(-\lambda pt(1-s)).
\end{aligned}$$

So, Z has a $Poisson(\lambda pt)$ distribution.

7.

$$\begin{aligned}\Pi_N(s) &= \exp(-\mu(1-s)), \quad \Pi_X(s) = (q+ps)^n \\ \Pi_Z(s) &= \exp(-\mu(1-\Pi_X(s))) = \exp(-\mu(1-(q+ps)^n))) \\ \Pi_Z'(s) &= \mu np(q+ps)^{n-1}\exp(-\mu(1-(q+ps)^n)))\end{aligned}$$

So $\mathrm{E}(Z) = \Pi_Z'(1) = \mu np$.

8. A rv with $G_0(p)$ distribution has mean p/q and variance p/q^2.

$$\mu_n = \mu^n; \quad \sigma_n^2 = \begin{cases} \mu^{n-1}\sigma^2\frac{\mu^n-1}{\mu-1} & \mu \neq 1 \\ n\sigma^2 & n = 1 \end{cases}$$

	(a)	(b)	(c)
μ_5	0.031	1	32
σ_5	0.091	10	2976

9. $\Pi(s) = \exp(-0.5(1-s)) \Rightarrow \theta_1 = p(0) = \Pi(0) = \exp(-0.5) = 0.6065$.

   other relationships found from

   $\theta_n = \Pi(\theta_{n-1}) \;\Rightarrow\; \theta_2 = 0.8215,\ \theta_3 = 0.9146,\ \theta_4 = 0.9582,\ \theta_5 = 0.9793$.

10. Using $\theta_n = \Pi(\theta_{n-1})$

    (a) 0.646.

    (b) P(extinction by 5th generation) $= 0.635$.

    Hence P(extinction at 6th generation) $= 0.646 - 0.635 = 0.011$

11. pgf is $\Pi(s) = r + qs + ps^2$. Solutions of $\Pi(\theta) = \theta$ are r/p and 1.

    So

    (a) if $p > r$ extinction is not certain and its probability of occurring is r/p.

    (b) if $p = r$ extinction is certain.

    (c) if $p < r$ extinction is certain.

12.

$$\Pi(s) = \frac{q}{1-ps}.$$

$$\theta = \Pi(\theta) \Rightarrow (p\theta - q)(\theta - 1) = 0 \Rightarrow \theta = 1 \quad \text{or} \quad \theta = \frac{q}{p}.$$

So, if $p \le q$ ultimate extinction has probability 1 and if $p > q$ ultimate extinction has probability q/p.