M3S4/M4S4: Applied probability: 2007-8
Solutions 4: Random Walks

1. We need to handle cases $p = q$ and $p \neq q$ separately. Let $q_{A,j}$ and $q_{B,a-j}$ be the probability of A and B losing all their money when they start with £j and £$(a-j)$ respectively.
   To show that the match must eventually end, we must show $q_{A,j} + q_{B,a-j} = 1$.

   case 1: If the roots are distinct $(p \neq q)$

$$q_{A,j} = \frac{(q/p)^j - (q/p)^a}{1-(q/p)^a}$$

   So, by interchanging p and q and j and $a-j$, we can evaluate the probability that B will lose all his money:

$$q_{B,a-j} = \frac{(p/q)^{a-j} - (p/q)^a}{1-(p/q)^a}$$

$$\begin{aligned} q_{A,j} + q_{B,a-j} &= \frac{((q/p)^j - (q/p)^a)(1-(p/q)^a) + ((p/q)^{a-j} - (p/q)^a)(1-(q/p)^a)}{(1-(q/p)^a)(1-(p/q)^a)} \\ &= \frac{(q/p)^j - (q/p)^a - (q/p)^{j-a} + (q/p)^{a-a} + (q/p)^{j-a} - (q/p)^{-a} - (q/p)^{j-a+a} + (q/p)^{a-a}}{1-(q/p)^a - (q/p)^{-a} + (q/p)^{a-a}} \\ &= \frac{2-(q/p)^a - (q/p)^{-a}}{2-(q/p)^a - (q/p)^{-a}} = 1. \end{aligned}$$

   as required.

   case 2: If the roots are equal $(p = q)$

$$q_{A,j} + q_{B,a-j} = 1 - \frac{j}{a} + 1 - \frac{a-j}{a} = 1.$$

   as required.

2. From lectures:

$$D_j = j(a-j).$$

Setting $j = 500, a = 1000$ gives

$$D_{500} = 500(1000-500) = 250000 \text{ games}$$

3. (a)

$$\mathrm{E}(X_n) = 25(0.8-0.2) = 15, \quad \mathrm{var}(X_n) = 4 \times 25 \times 0.8 \times 0.2 = 16.$$

$$\begin{aligned} \mathrm{P}(X_{25} < -10) &\approx \mathrm{P}\left(Z < \frac{-9.5-15}{4}\right) = \Phi(-6.125) = 0 \\ \mathrm{P}(X_{25} > 10) &\approx \mathrm{P}\left(Z > \frac{9.5-15}{4}\right) = 1 - \Phi(1.38) = 1 - 0.9162 = 0.0838 \end{aligned}$$

(b)

$$\begin{aligned} 0.95 &= \mathrm{P}(-1.96 < Z < 1.96) \\ &= \mathrm{P}\left(n(p-q) - 1.96\sqrt{4npq} < X_n < n(p-q) + 1.96\sqrt{4npq}\right) \end{aligned}$$

So a suitable range is given by

$$100 \times (0.8-0.2) \pm 1.96 \times \sqrt{4 \times 100 \times 0.8 \times 0.2} = (44.32, 75.68)$$

4. (a)

$$\mathrm{E}(Z_i) = 2p-(1-p) = 3p-1, \quad \mathrm{var}(Z_i) = E(Z_i^2) - E^2(Z_i) = 4p+(1-p)-(3p-1)^2 = 9p(1-p).$$

(b)

$$\mathrm{E}(X_n) = n(3p-1), \quad \mathrm{var}(X_n) = 9np(1-p).$$

(c)

$$\mathrm{P}(X_n = k) = \begin{cases} \binom{n}{(n+k)/3} p^{(n+k)/3} q^{(2n-k)/3} & n, k \text{ s.t. } n+k \text{ is a multiple of 3, } -n \le k \le 2n \\ 0 & \text{otherwise} \end{cases}$$

(d) (i) $\mathrm{E}(X_{20}) = 20(3 \times (1/3) - 1) = 0$.

(ii) $\mathrm{var}(X_{20}) = 9 \times 20(1/3)(2/3) = 40$.

(iii) $\mathrm{P}(X_{20} = 0) = 0$.

(iv) $\mathrm{P}(X_{20} = 1) = \binom{20}{7}(1/3)^7(2/3)^{13} = 0.182$.

(v)

$$\mathrm{E}(X_{180}) = 180(3/6-1) = -90, \quad \mathrm{var}(X_{180}) = 9 \times 180 \times (1-(1/6)) = 225.$$

$$
\begin{aligned}
P(-70 < X_{180} < 70) &\approx P\left(\frac{-69.5 - E(X_{180})}{\sqrt{\text{var}(X_{180})}} < Z < \frac{69.5 - E(X_{180})}{\sqrt{\text{var}(X_{180})}}\right) \\
&= P(-10.64 < Z < -1.37) \approx \Phi(-1.37) = 1 - 0.9147 = 0.0853.
\end{aligned}
$$

5. We have

$$F(s) = 1 - (1 - 4pqs^2)^{\frac{1}{2}}.$$

The term in s^{2m} is (from problem class)

$$f_{2m}s^{2m} = \frac{1}{2m-1}\binom{2m}{m}p^m q^m s^{2m}$$

Hence, with $p = q = 1/2$,

$$f_{2m} = \frac{1}{2m-1}\binom{2m}{m}\left(\frac{1}{2}\right)^{2m}$$

giving

(a) $f_4 = 0.125$.

(b) $f_{10} = 0.027$.