M3S4/M4S4: Applied probability: 2007-8
Solutions 5: Markov Chains

1. (a)

$$P = \begin{array}{c} 1 \\ 2 \\ 3 \\ \vdots \\ \vdots \\ a-1 \\ a \end{array} \begin{pmatrix} q & p & 0 & 0 & 0 & \dots & 0 \\ q & 0 & p & 0 & 0 & \dots & 0 \\ 0 & q & 0 & p & 0 & \dots & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \\ \vdots & & & \ddots & \ddots & \ddots & \\ 0 & \dots & & 0 & q & 0 & p \\ 0 & \dots & & 0 & 0 & 0 & 1 \end{pmatrix}$$

(b)

$$P = \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{6} & \frac{5}{6} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{6} & \frac{4}{6} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{3}{6} & \frac{3}{6} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{4}{6} & \frac{2}{6} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{5}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

2. $\mathrm{P}(X_n = j) = a_j^{(n)}$, then

$$\begin{aligned} \mathrm{P}(X_{n+1} = j) &= \sum_i \mathrm{P}(X_{n+1} = j \,|\, X_n = i)\mathrm{P}(X_n = i) \\ \Rightarrow a_j^{(n+1)} &= \sum_i a_i^{(n)} p_{ij} \quad \Rightarrow \boldsymbol{a}^{(n+1)} = \boldsymbol{a}^{(n)} P. \end{aligned}$$

3. We have

$$\begin{aligned} \pi_0 &= \pi_0(1-\alpha) + \pi_1\beta \\ \pi_0 + \pi_1 &= 1 \end{aligned}$$

So,

$$\pi_0 = \frac{\beta}{\alpha+\beta} \qquad \pi_1 = \frac{\alpha}{\alpha+\beta}$$

4. From $\boldsymbol{\pi} = \boldsymbol{\pi} P$ and $\sum \pi_j = 1$, we have

$$\begin{aligned}\pi_0 &= 0.5\pi_0 + 0.2\pi_1 + 0.1\pi_2 \\ \pi_1 &= 0.4\pi_1 + 0.6\pi_1 + 0.2\pi_2 \\ 1 &= \pi_0 + \pi_1 + \pi_2\end{aligned}$$

solving these equations gives $\pi_0 = 0.235$.

5. Classes are

$$\begin{array}{rl} \{0\} \quad \{a\} & \text{closed} \\ \{1, 2, 3, \dots a-1\} & \text{open} \end{array}$$

6. (a) i. only 1 closed communicating class: $\{3\}$, so yes.
   ii. $\boldsymbol{\pi} = (0\ 0\ 0\ 1)$.
   (b) i. 2 closed communicating classes $\{0, 1\}$ and $\{2, 3\}$ so no.
   ii. $\boldsymbol{\pi} = (a\ a\ b\ b)$ with $a + b = 0.5$.
   (c) i. only 1 closed communicating class: $\{0, 1, 2\}$ so yes.
   ii. $\boldsymbol{\pi} = (\frac{1}{4}\ \frac{5}{12}\ \frac{1}{3}\ 0)$.

7. (a) i. irreducible period 2.
   ii. unique stationary distribution exists (not limiting).
   (b) i. irreducible aperiodic.
   ii. unique stationary distribution exists (also limiting).
   (c) i. Classes: $\{0, 4\}$ period 2 closed
   $\{1\}$ aperiodic open
   $\{2, 3\}$ aperiodic open
   ii. a unique stationary distribution exists (not limiting).
   (d) i. Classes: $\{0, 3\}$ aperiodic closed
   $\{2, 4, 6\}$ period 3 closed
   $\{1, 5\}$ aperiodic open
   ii. more than one closed class - a unique stationary distribution does not exist.

8. (a)

$$\begin{aligned} f_{00}^{(n)} &= \text{P(first return to 0 in } n \text{ steps)} \\ &= p^{n-1}q \end{aligned}$$

(b) Must show $f_{00} = 1$

$$\begin{aligned} f_{00} &= \sum_{n=1}^{\infty} f_{00}^{(n)} \\ &= q + pq + p^2q + p^3q + \ldots \\ &= \frac{q}{1-p} = \frac{q}{q} = 1, \end{aligned}$$

so state 0 is recurrent.

9. The mean return time to state 0 is $\mu_0 = F'_{00}(1)$, where

$$F_{00}(s) = \sum_{n=0}^{\infty} f_{00}^{(n)} s^n$$

We have (from lectures)

$$\begin{aligned} f_{00}^{(1)} &= 1 - \alpha \\ f_{00}^{(n)} &= \alpha\beta(1-\beta)^{n-2} \end{aligned}$$

So,

$$\begin{aligned} F_{00}(s) &= (1-\alpha)s + \sum_{n=2}^{\infty} \alpha\beta(1-\beta)^{n-2}s^n \\ &= (1-\alpha)s + s^2 \sum_{n=0}^{\infty} \alpha\beta(1-\beta)^n s^n \\ &= (1-\alpha)s + \frac{s^2\alpha\beta}{1-(1-\beta)s} \\ \Rightarrow F'_{00}(s) &= (1-\alpha) + \frac{2\alpha\beta s(1-(1-\beta)s) + \alpha\beta s^2(1-\beta)}{(1-(1-\beta)s)^2} \\ F'_{00}(1) &= (1-\alpha) + \frac{2\alpha\beta^2 + \alpha\beta(1-\beta)}{\beta^2} \\ &= \frac{\alpha+\beta}{\beta} \end{aligned}$$