M3S4/M4S4: Applied probability: 2007-8
Solutions 6: Continuous time Markov processes

1. We have,

$$Q = \begin{pmatrix} 0 & 0 & 0 & & & & \\ \nu & -\nu & 0 & 0 & & & \\ 0 & 2\nu & -2\nu & 0 & 0 & & \\ & & & & & & \\ & & & i\nu & -i\nu & 0 & \\ & & & & & & \end{pmatrix}$$

We calculate $p_{0j}(t)$, $0 < j$, by solving

$$\frac{d}{dt}P(t) = P(t)Q$$

i.e.

$$\begin{pmatrix} & \\ \frac{d}{dt}p_{ij}(t) & \\ & \end{pmatrix} = \begin{pmatrix} & \\ p_{ij}(t) & \\ & \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & & & & \\ \nu & -\nu & 0 & 0 & & & \\ 0 & 2\nu & -2\nu & 0 & 0 & & \\ & & & & & & \\ & & & i\nu & -i\nu & 0 & \\ & & & & & & \end{pmatrix}$$

Giving

$$\frac{d}{dt}p_{ij}(t) = -j\nu p_{ij}(t) + (j+1)\nu p_{i,j+1}(t)$$

Multiply by s^j and sum over j to give

$$\frac{\partial}{\partial t}\sum_{j=0}^{\infty} p_{ij}(t)s^j = -\nu \sum_{j=1}^{\infty} jp_{ij}(t)s^j + \nu \sum_{j=0}^{\infty}(j+1)p_{i,j+1}(t)s^j$$

Note that,

$$\frac{\partial}{\partial s}\Pi_i(s,t) = \frac{\partial}{\partial s}\sum_{j=0}^{\infty} p_{ij}(t)s^j = \sum_{j=1}^{\infty} jp_{ij}(t)s^{j-1}.$$

So, we have

$$\begin{aligned} \frac{\partial}{\partial t}\sum_{j=0}^{\infty} p_{ij}(t)s^j &= -\nu s\sum_{j=1}^{\infty} jp_{ij}(t)s^{j-1} + \nu \sum_{j=1}^{\infty} jp_{ij}(t)s^{j-1} \\ \frac{\partial}{\partial t}\Pi_i(s,t) &= \nu(1-s)\frac{\partial}{\partial s}\Pi_i(s,t) \end{aligned}$$

as required.

2. From lectures the differential difference equations for general birth and death process are given by,

$$\begin{aligned}\frac{d}{dt}p_0(t) &= -\beta_0 p_0(t) + \nu_1 p_1(t) \\ \frac{d}{dt}p_j(t) &= \beta_{j-1}p_{j-1}(t) - (\beta_j + \nu_j)p_j(t) + \nu_{j+1}p_{j+1}(t) \qquad j \geq 1\end{aligned}$$

(a) When $\beta_n = \lambda,\ \nu_n = 0$, we have for $j > 0$:

$$\frac{d}{dt}p_j(t) = \lambda p_{j-1}(t) - \lambda p_j(t).$$

(b) When $\beta_n = \beta n,\ \nu_n = 0$, we have for $j > 0$:

$$\frac{d}{dt}p_j(t) = (j-1)\beta p_{j-1}(t) - j\beta p_j(t).$$

3. (a) $\frac{1}{\lambda}$.

   (b) $\frac{3}{\lambda} + \frac{2}{\nu}$.

   (c) $\frac{1}{\lambda^2}$.

   (d) $\frac{3}{\lambda^2} + \frac{2}{\nu^2}$.

4. (a)

$$\begin{aligned}\text{P(wake during } \delta t) &= \beta\delta t + o(\delta t) \\ \text{P(sleep during } \delta t) &= \nu\delta t + o(\delta t)\end{aligned}$$

If there are i awake then $N - i$ are asleep.

$$\begin{aligned}i \to i+1 &= (N-i)\beta & &\text{(one of the } (N-i) \text{ wake)} \\ i \to i-1 &= \nu i & &\text{(one of the } i \text{ sleep)}\end{aligned}$$

Giving

$$Q = \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ \vdots \\ N \end{array} \begin{pmatrix} -N\beta & N\beta & 0 & 0 & & \\ \nu & -\nu - \beta(N-1) & \beta(N-1) & 0 & & \\ 0 & 2\nu & -2\nu - \beta(N-2) & \beta(N-2) & & \\ & & \ddots & \ddots & \ddots & \\ & & & & & \\ & & & & N\nu & -N\nu \end{pmatrix}$$

(b) For the stationary distribution, solve $\boldsymbol{\pi} Q = 0$, from notes, for a general birth and death process we have

$$\begin{aligned}
\pi_n &= \frac{\beta_{n-1}\beta_{n-2}\dots\beta_0}{\nu_n\dots\nu_1}\pi_0 \quad n \geq 1 \\
\pi_n &= \frac{(N\beta)(\beta(N-1))(\beta(N-2))\dots(\beta(N-(n-1)))}{\nu(2\nu)(3\nu)\dots(n\nu)}\pi_0 \\
&= \frac{\beta^n}{\nu^n}\frac{N!}{n!(N-n)!}\pi_0 \\
\pi_n &= \binom{N}{n}\left(\frac{\beta}{\nu}\right)^n \pi_0 \quad n \geq 1
\end{aligned}$$

also, $\sum_{n=1}^{N} \pi_n = 1$, giving

$$\pi_0 = \frac{1}{1 + \sum_{n=1}^{N} \binom{N}{n}\left(\frac{\beta}{\nu}\right)^n}$$

(c) For one individual we have

$$Q = \begin{array}{c} s \\ w \end{array} \begin{pmatrix} -\beta & \beta \\ \nu & -\nu \end{pmatrix}$$

(d) From the Forward Differential Equations:

$$\begin{aligned}
\frac{d}{dt}P(t) &= P(t)Q \\
\frac{d}{dt}(1 - p_{sw}(t)) &= -\beta(1 - p_{sw}(t)) + \nu p_{sw}(t) \\
\Rightarrow -\frac{d}{dt}p_{sw}(t) &= p_{sw}(t)(\beta + \nu) - \beta \\
\Rightarrow \int \frac{dp_{sw}(t)}{p_{sw}(t)(\beta+\nu) - \beta} &= \int -1 dt \\
\Rightarrow \frac{\log(p_{sw}(t)(\beta+\nu) - \beta)}{\beta + \nu} &= -t + c
\end{aligned}$$

From $p_{sw}(0) = 0$ we find $c = \frac{\log(-\beta)}{(\beta+\nu)}$ giving

$$p_{sw}(t) = \frac{\beta}{\beta+\nu}(1 - e^{-(\beta+\nu)t})$$

Also

$$\frac{d}{dt}(1 - p_{ww}(t)) = -\beta(1 - p_{ww}(t)) + \nu p_{ww}(t)$$

same solution as above, except we have $p_{ww}(0) = 1$ giving

$$p_{ww}(t) = \frac{\beta + \nu e^{-(\beta+\nu)t}}{\beta + \nu}$$

(e) From the hint, we have

$$\begin{aligned}
\mathrm{E}(X_m(t)) &= mp_{ww} + (N-m)p_{sw} \\
&= m\left(\frac{\beta + \nu e^{-(\beta+\nu)t}}{\beta+\nu}\right) + (N-m)\left(\frac{\beta}{\beta+\nu}(1-e^{-(\beta+\nu)t})\right) \\
&= me^{-(\beta+\nu)t} + \frac{N\beta}{\beta+\nu}(1-e^{-(\beta+\nu)t})
\end{aligned}$$

5. (a) The Backward Differential Equations are given by,

$$\frac{d}{dt}P(t) = QP(t)$$

For the linear birth and death process we have

$$Q = \begin{pmatrix}
0 & 0 & 0 & \dots & & \\
\nu & -(\nu+\beta) & \beta & 0 & & \\
0 & 2\nu & -(2\nu+2\beta) & 2\beta & 0 & \\
0 & 0 & 3\nu & -(3\nu+3\beta) & 3\beta & \\
 & & & \ddots & \ddots & \ddots \\
 & & & & &
\end{pmatrix}.$$

Thus,

$$\begin{aligned}
\frac{d}{dt}p_{0j}(t) &= 0 \quad \forall j \\
\frac{d}{dt}p_{ij}(t) &= i\nu p_{i-1,j}(t) - i(\nu+\beta)p_{ij}(t) + i\beta p_{i+1,j}(t) \quad \forall j \ (i>0)
\end{aligned}$$

Multiply by s^j and sum over j to give

$$\frac{\partial}{\partial t}\Pi_i(s,t) = i\nu\Pi_{i-1}(s,t) - i(\nu+\beta)\Pi_i(s,t) + i\beta\Pi_{i+1}(s,t) \quad i>0.$$

(b) $\Pi_i(s,t)$ is the pgf of $X_i(t)$ – the number of individuals at time t given that $X(0) = i$. We can write

$$X_i(t) = \underbrace{X_1(t) + X_1(t) + \ldots + X_1(t)}_{i \text{ times}}$$

(a colony of size i can be thought of as i colonies of size 1).

So, by standard pgf results, we have

$$\Pi_i(t) = [\Pi_1(t)]^i$$

(c) When $i = 1$,

$$\frac{\partial}{\partial t}\Pi_1(s,t) = \nu\Pi_0(s,t) - (\nu+\beta)\Pi_1(s,t) + \beta\left[\Pi_1(s,t)\right]^2.$$

$$\begin{aligned}\Pi_0(s,t) &= p_{00}(t) + p_{01}(t)s + p_{02}(t)s^2 + \dots \\ &= 1 \qquad (\text{as } p_{00}(t) = 1 \text{ and } p_{0j}(t) = 0\ \forall\, j).\end{aligned}$$

Let $y = \Pi_1(s,t)$,

$$\frac{dy}{\partial t} = \nu - (\nu+\beta)y + \beta y^2 = (\beta y - \nu)(y-1).$$

Case $\beta = \nu$:

$$\begin{aligned}\frac{dy}{dt} &= \beta(y-1)^2 \\ \int \frac{dy}{(y-1)^2} &= \int \beta\, dt \\ \frac{-1}{y-1} &= \beta t + c \Rightarrow y = 1 - \frac{1}{\beta t + c} \\ \Rightarrow \Pi_1(s,t) &= \frac{\beta t + c - 1}{\beta t + c}\end{aligned}$$

Initial condition $\Pi_1(s,0) = s$ gives $s = (c-1)/c$ and

$$\Pi_1(s,t) = \frac{\beta t + s(1-\beta t)}{\beta t + 1 - s\beta t}$$

which agrees with the lecture notes.

Case $\beta \neq \nu$:

$$\begin{aligned}\frac{dy}{dt} &= (\beta y - \nu)(y-1) \\ \int \left(\frac{\beta}{\nu - \beta y} - \frac{1}{1-y}\right) dy &= \int (\beta - \nu) dt \\ -\log(\nu - \beta y) + \log(1-y) &= (\beta - \nu)t + c\end{aligned}$$

$$\Pi_1(s,0) = s \Rightarrow c = -\log(\nu - \beta s) + \log(1-s)$$

So (after some algebra!)

$$\Pi_1(s,t) = \frac{\nu(1-s) - (\nu - \beta s)e^{(\nu-\beta)t}}{\beta(1-s) - (\nu-\beta s)e^{(\nu-\beta)t}}$$

again in accordance with lecture notes.