M3S4/M4S4 Revision Notes

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1. Poisson Process

2. Branching Processes

3. Random Walks

4. Discrete time Markov Chains

5. Continuous time Markov processes – Birth, Death and Immigration

1 Poisson Process

Recall the **Bernoulli process**: sequence of independent Bernoulli trials, same p.

Poisson Process: continuous time analogue.

AXIOMS:

I P(exactly 1 event in a time interval of length δt) $= \lambda\delta t + o(\delta t)$

II P(two or more events in a time interval of length δt) $= o(\delta t)$

III Occurrence of events after time t is independent of events before time t

number of events by time $t = X(t) \sim Poisson(\lambda t)$.

Distribution of time, T, between $(k-1)$th and kth $\Rightarrow \mathrm{P}(T \leq t) = 1 - e^{-\lambda t}$.

$\Rightarrow T \sim Exponential(\lambda)$.

Stationary: distribution of number of events in $(u, u+t]$ is same as distribution in $(0, t]$ for all $t, u > 0$.

Non-homogeneous Poisson: $\lambda = \lambda(t)$,

$\Rightarrow$ Number of events in $(0, t] = X(t) \sim Poisson(\int_0^t \lambda(t)\, dt)$.

Compound Poisson: Each Poisson event associated with Y "occurrences", Y a random variable.

$$\mathrm{E}(X) = E_Y\left[E(X \mid Y)\right]$$

Doubly Stochastic: $\lambda(t)$ is a random variable.

Deterministic model: number of events in interval of length $t+\delta t$ is approximated by expected value - formulate differential equation for $D(t)$ – number of events in $(0, t]$ in deterministic approximation.

Mean Time Function: $\mu(t) = \mathrm{E}(X(t))$.

Variance Time Function: $\sigma^2(t) = \mathrm{var}(X(t))$.

Index of Dispersion: $I(t) = \frac{\sigma^2(t)}{\mu(t)}$

2 Branching Processes

Organise by **generations**: Discrete time.

If P(no offspring) $\neq 0$ there is a probability that the process will die out.

Let X = number of offspring of an individual

$$p(x) = \mathrm{P}(X = x) = \quad \text{"offspring prob. function"}$$

Galton-Watson process:

(i) p same for all individuals

(ii) individuals reproduce independently

Define:

$$\begin{aligned} Z_n &= \text{number of individuals at time } n \quad (\text{start with } Z_0 = 1) \\ T_n &= \text{total number born up to and including generation } n \end{aligned}$$

Probability generating functions:

The p.g.f. is

$$\Pi_X(s) = \mathrm{E}(s^X) = \sum_{x=0}^{\infty} p(x) s^x$$

$$\begin{aligned} \text{Note:} \quad \Pi(0) &= p(0) \\ \Pi(1) &= \sum p(x) = 1 \end{aligned}$$

$$E(X) = \mu = \Pi^{'}(1); \quad \mathrm{var}(X) = \sigma^2 = \Pi^{''}(1) + \mu - \mu^2$$

Sums of a random number of rvs

$$Z = X_1 + X_2 + \ldots + X_N$$

i.e. Z is the sum of N independent discrete rvs $X_1, X_2, \ldots, X_N$. If X_i has range $\{0, 1, 2, \ldots\}$ and pgf $\Pi_X(s)$, N is a rv with range $\{0, 1, 2, \ldots\}$ and pgf $\Pi_N(s)$ then

$$\Pi_Z(s) = \Pi_N\left[\Pi_X(s)\right]$$

and Z has a compound distribution.

For nth generation, we have

$$Z_n = X_1 + X_2 \ldots + X_{Z_{n-1}},$$

where X_i is the number born to the ith member of generation $n-1$. so,

$$\Pi_n(s) = \Pi_{n-1}\left[\Pi(s)\right]$$

where $\Pi_n(s)$ is the pgf of Z_n and $\Pi(s)$ is the offspring pgf.
Given $\mathrm{E}(X) = \mu$ and $\mathrm{var}(X) = \sigma^2$, use pgfs to derive:

$$\begin{aligned} \mathrm{E}(Z_n) &= \mu_n = \mu^n \\ \mathrm{var}(Z_n) &= \sigma_n^2 = \begin{cases} \mu^{n-1}\sigma^2 \dfrac{1-\mu^n}{1-\mu} & \mu \neq 1 \\ n\sigma^2 & \mu = 1 \end{cases} \end{aligned}$$

Probability of ultimate extinction, θ^*

Must have $\mathrm{P}(X=0) = p(0) \neq 0$.

1. $\mu \leq 1 \Rightarrow \theta^* = 1 \Rightarrow$ ultimate extinction certain.

2. $\mu > 1 \Rightarrow \theta^* < 1 \Rightarrow$ ultimate extinction not certain

$$\theta^* = \text{smallest positive solution of } \theta = \Pi(\theta)$$

Hint: remember $\theta = 1$ is always a solution.

3 Random Walks

Consider a particle at some position on a line, moving with the following transition probabilities:

- with prob p it moves 1 unit to the right.
- with prob q it moves 1 unit to the left.
- with prob r it stays where it is.

Position at time n is given by,

$$X_n = Z_1 + \ldots + Z_n \qquad Z_n = \begin{cases} +1 \\ -1 \\ \;\;0 \end{cases}$$

random walks satisfy the Markov property.
i.e. the distribution of X_n is determined by the value of X_{n-1}
Looked at reflecting and absorbing barriers.

Gambler's ruin

Two players A and B.
A starts with £j, B with £$(a-j)$.
Play a series of indep. games until one or other is ruined.
Z_i = amount A wins in ith game = ± 1.

$$\mathrm{P}(Z_i = 1) = p \quad \mathrm{P}(Z_i = -1) = 1 - p = q.$$

Stop if $X_{n-1} = 0$ (A ruined) or $X_{n-1} = a$: (A wins and B ruined).

Unrestricted RW with $p+q=1$

– recurrence (persistence) i.e. return to origin certain

– transience i.e. pos prob of never returning

Position after n steps

– use binomial link to find exact distribution

– CLT for large n

– find mean/variance then $X_n \sim N(\mathrm{E}(X_n), \mathrm{var}(X_n))$ approx.

Return Probabilities

$$\begin{aligned} f_n &= \mathrm{P}(\underline{\text{FIRST}} \text{ return at } n) \\ u_n &= \mathrm{P}(\text{some return at } n) \end{aligned}$$

$$F(s) = \sum_0^\infty f_n s^n \qquad U(s) = \sum_0^\infty u_n s^n \qquad U(s) = 1 + F(s)U(s)$$

RW recurrent iff $\sum u_n = \infty, \sum f_n = 1$.

Probability of ultimate ruin

Define: R_j = event A loses if he starts with £j, W = event A wins first game.

TRICK: condition on first game

$$\mathrm{P}(R_j) = \mathrm{P}(R_j \,|\, W)\mathrm{P}(W) + \mathrm{P}(R_j \,|\, \bar{W})\mathrm{P}(\bar{W})$$

form recurrence relation (difference equation)and solve to calculate $q_j = \mathrm{P}(R_j)$, use same trick to calculate expected duration of game.

4 Markov Chains

$$\mathrm{P}(X_{n+1} = j \,|\, X_n = i \text{ and } A) = \mathrm{P}(X_{n+1} = j \,| X_n\, i) = p_{ij}$$

where A is any event depending only on $\{X_{n-1}, X_{n-2}, \ldots, X_0\}$

TRANSITION MATRIX P with element p_{ij} and $\sum_j p_{ij} = 1 \; \forall i$

n-step transition $P^{(n)}$

Chapman-Kolmogorov equations:

$$\begin{aligned} p_{ij}^{(m+n)} &= \sum_k p_{ik}^{(m)} p_{kj}^{(n)} \\ \Rightarrow P^{(n)} &= P^n \end{aligned}$$

Sometimes MCs converge to an equilibrium distribution

$$\boldsymbol{\pi} = (\pi_1, \ldots, \pi_s); \qquad \boldsymbol{\pi} = \boldsymbol{\pi} P; \quad \pi_j \geq 0 \;\forall\, j; \quad \sum \pi_j = 1$$

Communicating Classes

$i \leftrightarrow j$ for all states in a class (there is a path of non-zero probability from i to j and back).

closed class – impossible to leave

one class – irreducible

Every Markov Chain with a finite state space has a unique stationary distribution unless the chain has two or more closed communicating classes.

Periodicity

The periodicity of state i is defined as

$$\gcd\{n \geq 1 \,:\, p_{ii}^{(n)} > 0\}.$$

All states in the same c.c. are either:

– aperiodic (period 1)

– periodic with same period

MC which is:

- irreducible
- finite state space
- aperiodic

Then $p_{ij}^{(n)} \to \pi_j,\ j = 1, \ldots, s$ as $n \to \infty$

where $\boldsymbol{\pi}$ is the unique stationary distribution, which is also limiting.

Return probabilities and return times c.f RWs

mean first return time μ_i:

NULL RECURRENT: infinite mean recurrence time

POSITIVE RECURRENT: finite mean recurrence time

For a recurrent, irreducible, aperiodic MC:

$$\lim_{n\to\infty} = \frac{1}{\mu_i} = \pi_i$$

5 Continuous time Markov Chains

$X(t)$ is the state of the chain at time t. Transition matrix $P(t)$ with elements $p_{ij}(t)$ where

$$p_{ij}(t) = \mathrm{P}(X(t) = j \mid X(0) = i)$$

Define

$$q_{ij} = \frac{d}{dt} p_{ij}(t)\Big|_{t=0}$$

$$p_{ij}(\delta t) = \begin{cases} 1 + \delta t q_{ii} + o(\delta t) & i = j \\ \delta t q_{ij} + o(\delta t) & i \neq j \end{cases}$$

Description of process:

remains in state i for a period exponentially distributed with mean $-1/q_{ii}$, and then jumps to another state. The state is j ($\neq i$) with probability $-q_{ij}/q_{ii}$....

Forward and Backward Equations

$$\frac{\mathrm{d}}{\mathrm{d}t} P(t) = P(t)Q \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t} p_{ij}(t) = \sum_k p_{ik}(t) q_{kj} \ \ \forall i, j \ \ \text{(FORWARD EQNS)}$$
$$\frac{\mathrm{d}}{\mathrm{d}t} P(t) = QP(t) \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t} p_{ij}(t) = \sum_k q_{ik} p_{kj}(t) \ \ \forall i, j \ \ \text{(BACKWARD EQNS)}$$

Stationary Distribution

$$\boldsymbol{\pi} = \boldsymbol{\pi} P(t) \text{ or } \boldsymbol{\pi} Q = \mathbf{0}; \quad \sum_j \pi_j = 1, \quad \pi_j \geq 0 \ \forall j$$

For an irreducible process $\boldsymbol{\pi}$ is unique if it exists and if it does exist then the process is positive persistent.