EE2: Green's, Divergence & Stokes' Theorems plus Maxwell's Equations

Green's Theorem in a plane: Let $P(x,y)$ and $Q(x,y)$ be arbitrary functions in the x,y plane in which there is a *closed* boundary C enclosing[1] a region R. Green's Theorem connects behaviour at the boundary with what is happening inside

$$\oint_C (Pdx + Qdy) = \int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy \tag{1}$$

In two-dimensions (2D), Green's Theorem can be converted into 2D-versions of the Divergence and Stokes' Theorems respectively. To do this think of a point on the closed boundary C and draw the unit normal vector $\hat{\boldsymbol{n}}$ and the unit tangent vector $\hat{\boldsymbol{t}}$. As I showed in the lectures

$$\hat{\boldsymbol{t}} = \frac{d\boldsymbol{r}}{ds} = \hat{\boldsymbol{\imath}}\frac{dx}{ds} + \hat{\boldsymbol{\jmath}}\frac{dy}{ds} \tag{2}$$

where ds is a small element of arc-length. Obviously, $\hat{\boldsymbol{t}} \cdot \hat{\boldsymbol{n}} = 0$ so we can work out $\hat{\boldsymbol{n}}$ from[2] (2)

$$\hat{\boldsymbol{n}} = \hat{\boldsymbol{\imath}}\frac{dy}{ds} - \hat{\boldsymbol{\jmath}}\frac{dx}{ds} \tag{3}$$

The Divergence Theorem: Define the 2D-vector

$$\boldsymbol{u}(x,y) = \hat{\boldsymbol{\imath}}Q(x,y) - \hat{\boldsymbol{\jmath}}P(x,y) \tag{4}$$

which means that Green's Theorem in (1) converts to the 2D-Divergence Theorem (also known as Gauss' Theorem)

$$\oint_C \boldsymbol{u} \cdot \hat{\boldsymbol{n}}\, ds = \int\int_R \operatorname{div} \boldsymbol{u}\, dxdy. \tag{5}$$

The 3D-version uses an arbitrary $3D$ vector field $\boldsymbol{u}(x,y,z)$ that lives in some finite, simply connected volume V whose surface is S: dA is some small element of area on the curved surface S

$$\int\int_S \boldsymbol{u} \cdot \hat{\boldsymbol{n}}\, dA = \int\int\int_V \operatorname{div} \boldsymbol{u}\, dV. \tag{6}$$

Stokes' Theorem: Define the 2D-vector

$$\boldsymbol{v}(x,y) = \hat{\boldsymbol{\imath}}P(x,y) + \hat{\boldsymbol{\jmath}}Q(x,y) \tag{7}$$

Note that $\hat{\boldsymbol{k}} \cdot \operatorname{curl} \boldsymbol{v} = Q_x - P_y$ and $Pdx + Qdy = \boldsymbol{v} \cdot d\boldsymbol{r}$. Then Green's Theorem in (1) converts to 2D-Stokes' Theorem

$$\oint_C \boldsymbol{v} \cdot d\boldsymbol{r} = \oint_C \boldsymbol{v} \cdot \hat{\boldsymbol{t}}\, ds = \int\int_R \hat{\boldsymbol{k}} \cdot \operatorname{curl} \boldsymbol{v}\, dxdy. \tag{8}$$

The 3D-version of Stokes' Theorem uses an arbitrary $3D$ vector field $\boldsymbol{v}(x,y,z)$ that lives in V

$$\oint_C \boldsymbol{v} \cdot d\boldsymbol{r} = \int\int_S \hat{\boldsymbol{n}} \cdot \operatorname{curl} \boldsymbol{v}\, dA. \tag{9}$$

$\hat{\boldsymbol{n}}$ is the unit normal vector to the complete surface S of the volume V. For Stokes' Theorem, C is some closed circuit enscribed on the surface of the volume and S is the surface of the 'cap' above C.

[1] $P(x,y)$ and $Q(x,y)$ need to be continuous and the two partial derivatives on the RHS of (1) need to exist everywhere in R.

[2] A $\pm$ sign in front of (3) would account for outer and inner normals.

Maxwell's Equations

Now consider a $3D$ volume with a closed circuit C on its surface. Let's use the $3D$ versions of the Divergence and Stokes' Theorems to derive some relationships between various electro-magnetic variables.

1. Using the charge density ρ, the total charge within the volume V must be equal to surface area integral of the electric flux density $\boldsymbol{D}$ through the surface S (recall that $\boldsymbol{D} = \epsilon \boldsymbol{E}$ where $\boldsymbol{E}$ is the electric field).

$$\int\int\int_V \rho \, dV = \int\int_S \boldsymbol{D} \cdot \hat{\boldsymbol{n}} \, dA \tag{10}$$

   Using the 3D-Divergence Theorem (6) above on the RHS of (10

$$\int\int\int_V \rho \, dV = \int\int\int_V \operatorname{div} \boldsymbol{D} \, dV. \tag{11}$$

   Hence we have the first of Maxwell's equations

$$\operatorname{div} \boldsymbol{D} = \rho. \tag{12}$$

   This is also known as Gauss's Law.

2. Following the above in the same manner for the magnetic flux density $\boldsymbol{B}$ (recall that $\boldsymbol{B} = \mu \boldsymbol{H}$ where $\boldsymbol{H}$ is the magnetic field) but noting that there are no magnetic sources (so $\rho_{mag} = 0$), we have the 2nd of Maxwell's equations

$$\operatorname{div} \boldsymbol{B} = 0. \tag{13}$$

3. Faraday's Law says that the rate of change of magnetic flux linking a circuit C is proportional to the electromotive force (in the negative sense). Mathematically this is expressed as

$$\frac{d}{dt}\int\int_S \boldsymbol{B} \cdot \hat{\boldsymbol{n}} \, dA = -\oint_C \boldsymbol{E} \cdot d\boldsymbol{r} \tag{14}$$

   Using 3D-Stokes' Theorem (9) on the RHS of (13), and taking the time derivative through the surface integral (thereby making it a partial derivative) we have the 3rd of Maxwell's equations

$$\operatorname{curl} \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = 0 \tag{15}$$

4. Ampere's (Biot-Savart) Law expressed mathematically (the line integral of the magnetic field around a circuit C is equal to the current enclosed) is

$$\int\int_S \boldsymbol{J} \cdot \hat{\boldsymbol{n}} \, dA = \oint_C \boldsymbol{H} \cdot d\boldsymbol{r} \tag{16}$$

   where $\boldsymbol{J}$ is the current density and $\boldsymbol{H}$ is the magnetic field. Using 3D-Stokes' Theorem (9) on the RHS we find that we have $\operatorname{curl} \boldsymbol{H} = \boldsymbol{J}$ and therefore $\operatorname{div} \boldsymbol{J} = 0$, which is inconsistent with the first three of Maxwell's equations. Why? The continuity equation for the total charge is

$$\frac{d}{dt}\int\int\int_V \rho \, dV = -\int\int_S \boldsymbol{J} \cdot \hat{\boldsymbol{n}} \, dA. \tag{17}$$

Using the Divergence Theorem on the LHS we obtain

$$\operatorname{div} \boldsymbol{J} + \frac{\partial \rho}{\partial t} = 0. \tag{18}$$

If $\operatorname{div} \boldsymbol{J} = 0$ then ρ would have to be independent of t. To get round this problem we use Gauss's Law $\rho = \operatorname{div} \boldsymbol{D}$ expressed above to get

$$\operatorname{div} \left\{ \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} \right\} = 0 \tag{19}$$

and so this motivates us to replace $\boldsymbol{J}$ in $\operatorname{curl} \boldsymbol{H} = \boldsymbol{J}$ by $\boldsymbol{J} + \partial \boldsymbol{D}/\partial t$ giving the 4th of Maxwell's equations

$$\operatorname{curl} \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}. \tag{20}$$

5. We finally note that because $\operatorname{div} \boldsymbol{B} = 0$ then $\boldsymbol{B}$ is a solenoidal vector field: there must exist a vector potential $\boldsymbol{A}$ such that $\boldsymbol{B} = \operatorname{curl} \boldsymbol{A}$. Using this in the 3rd of Maxwell's equations, we have

$$\operatorname{curl} \left[\boldsymbol{E} + \frac{\partial \boldsymbol{A}}{\partial t} \right] = 0 \tag{21}$$

This means that there must also exist a scalar potential ϕ that satisfies

$$\boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t} - \nabla \phi. \tag{22}$$